# MA/CSC 427: INTRODUCTION TO NUMERICAL ANALYSIS I

## SOLUTIONS TO SELECTED REVIEW PROBLEMS

PREPARED BY KARTIK

(1) (a) We have

$$2 = \frac{2}{2} \times 2^1 = 1 \times 2^1$$

$$= (1.00)_2 \times 2^1$$

MANTISSA = $00 \ldots 0$ (23 BITS)

BIASED EXPONENT = $1 + 127 = 128 = 1000000$[illegible]

SIGN BIT = 0

| | | |
|---|---|---|
| 0 | 10000000 | 000 – – – – – 0 (ALL ZEROS) |
| SIGN BIT | BIASED EXPONENT | MANTISSA |

$2 =$ the above.

NEXT NUMBER IS

| | | |
|---|---|---|
| 0 | 10000000 | 000 – – – – 1 |
| SIGN BIT | BIASED EXPONENT | MANTISSA |

$$= (1.00 \ldots 1)_2 \times 2^1$$

$$= (1 + 2^{-23}) \times 2 = (2 + 2^{-22})$$

$\therefore$ GAP IS $2^{-22}$

1(b) We have

$$1024 = \frac{2^{10}}{2^{10}} \times 2^{10} = 1 \times 2^{10}$$

$$= (1.00)_2 \times 2^{10}$$

MANTISSA $= 0 - - - 0$
(23 BITS)

BIASED EXPONENT $= 10 + 127 = 137$
$= (10001001)_2$

1024 =

| | | ALL ZEROS |
|---|---|---|
| 0 | 10001001 | 000 - - 0 |
| SIGN BIT | BIASED EXPONENT | MANTISSA |

Next number is

| | | |
|---|---|---|
| 0 | 1001001 | 00 - - 01 |
| SIGN BIT | BIASED EXPONENT | MANTISSA |

$$= (1.00 - - 1)_2 \times 2^{10}$$

$$= (1 + 2^{-23}) \times 2^{10}$$

$$= (2^{10} + 2^{-13})$$

GAP is $\underline{\underline{2^{-13}}}$

2(b) WORK OUT 2(a) YOURSELF

I WILL DO PART 2(b) ONLY here.

Show that the truncation error in the following 3 point forward difference formula

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2})}{h^2}$$

is $O(h)$

Note that the points are equally spaced. So, $x_{i+1} = x_i + h$, $x_{i+2} = x_i + 2h$ etc

Using the mean value theorem, we have

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + \frac{h^3}{6} f'''(\xi_1) \quad \text{where } \xi_1 \in (x_i, x_{i+1})$$

SIMILARLY

$$f(x_{i+2}) = f(x_i) + (2h) f'(x_i) + \frac{4h^2}{2} f''(x_i) + \frac{8h^3}{6} f'''(\xi_2) \quad \text{where } \xi_2 \in (x_i, x_{i+2})$$

$\therefore f(x_i) - 2f(x_{i+1}) + f(x_{i+2})$

$$= \left(f(x_i) - 2f(x_i) + f(x_i)\right) - 2hf'(x_i) + 2hf'(x_i) - h^2 f''(x_i) + 2h^2 f''(x_i) - \frac{2h^3}{6}\left(f'''(\epsilon_1) + 4f'''(\epsilon_2)\right)$$

$$\therefore \frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2})}{h^2} - f''(x_i) = -\frac{2h^3}{6h^2}\left(f'''(\epsilon_1) + 4f'''(\epsilon_2)\right)$$

TRUNACTION ERROR =

$$\left|\frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2})}{h^2} - f''(x_i)\right| = \frac{h}{3}\left(f'''(\epsilon_1) + 4f'''(\epsilon_2)\right)$$

i.e the truncation error is $O(h)$ (in big O notation)

(3) LOOK AT THE DISCUSSION IN PAGES 189-190 to derive the Simpson formula. Work out all the steps clearly.

(5)

We have:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi)$$

Since the error term $\frac{h^5}{90} f^{(4)}(\xi)$ involves the $4^{th}$ derivative of $f$, the degree of accuracy of Simpson's rule is 3

i.e it gives the exact result when applied to any polynomial of degree three or less.

