

MA/CSC 427: INTRODUCTION TO  
NUMERICAL ANALYSIS I

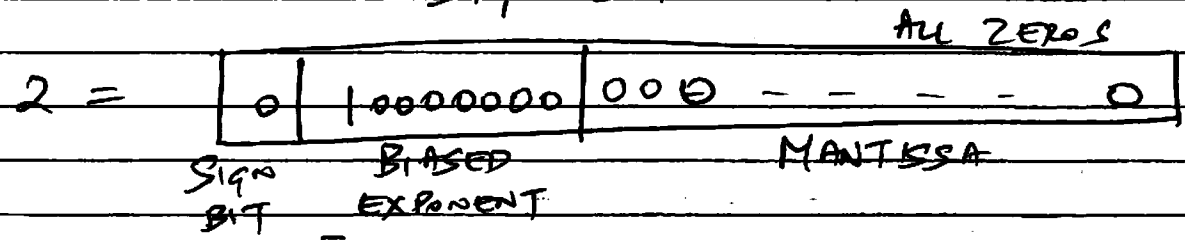
SOLUTIONS TO SELECTED REVIEW PROBLEMS  
PREPARED BY KARTIK

(1) We have

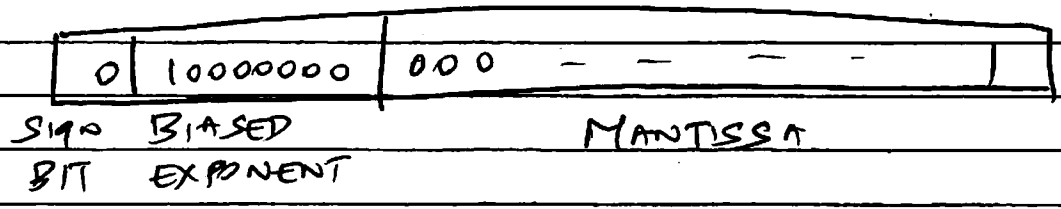
$$2 = \frac{2}{2} \times 2^1 = 1 \times 2^1$$
$$= (1.00)_2 \times 2^1$$

MANTISSA = 00...0  
(23 BITS)

BIASED EXPONENT = 1 + 127 = 128 = 1000 0000  
SIGN BIT = 0



NEXT NUMBER IS



$$= (1.00...1)_2 \times 2^1$$

$$= (1 + 2^{-23}) \times 2 = (2 + 2^{-22})$$

$\therefore$  GAP IS  $2^{-22}$

(2)

(b) We have

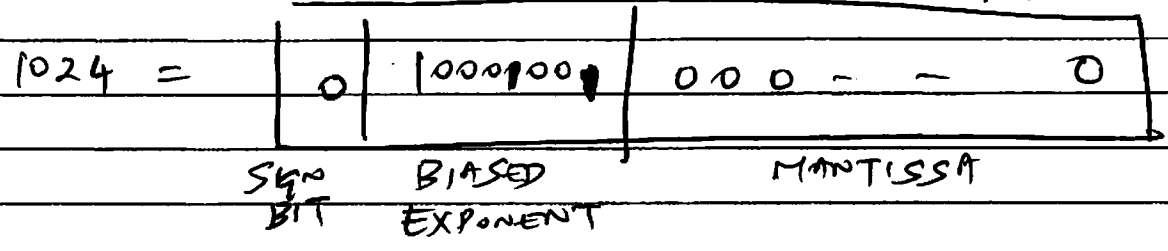
$$1024 = \frac{2^{10}}{2^{10}} \times 2^{10} = 1 \times 2^{10}$$

$$= (1.00)_2 \times 2^{10}$$

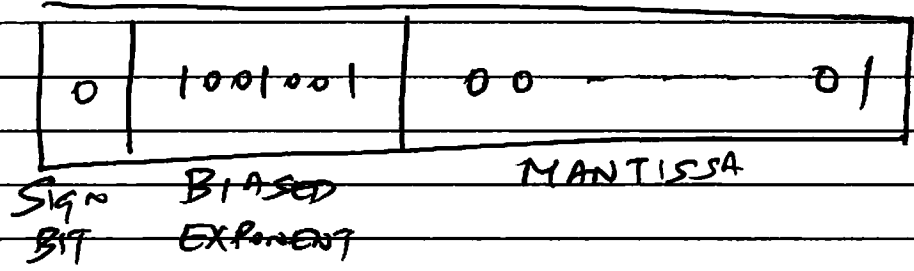
MANTISSA = 0 - - - 0  
(23 BITS)

