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DANTZIG WOLFE DECOMPOSITION

WE WANT TO SOLVE THE BLOCK ANGULAR LP (26.8) ON PAGE 435 OF CHVATAL

THE LP IS

$$\text{Max } c^T x, \quad \text{s.t. } Ax = b, \ x \geq 0$$

where

$$A = \left[\begin{array}{cccc|cccc|cc} 1 & 2 & 0 & 0 & 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 2 & 3 & 1 & 0 & & & & & & \\ 1 & 1 & 0 & 1 & & & & & & \\ \hline & & & & 1 & 3 & 1 & 0 & & \\ & & & & 2 & 1 & 0 & 1 & & \end{array}\right], \quad b = \begin{bmatrix} 225 \\ 114 \\ 120 \\ 50 \\ 150 \\ 100 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c} A_1 & A_2 & D \\ \hline B_1 & & \\ \hline & B_2 & \end{array}\right]$$

and $c = [3 \ 4 \ 0 \ 0 \ 4 \ 7 \ 0 \ 0 \ 0 \ 0]^T \longrightarrow (1)$

The LP has the following structure

$$\text{Max } c_1^T x_1 + c_2^T x_2 + d^T w$$

$$\begin{aligned} \text{s.t. } \ A_1 x_1 + A_2 x_2 + Dw &= b \longrightarrow (2) \\ B_1 x_1 \qquad\qquad &= d_1 \\ B_2 x_2 &= d_2 \end{aligned}$$

$x_1 \geq 0, \ x_2 \geq 0, \ w \geq 0$

Let us develop our decomposition algorithm on LP (2)

Define

$$P_1 = \left\{ x_1 \;\middle|\; \begin{array}{l} B_1 x_1 = d_1 \\ x_1 \geq 0 \end{array} \right\}$$

$$P_2 = \left\{ x_2 \;\middle|\; \begin{array}{l} B_2 x_2 = d_2 \\ x_2 \geq 0 \end{array} \right\}$$

We will assume that P_1 and P_2 are nonempty and bounded

Using the "representation theorem" for bounded polyhedral sets (polytopes) we have

$$x_1 = \sum_{j \in J_1} (\lambda_1^j)\, x_1^j, \quad \sum_{j \in J_1} (\lambda_1^j) = 1, \quad \lambda_1^j \geq 0, \ \forall j \in J_1$$

$$x_2 = \sum_{j \in J_2} (\lambda_2^j)\, x_2^j, \quad \sum_{j \in J_2} (\lambda_2^j) = 1, \quad \lambda_2^j \geq 0, \ \forall j \in J_2$$

Plugging in these expressions for x_1 and x_2 in the objective function and coupling constraints of (2) gives

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$$\text{Max} \quad c_1^T\left(\sum_{j\in J_1} \lambda_1^j x_1^j\right) + c_2^T\left(\sum_{j\in J_2} \lambda_2^j x_2^j\right) + d^T w$$

$$\text{s.t.} \quad A_1\left(\sum_{j\in J_1} \lambda_1^j x_1^j\right) + A_2\left(\sum_{j\in J_2} \lambda_2^j x_2^j\right) + Dw = b$$

$$\sum_{j\in J_1} \lambda_1^j = 1$$

$$\sum_{j\in J_2} \lambda_2^j = 1$$

$$\lambda_1^j \geq 0 \quad \forall j \in J_1$$

$$\lambda_2^j \geq 0 \quad \forall j \in J_2$$

Which can also be written as

$$\text{Max} \quad \sum_{j\in J_1} (c_1^T x_1^j)\lambda_1^j + \sum_{j\in J_2} (c_2^T x_2^j)\lambda_2^j + d^T w$$

$$\text{s.t.} \quad \sum_{j\in J_1} \begin{bmatrix} A_1 x_1^j \\ 1 \\ 0 \end{bmatrix} \lambda_1^j + \sum_{j\in J_2} \begin{bmatrix} A_2 x_2^j \\ 0 \\ 1 \end{bmatrix} \lambda_2^j + \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} w = \begin{bmatrix} b \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_1^j \geq 0 \quad \forall j \in J_1$$