(4)

Determine the values of n and h to approximate

$$I = \int_0^2 \frac{dx}{x+4}$$

to within $10^{-5}$ and compute the approximation (using the composite Simpson rule)

The truncation error in the composite Simpson rule in evaluating $\int_a^b f(x)\,dx$

is $\dfrac{b-a}{180} h^4 f^{(4)}(\mu)$

where $h = \left(\dfrac{b-a}{n}\right)$ and $\mu \in (a,b)$

In our example,
We have

$$\text{ERROR} = \left|\left(\frac{b-a}{180}\right) h^4 f^4(\mu)\right| = \left|\left(\frac{2}{180}\right)\left(\frac{2}{n}\right)^4 f^4(\mu)\right|$$

where $\mu \in (0,2)$

$$f(x) = \frac{1}{(x+4)} \qquad \therefore f^{(4)}(x) = \frac{24}{(x+4)^5}$$

$\therefore$ ERROR

$$= \left|\left(\frac{32}{180}\right)\frac{1}{n^4}\frac{24}{(\mu+4)^5}\right|$$

$$= \left|\frac{64}{15n^4(\mu+4)^5}\right| \le \left|\frac{64}{15n^4(4)^5}\right|$$

[Since the max value of $\dfrac{1}{(\mu+4)^5}$ over $\mu \in (0,2)$ is $\dfrac{1}{(4)^5}$]

We want

$$\left|\frac{64}{15n^4(4)^5}\right| < 10^{-5} \quad \therefore n^4 > \frac{10^5}{15 \times 16} \quad \text{i.e. } n > 4.51$$

$\frac{\cdot}{\cdot}$ — $\frac{1}{3}$ — $\frac{2}{3}$ — $\frac{1}{\cdot}$ — $\frac{4}{3}$ — $\frac{5}{3}$ , 2

$x_0$ $x_1$ $x_2$ $x_3$ $x_4$ $x_5$ $x_6$ (7)

$\therefore n > 4.51$

Remembering that the composite Simpson rule requires an even no of intervals we choose $n = 6$

$$h = \left(\frac{b-a}{6}\right) = \left(\frac{2}{6}\right) = \frac{1}{3}$$

$$\therefore \int_0^2 \frac{1}{x+4}\,dx = \frac{1}{9}\left[\frac{1}{(0+4)} + 2\left(\frac{1}{2/3+4}\right) + 2\left(\frac{1}{4/3+4}\right) + 4\left(\frac{1}{1/3+4}\right) + 4\left(\frac{1}{1+4}\right) + 4\left(\frac{1}{5/3+4}\right) + \frac{1}{(2+4)}\right]$$

(USING FORMULA IN THM (4.4) on page 199)

$$= 0.405466$$

(5) (a) I WILL DO ONLY 6(b)

$$p_n = 1/n^2 \quad n \geq 1$$

We have $p_{n+1} = \dfrac{1}{(n+1)^2}$

Consider $\displaystyle\lim_{n \to \infty} \left|\frac{p_{n+1}}{p_n}\right| = \lim_{n \to \infty} \frac{n^2}{(n+1)^2}$

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$\therefore \lim_{n\to\infty} \left|\frac{p_{n+1}}{p_n}\right| = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2}$

$= 1$

$\therefore$ By def$^n$ (2-6) on page 75, the sequence
$\{p_n\} = \{\frac{1}{n^2}\}$ converges linearly to zero with asymptotic error constant 1

We have
$|p_n - p| = \left|\left(\frac{1}{n^2} - 0\right)\right| = \frac{1}{n^2}$

$\therefore |p_n - p| \leq 5 \times 10^{-2}$ if $\frac{1}{n^2} \leq 5 \times 10^{-2}$

$n^2 \geq \frac{1}{5 \times 10^{-2}} = \frac{100}{5}$

$\therefore n^2 \geq 20$

$n = 5$ OR LARGER

5 (b) I WILL DO ONLY part 8(a)

We have $p_n = 10^{-2^n}$

$p_{n+1} = 10^{-2^{(n+1)}}$

$\therefore \lim_{n\to\infty} \left|\frac{p_{n+1}}{p_n^2}\right| = \lim_{n\to\infty} \frac{10^{-2(n+1)}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2(n+1)}}{10^{-2(n+1)}}$