BIASED

EXPONENT = 10 + 127 = 137  
= (10001001)<sub>2</sub>  
All zeros



Next number is



$$= (1.00 - - - 1)_2 \times 2^{10}$$

$$= (1 + 2^{-23}) \times 2^{10}$$

$$= (2^{10} + 2^{-13})$$

GAP is 2<sup>-13</sup>

(3)

2(b)

WORK OUT 2(a) YOURSELF

I WILL DO PART 2(b) ONLY  
here.

Show that the truncation error  
in the following 3 point forward  
difference formula:

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2}))}{h^2}$$

is  $O(h)$

Note that the points are  
equally spaced. So,  $x_{i+1} = x_i + h$   
 $x_{i+2} = x_i + 2h$  etc

Using the mean value theorem,  
we have

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + \frac{h^3}{6} f'''(\xi_1) \quad \text{where } \xi_1 \in (x_i, x_{i+1})$$

SIMILARLY

$$f(x_{i+2}) = f(x_i) + (2h)f'(x_i) + \frac{4h^2}{2} f''(x_i) + \frac{8h^3}{6} f'''(\xi_2) \quad \text{where } \xi_2 \in (x_i, x_{i+2})$$

(2)

$$\therefore f(x_i) - 2f(x_{i+1}) + f(x_{i+2})$$

$$= (f(x_i) - 2f(x_i) + f(x_i)) - 2hf'(x_i) + 2hf'(x_i) \\ - h^2 f''(x_i) + 2h^2 f''(x_i) \\ - \frac{2h^3}{6} (f'''(\xi_1) + 4f'''(\xi_2))$$

$$\therefore \frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2})}{h^2} - f''(x_i)$$

$$= \frac{-2h^3}{6h^2} (f'''(\xi_1) + 4f'''(\xi_2))$$

TRUNCATION ERROR =

$$\left| \frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2})}{h^2} - f''(x_i) \right|$$

$$= \frac{h}{3} (f'''(\xi_1) + 4f'''(\xi_2))$$

i.e. the truncation error is  $O(h)$   
(in big O notation)

(3)

LOOK AT THE DISCUSSION IN PAGES  
189-190 to derive the Simpson  
formula. Work out all the  
steps clearly.

(5)

We have:

$$\int_{x_0}^{x_2} f(x) dx = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)]$$
$$= \frac{h^5}{90} f^{(4)}(\xi)$$

Since the error term  $\frac{h^5}{90} f^{(4)}(\xi)$  involves the 4<sup>th</sup> derivative of  $f$ ,

the degree of accuracy of Simpson's rule is 3. i.e. it gives the exact result when applied to any polynomial of degree three or less.

(4)

Determine the values of  $n$  and  $h$  to approximate

$$I = \int_{-2}^2 \frac{dx}{x+4}$$

to within  $10^{-5}$  and compute the approximation (using the composite Simpson rule)

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The truncation error in the composite Simpson rule in evaluating

$$\int_a^b f(x) dx \approx \frac{b-a}{180} h^4 f^{(4)}(\mu)$$

where  $h = \frac{b-a}{n}$  and  $\mu \in (a, b)$

In our example,  
we have

$$\text{ERROR} = \left| \left( \frac{b-a}{180} \right) h^4 f^{(4)}(\mu) \right| = \left| \left( \frac{2}{180} \right) \left( \frac{2}{n} \right)^4 f^{(4)}(\mu) \right|$$

where  $\mu \in (0, 2)$

$$f(x) = \frac{1}{(x+4)} \quad \therefore f^{(4)}(x) = \frac{24}{(x+4)^5}$$

$\therefore$  ERROR

$$= \left| \left( \frac{32}{180} \right) \frac{1}{n^4} \frac{24}{(\mu+4)^5} \right|$$

$$\leq \left| \frac{64}{15n^4(4)^5} \right|$$

[Since the max value of  $\frac{1}{(\mu+4)^5}$  over  $\mu \in (0, 2)$  is  $\frac{1}{(4)^5}$ ]