$$\lambda_2^j \geq 0 \quad \forall j \in J_2$$

The dual is

$$\text{MIN} \quad b^T y + z_1 + z_2$$

s.t

$$S_1^j = (c_1 - A_1^T y)^T x_1^j - z_1 \le 0, \quad \forall j \in J_1$$

$$S_2^j = (c_2 - A_2^T y)^T x_2^j - z_2 \le 0 \quad \forall j \in J_2$$

$$S_3 = d - D^T y \le 0$$

To find the entering nonbasic variable we solve the following 2 subproblems

$$\max_{j \in J_1} s_1^j \quad \text{and} \quad \max_{j \in J_2} s_2^j$$

Now

$$\max_{j \in J_1} s_1^j = \max_{j \in J_1} (c_1 - A_1^T y)^T x_1^j - z_1$$

$$= \left(\max_{j \in J_1} (c_1 - A_1^T y)^T x_1^j \right) - z_1$$

$$= \left(\begin{array}{ll} \text{Max} & (c_1 - A_1^T y)^T x_1 \\ \text{s.t} & B_1 x_1 = d_1 \\ & x_1 \ge 0 \end{array} \right) - z_1$$

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Similarly

$$\underset{j \in J_2}{\text{Max}} \; s_2^j = \begin{pmatrix} \text{Max} & (c_2 - A_2^T y)^T x_2 \\ \text{s.t.} & B_2 x_2 = d_2 \\ & x_2 \geq 0 \end{pmatrix} - z_2$$

To find the entering nonbasic variable in Step 2 of the revised simplex method

We compute

$$s_1^* = \theta_1 - z_1$$

where $$\theta_1 = \begin{array}{ll} \text{Max} & (c_1 - A_1^T y)^T x_1 \\ \text{s.t} & B_1 x_1 = d_1 \\ & x_1 \geq 0 \end{array}$$

Similarly $s_2^* = \theta_2 - z_2$

where $$\theta_2 = \begin{array}{ll} \text{Max} & (c_2 - A_2^T y)^T x_2 \\ \text{s.t} & B_2 x_2 = d_2 \\ & x_2 \geq 0 \end{array}$$

and $s_3^* = (d - D^T y) \in R^2$ (in the example)

Let

$$s^* = \text{Max} \{ s_1^*, s_2^*, (s_3^*)_1, (s_3^*)_2 \}$$

Let $j = \text{argmax} \{ s_1^*, s_2^*, (s_3^*)_1, (s_3^*)_2 \}$

Suppose $s^* > 0$

(a) If $j = 1$, then introduce the column $\begin{bmatrix} A_1 x_1^* \\ 1 \\ 0 \end{bmatrix}$ into the basis in the next iteration

(b) If $j = 2$, then introduce the column $\begin{bmatrix} A_2 x_2^* \\ 0 \\ 1 \end{bmatrix}$ into the basis in the next iteration

(Note that

$x_1^* = \arg\max (c_1 - A_1^T y)^T x_1$ s.t. $B_1 x_1 = d_1$, $x_1 \geq 0$ and $x_2^* = \arg\max (c_2 - A_2^T y)^T x_2$ s.t. $B_2 x_2 = d_2$, $x_2 \geq 0$

respectively)

(c) If $j = 3$, then introduce the column $\begin{bmatrix} D(:,1) \\ 0 \\ 0 \end{bmatrix}$ into the basis ($D(:,1)$ is the 1st column of D)

(d) Finally, if $j = 4$ then introduce the column $\begin{bmatrix} D(:,2) \\ 0 \\ 0 \end{bmatrix}$ into the basis

On the other hand if $s^* \leq 0$, then we are OPTIMAL!

Once we know how to bring in a column of A corresponding to the entering nonbasic variable, we can give

the complete REVISED SIMPLEX method with COLUMN GENERATION :-

1. Step 1 :- Choose an initial basis matrix A_B so that

$$X_B = A_B^{-1} b \geq 0$$

(c_B can be computed similarly!)