$= 1$

$\therefore$ By defn (2.6) on page 75,

the sequence $\{10^{-2^n}\}$

converges to 0 of order 2 (quadratically)

with asymptotic error constant 1

5(c) We have

$$|p_n - p| \leq \frac{b-a}{2^n}$$

Let us assume $p = 0$

$$\therefore |p_n| \leq \frac{b-a}{2^n}$$

We have $|p_{n+1}| \leq \dfrac{b-a}{2^{n+1}}$

Take $|p_n| \approx \dfrac{b-a}{2^n}$ and $|p_{n+1}| \approx \dfrac{b-a}{2^{n+1}}$

$$\therefore \lim_{n\to\infty} \left|\frac{p_{n+1}}{p_n}\right| = \lim_{n\to\infty} \frac{\left(\frac{b-a}{2^{n+1}}\right)}{\left(\frac{b-a}{2^n}\right)}$$

$$= \left(\frac{1}{2}\right)$$

$\therefore$ The sequence $\{p_n\}$ converges to 0 linearly with asymptotic error constant $1/2$

(6) $$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

We have

$$P_n(x_2) = f[x_2] = f[x_0] + f[x_0, x_1](x_2 - x_0) + a_2(x_2-x_0)(x_2-x_1)$$

$$\therefore f[x_2] = f[x_0] + f[x_1] - f[x_1] + f[x_0, x_1](x_2 - x_0) + a_2(x_2-x_0)(x_2-x_1)$$

$$\frac{f[x_2] - f[x_1]}{(x_2-x_1)} = \frac{f[x_0] - f[x_1]}{(x_2-x_1)} + f[x_0,x_1]\left(\frac{x_2-x_0}{x_2-x_1}\right) + a_2(x_2-x_0)$$ Since $x_1 \neq x_2$

$$f[x_1, x_2] = \frac{-f[x_0, x_1]}{(x_2-x_1)}(x_1-x_0) + f[x_0,x_1]\left(\frac{x_2-x_0}{x_2-x_1}\right) + a_2(x_2-x_0)$$

$$\therefore f[x_1, x_2] = f[x_0, x_1]\left[\frac{x_0-x_1}{x_2-x_1} + \frac{x_2-x_0}{x_2-x_1}\right] + a_2(x_2-x_0)$$

$$\therefore f[x_1, x_2] = f[x_0, x_1] + a_2(x_2 - x_0)$$

$$\therefore a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$= f[x_0, x_1, x_2] \rightarrow \text{2nd divided difference}$$

(See last equation on page 119)

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| $x$ | $f(x)$ |
|---|---|
| $-2$ | $0$ |
| $0$ | $1$ |
| $1$ | $-1$ |

a Lagrange Interpolation:–

3 points

We have

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{x(x-1)}{6}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = -\frac{(x+2)(x-1)}{2}$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x+2)}{3}$$

∴ The Lagrange interpolating polynomial is

$$P_2(x) = f(x_0)\,L(x_0) + f(x_1)\,L(x_1) + f(x_2)\,L(x_2)$$

$$= -\frac{1}{6}\left[5x^2 + 7x - 6\right] \rightarrow (1)$$

(b)

NEWTON'S DIVIDED difference :-

| $x$ | $f(x)$ | | |
|---|---|---|---|
| $-2$ | $f[x_0] = 0$ | $f[x_0, x_1] = \frac{1}{2}$ | $f[x_0, x_1, x_2]$ |
| $0$ | $f[x_1] = 1$ | $f[x_1, x_2] = -2$ | $= -5/6$ |
| $1$ | $f[x_2] = -1$ | | |

(SEE TABLE 3.7) on page 120

$$\therefore P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= -\frac{5x^2}{6} + 1 - \frac{7}{6}x \rightarrow (2)$$

Note that (1) and (2) are identical

(c) Consider $g(x) = \dfrac{f(x)}{(f'(x))^{1/2}}$

We have

$$g'(x) = \frac{f'(x)^2 - \dfrac{f(x)\,f''(x)}{2}}{(f'(x))^{3/2}}$$

→ CARRY OUT THIS CALCULATION

The $(n+1)$th iterate in Newton's method is obtained as

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$= x_n - \frac{\dfrac{f(x_n)}{(f'(x_n))^{1/2}}}{\dfrac{\left((f'(x_n))^2 - \dfrac{f(x_n)f''(x_n)}{2}\right)}{(f'(x_n))^{3/2}}}$$

$$= x_n - \frac{f(x_n)\,f'(x_n)}{\left((f'(x_n))^2 - \dfrac{f(x_n)f''(x_n)}{2}\right)}$$