We want

$$\left| \frac{64}{15n^4(4)^5} \right| < 10^{-5} \quad \therefore n^4 > \frac{10^5}{15 \times 16} \quad \text{i.e. } n > 4.51$$

$$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$$

(7)

$$\therefore n > 4.5$$

Remembering that the composite Simpson rule requires an even no of intervals we choose  $h=6$

$$h = \left( \frac{b-a}{6} \right) = \left( \frac{2}{6} \right) = \frac{1}{3}$$

$$\int_0^2 \frac{1}{x+4} dx = \frac{1}{9} \left[ \frac{1}{(0+4)} + 2 \left( \frac{1}{2/3+4} \right) \right.$$

$$\left. + 2 \left( \frac{1}{4/3+4} \right) \right.$$

(Using Formula  
In Form (44)  
or page 199)

$$+ 4 \left( \frac{1}{1/3+4} \right)$$

$$+ 4 \left( \frac{1}{1+4} \right)$$

$$+ 4 \left( \frac{1}{5/3+4} \right) + \frac{1}{(2+4)} \left. \right]$$

$$= 0.405466$$

(5) (a) I will do only 6(b)  
 $p_n = 1/n^2 \quad n \geq 1$

$$\text{We have } p_{n+1} = \frac{1}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} \left| \frac{p_{n+1}}{p_n} \right| = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2}$$

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$$\therefore \lim_{n \rightarrow \infty} \left| \frac{p_{n+1}}{p_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^2}$$

$$= 1$$

$\therefore$  By def<sup>n</sup> (2.6) on page 75, the sequence

$\{p_n\} = \left\{ \frac{1}{n^2} \right\}$  converges linearly to zero with asymptotic error constant 1

We have

$$|p_n - p| = \left| \left( \frac{1}{n^2} - 0 \right) \right| = \frac{1}{n^2}$$

$$\therefore |p_n - p| \leq 5 \times 10^{-2} \quad \forall \quad \frac{1}{n^2} \leq 5 \times 10^{-2}$$

$$n^2 \geq \frac{1}{5 \times 10^{-2}} = \frac{100}{5}$$

$$\therefore n^2 \geq 20$$

$$n = 5 \text{ OR LARGER}$$

5 (b) I will do ONLY part 8(a)

We have  $p_n = 10^{-2^n}$

$$p_{n+1} = 10^{-2^{(n+1)}}$$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{p_{n+1}}{p_n^2} \right| = \lim_{n \rightarrow \infty} \frac{10^{-2^{(n+1)}}}{\left(10^{-2^n}\right)^2} = \lim_{n \rightarrow \infty} \frac{10^{-2^{(n+1)}}}{10^{-2 \cdot 2^n}}$$

$$= 1$$

9

∴ By defn (2.6) on page 75,

the sequence  $\{10^{-2^n}\}$

converges to 0 of order 0  
(quadratically)  
with asymptotic error  
constant 1

5(c) We have

$$|p_n - p| \leq \frac{b-a}{2^n}$$

let us assume  $p = 0$

$$|p_n| \leq \frac{b-a}{2^n}$$

We have  $|p_{n+1}| \leq \frac{b-a}{2^{n+1}}$

Take  $|p_n| \approx \frac{b-a}{2^n}$  and  $|p_{n+1}| \approx \frac{b-a}{2^{n+1}}$

$$\lim_{n \rightarrow \infty} \left| \frac{p_{n+1}}{p_n} \right| = \lim_{n \rightarrow \infty} \frac{\left( \frac{b-a}{2^{n+1}} \right)}{\left( \frac{b-a}{2^n} \right)}$$

$$= \left( \frac{1}{2} \right)$$

∴ The sequence  $\{p_n\}$  converges to 0  
linearly with asymptotic error constant  $1/2$

$$(6) \quad P_n(x) = f[x_0] + f[x_0, x_1](x-x_0) \\ + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) \\ + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