(WE WILL ILLUSTRATE HOW STEP 1 IS DONE ON THE EXAMPLE FROM CHVATAL)

2. Step 2 :- Solve for $\begin{bmatrix} y^* \\ z_1^* \\ z_2^* \end{bmatrix}$ using

$$A_B^T \begin{bmatrix} y^* \\ z_1^* \\ z_2^* \end{bmatrix} = c_B$$

Compute $\theta_1^* = \underset{\text{s.t}}{\text{Max}} \; (c_1 - A_1^T y^*)^T x_1$

$$B_1 x_1 = d_1$$

$$x_1 \geq 0$$

and let x_1^* be the optimal solution to this problem

$$\theta_2^* = \text{Max } (c_2 - A_2^T y^*)^T x_2$$
$$\text{s.t. } B_2 x_2 = d_2$$
$$x_2 \geq 0$$

and let x_2^* be the corresponding optimal solution

Compute

$$S_1^* = \theta_1^* - z_1^*$$
$$S_2^* = \theta_2^* - z_2^*$$
$$S_3^* = d - D^T y^*$$

(Note that S_3 is a 2 dimensional vector in CHVATAL's example)

Let $S^* = \text{Max} \{ S_1^*, S_2^*, (S_3)_1^*, (S_3)_2^* \}$

and let

$$j = \text{argmax} \{ S_1^*, S_2^*, (S_3)_1^*, (S_3)_2^* \}.$$

(i)

If $j = 1$, then $a_N = \begin{bmatrix} A_1 x_1^* \\ 1 \\ 0 \end{bmatrix}$

(ii) If $j=2$, then $a_N = \begin{bmatrix} A_2 x_2^* \\ 0 \\ 1 \end{bmatrix}$

(iii) If $j=3$, then $a_N = \begin{bmatrix} D(:,1) \\ 0 \\ 0 \end{bmatrix}$

where $D(:,1)$ is the 1st column of D

(iv) If $j=4$, then $a_N = \begin{bmatrix} D(:,2) \\ 0 \\ 0 \end{bmatrix}$

(3) Step 3 :- Solve

$$A_B d = a_N$$

If $d \le 0$ then our LP is unbounded

Else let $\theta = \min_{\substack{i=1,2,\dots,m \\ d_i > 0}} \left\{ \frac{x_{B(i)}}{d_i} \right\}$

and let $k = \arg\min_{\substack{i=1,2,\dots,m \\ d_i > 0}} \left\{ \frac{x_{B(i)}}{d_i} \right\}$

(4) Step 4 :- Update A_B, x_B and c_B.

We have

$$A_B(:,k) = a_N$$

$$C_B(k) = (c_1^T x_1^*) \quad \text{If } j = 1$$
$$= (c_2^T x_2^*) \quad \text{If } j = 2$$
$$= d(1) \quad \text{If } j = 3$$
$$= d(2) \quad \text{If } j = 4$$

Finally,
$$X_B(k) = \theta$$
$$X_B(l) = X_B(l) - \theta d_l$$
$$l = 1, 2, \ldots, m$$
$$l \neq k.$$

