MA/CSC 427-001: Introduction to Numerical Analysis I
Midterm Exam - October 8, 2007
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SOLUTIONS TO MIDTERM EXAM

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INSTRUCTIONS

1. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
2. TIME LIMIT: 50 minutes
3. There are 4 pages and 3 questions on the exam. Each question appears on a different page. Read each question carefully.
4. The exam is worth 100 points. The distribution of these points is clearly indicated on the exam.
5. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
6. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
7. This is a closed book exam. You may use three crib sheets of formulas on the exam.
8. You can also use an electronic calculator on the exam.

1. [30 points] Consider the problem for calculating the square root of a positive number, i.e., $x = (R)^{\frac{1}{2}}$ where $R > 0$.

   (a) [10 points] Rewrite this problem as finding the roots of a nonlinear equation.

   (b) [10 points] We will apply Newton's method to solve the nonlinear equation. Write down the Newton iteration formula for this problem.

   (c) [10 points] Suppose $R = 17$. What is a suitable starting point x_0? Calculate the first two Newton iterates x_1 and x_2.

1(a) We have $X = (R)^{1/2}$ where $R > 0$

$\therefore X^2 = R$ $\quad\therefore$ We have to solve

$$f(x) = x^2 - R = 0$$

1(b)

$$f(x) = x^2 - R$$
$$f'(x) = 2x$$

NEWTON ITERATION FRMULA IS

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \therefore x_{n+1} = x_n - \frac{(x_n^2 - R)}{2x_n}$$

$$\therefore \boxed{x_{n+1} = \frac{2x_n^2 - x_n^2 + R}{2x_n} = \frac{x_n^2 + R}{2x_n}}$$

1(c) Suppose $R = 17$

Let $x_0 = 4$ (Since $x_0^2 = 16$)

$$X_1 = \frac{x_0^2 + 17}{2x_0} = \frac{16 + 17}{2(4)} = \frac{33}{8} = \boxed{4.125}$$

$$X_2 = \frac{x_1^2 + 17}{2x_1} = \frac{(4.125)^2 + 17}{2(4.125)} = \boxed{4.1231}$$

2. [30 points] Let $f(x) = \frac{1}{x}$. Show that

(a) [15 points] The 2nd Newton divided difference

$$f[x_0, x_1] = \frac{-1}{x_0 x_1}.$$

(b) [15 points] The 3rd Newton divided difference

$$f[x_0, x_1, x_2] = \frac{1}{x_0 x_1 x_2}.$$

2(a) We have

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\frac{1}{x_1} - \frac{1}{x_0}}{x_1 - x_0} = \frac{\frac{x_0 - x_1}{x_1 x_0}}{x_1 - x_0}$$

$$= \frac{-1}{x_1 x_0}$$

2(b) $$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

We have $f[x_1, x_2] = \frac{-1}{x_1 x_2}$ and $f[x_0, x_1] = \frac{-1}{x_0 x_1}$

$$\therefore f[x_0, x_1, x_2] = \frac{\frac{-1}{x_1 x_2} + \frac{1}{x_1 x_0}}{x_2 - x_0} = \frac{\frac{x_2 - x_0}{x_0 x_1 x_2}}{x_2 - x_0} = \frac{1}{x_0 x_1 x_2}$$

3. [40 points] Show that the truncation error in the following central difference formula

$$f''(x_0) = \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}$$

for computing the 2nd derivative is $O(h^2)$.

Using the mean value theorem we have

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(\xi_1(x))$$

where $\xi_1(x) \in (x_0-h, x_0)$

Similarly

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(\xi_2(x))$$

where $\xi_2(x) \in (x_0, x_0+h)$

$$\therefore \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}$$

$$= \frac{f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(\xi_1(x)) - 2f(x_0) + f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{4}f^{(4)}(\xi_2(x))}{h^2}$$

$$\therefore \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}$$

$$= \frac{h^2 f''(x_0) + \frac{h^4}{24}\left(f^{(4)}(\xi_1(x)) + f^{(4)}(\xi_2(x))\right)}{h^2}$$

$$\therefore \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2} = f''(x_0) + \frac{h^2}{24}\left(f^{(4)}(\xi_1(x)) + f^{(4)}(\xi_2(x))\right)$$

$\therefore$ TRUNACTION ERROR $= \left| \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2} - f''(x_0) \right|$

$$= \frac{h^2}{24} \left| f^{(4)}(\xi_1(x)) + f^{(4)}(\xi_2(x)) \right|$$

$\therefore$ TRUNACTION ERRR IS $O(h^2)$ (IN BIG-O NOTATION)