We have

$$P_n(x_2) = f[x_2] = f[x_0] + f[x_0, x_1](x_2-x_0) \\ + a_2(x_2-x_0)(x_2-x_1)$$

$$\therefore f[x_2] = f[x_0] + f[x_1] - f[x_1] \\ + f[x_0, x_1](x_2-x_0) \\ + a_2(x_2-x_0)(x_2-x_1)$$

$$\frac{f[x_2] - f[x_1]}{(x_2-x_1)} = \frac{f[x_0] - f[x_1]}{(x_2-x_1)} + \frac{f[x_0, x_1](x_2-x_0)}{(x_2-x_1)} \\ + a_2(x_2-x_0) \quad \text{Since } x_1 \neq x_2$$

$$f[x_1, x_2] = \frac{f[x_0, x_1](x_1-x_0)}{(x_2-x_1)} + \frac{f[x_0, x_1](x_2-x_0)}{(x_2-x_1)} \\ + a_2(x_2-x_0)$$

$$= f[x_1, x_2] = f[x_0, x_1] \left[ \frac{x_0-x_1}{x_2-x_1} + \frac{x_2-x_0}{x_2-x_1} \right] + a_2(x_2-x_0)$$

$$\therefore f[x_1, x_2] = f[x_0, x_1] + a_2(x_2 - x_0)$$

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$= f[x_0, x_1, x_2] \rightarrow \text{2nd divided difference}$$

(See last equation on page 119)

7

x	f(x)
2	0
0	1
1	-1

8 Lagrange Interpolation

3 points

We have

$$h_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-1)}{6}$$

$$h_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-(x+2)(x-1)}{2}$$

$$h_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{x(x+2)}{3}$$

(2)

∴ The Lagrange interpolating polynomial is

$$P_2(x) = f(x_0) L(x_0) + f(x_1) L(x_1) + f(x_2) L(x_2)$$

$$= -\frac{1}{6} [5x^2 + 7x - 6] \rightarrow \textcircled{1}$$

(3)

NEWTON'S DIVIDED difference :-

$x$	$f(x)$		
-2	$f[x_0] = 0$	$f[x_0, x_1] = \frac{1}{2}$	$f[x_0, x_1, x_2]$
0	$f[x_1] = 1$	$f[x_1, x_2] = -2$	$= -5/6$
1	$f[x_2] = -1$		

(SEE TABLE 3.7)  
on page 120

$$\therefore P_2(x) = f[x_0]$$

$$+ f[x_0, x_1] (x - x_0) + f[x_0, x_1, x_2] (x - x_0)(x - x_1)$$

$$= -\frac{5x^2}{6} + 1 - \frac{7}{6}x \rightarrow \textcircled{2}$$

Note that  $\textcircled{1}$  and  $\textcircled{2}$  are identical

(13)

(c)

Consider

$$g(x) = f(x)$$

$$\frac{1}{(f'(x))^{1/2}}$$

We have

$$g'(x) = \frac{f'(x)^2 - f(x)f''(x)}{2}$$

$$\frac{1}{(f'(x))^{3/2}}$$

→ CARRY OUT THIS CALCULATION

The  $(n+1)$ th iterate in Newton's method is obtained as

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$= x_n - \frac{f(x_n)}{(f'(x_n))^{1/2}}$$

$$\frac{1}{\left( \frac{f'(x_n)^2 - f(x_n)f''(x_n)}{2} \right)}$$

$$(f'(x_n))^{3/2}$$

$$= x_n - \frac{f(x_n) f'(x_n)}{(f'(x_n))^2 - \frac{f(x_n)f''(x_n)}{2}}$$