Return to Step 2 and iterate

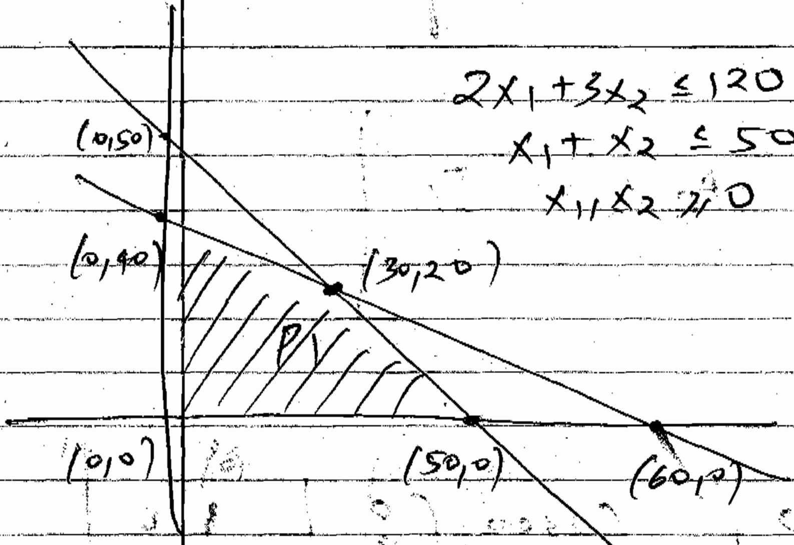
Now to our example :–

$$P_1 = \left\{ (x_1, x_2, x_3, x_4) \,\middle|\, \begin{array}{l} 2x_1 + 3x_2 + x_3 = 120 \\ x_1 + x_2 + x_4 = 50 \\ x_1, x_2, x_3, x_4 \geq 0 \end{array} \right\}$$

$$P_2 = \left\{ (x_5, x_6, x_7, x_8) \,\middle|\, \begin{array}{l} x_5 + 3x_6 + x_7 = 150 \\ 2x_5 + x_6 + x_8 = 100 \\ x_5, x_6, x_7, x_8 \geq 0 \end{array} \right\}$$

Sketch of P_1 in $x_1 - x_2$ space

$$2x_1 + 3x_2 \le 120$$
$$x_1 + x_2 \le 50$$
$$x_1, x_2 \ge 0$$

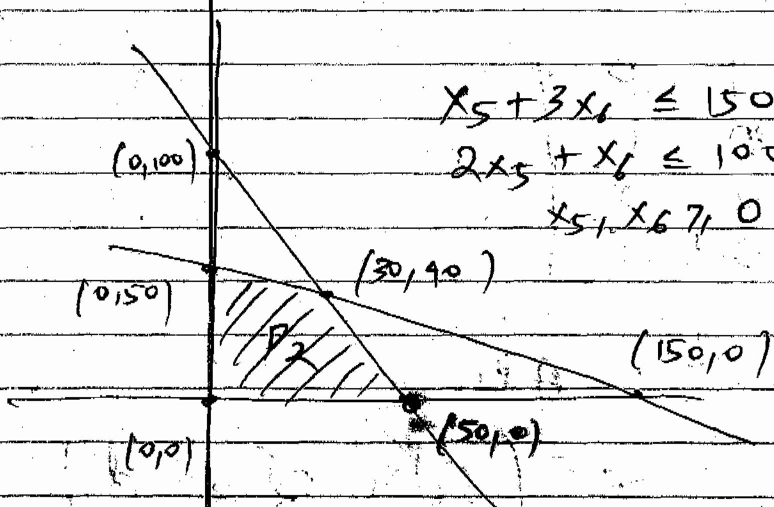


EXTREME POINTS OF P_1 are

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 120 \\ 50 \end{pmatrix}, \begin{pmatrix} 0 \\ 40 \\ 0 \\ 10 \end{pmatrix}, \begin{pmatrix} 30 \\ 20 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 50 \\ 0 \\ 20 \\ 0 \end{pmatrix} \right\}$$

Sketch of P_2 in $x_5 - x_6$ space

$$x_5 + 3x_6 \le 150$$
$$2x_5 + x_6 \le 100$$
$$x_5, x_6 \ge 0$$



EXTREME POINTS OF P_2 are

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 150 \\ 100 \end{pmatrix}, \begin{pmatrix} 50 \\ 0 \\ 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 50 \\ 0 \\ 50 \end{pmatrix}, \begin{pmatrix} 30 \\ 40 \\ 0 \\ 0 \end{pmatrix} \right\}$$

We will start with the following basis

$$A_B = \begin{bmatrix} A_1 x_1^1 & A_2 x_2^1 & D \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 120 \\ 50 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 150 \\ 100 \end{bmatrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ 1 & 0 & \\ 0 & 1 & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Also $b = \begin{bmatrix} 225 \\ 114 \\ 1 \\ 1 \end{bmatrix}$

Also

$$c_B = \begin{bmatrix} c_1^T x_1^1 & c_2^T x_2^1 & d \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$

(SEE THE CODE DANTZIG_WOLFE.M ON COURSE WEBPAGE FOR THE COMPUTATIONAL RESULTS)