MA/CSC 427-001: Introduction to Numerical Analysis I
Final Exam - December 14, 2007
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SOLUTIONS TO FINAL EXAM
PREPARED BY KARTIK

INSTRUCTIONS

1. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
2. TIME LIMIT: 3 hours
3. There are 6 pages and 5 questions on the exam. Each question appears on a different page. Read each question carefully.
4. The exam is worth 105 points including 5 extra credit points. The distribution of these points is clearly indicated on the exam.
5. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
6. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
7. This is a closed book exam. You may use five crib sheets of formulas on the exam.
8. You can also use an electronic calculator on the exam.

1. [25 points] Consider the nonlinear boundary value problem:

$$\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 + y = \ln x, \; 1 \le x \le 2, \; y(1) = 0 \text{ and } y(2) = \ln 2. \qquad (1)$$

(a) [10 points] Discretize the second order ODE using the central difference formulas

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

for approximating the 1st and 2nd order derivatives in (1), respectively. Use the step size $h = 0.5$ to approximate the solution to the boundary value problem. Compare your results to the actual solution $y = \ln x$.

(b) [15 points] Write down your MATLAB commands and functions to solve (1) using MATLAB's boundary value solver *bvp4c*?

1(a) We have

$$\left(\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}\right) + \left(\frac{w_{i+1} - w_{i-1}}{2h}\right)^2 + w_i = \ln x_i \qquad i = 1$$

where $x_1 = 1.5$, $w_0 = 0$, $w_2 = \ln 2$, and w_1 is the approximate solution at $x_1 = 1.5$, and $h = 0.5$

$$\therefore \left(\frac{\ln 2 - 2w_1}{h^2}\right) + \left(\frac{\ln 2}{2h}\right)^2 + w_1 = \ln(1.5)$$

$$\therefore (4h^2 - 8)w_1 = 4h^2\ln(1.5) - 4\ln 2 - (\ln 2)^2$$

$$\therefore w_1 = \frac{4h^2\ln(1.5) - 4\ln 2 - (\ln 2)^2}{(4h^2 - 8)} \Rightarrow w_1 = 0.406797$$

1(a) $\therefore$ OUR SOLUTION

is $w_0 = 0$

$w_1 = 0.406797$

$w_2 = \ln 2$

The actual solution at these points is $y_0 = 0$

$y_1 = 0.405465$

$y_2 = \ln 2.$

1(b) MATLAB COMMANDS

```
>> solinit = bvpinit(linspace(1,2,100),1,1);
>> sol = bvp4c(@odefun, @bvpfun,
                solinit)
```

The functions odefun and bvpfun are

```
function yprime = odefun(x,y)
    yprime = [y(2); -y(1)-y(2)^2+log(x)]
```

```
function res = bvpfun(ya,yb)
    res = [ya(1); yb(1)-log(2)];
```

2. [15 points] The lateral surface area S of a cone is given by:

$$S = \pi r\sqrt{r^2+h^2} \tag{2}$$

where r is the radius of the base and h is the height. Determine the radius of a cone that has surface area of 1200m^2 and a height of 20 m using Newton's method. Start with $r_0 = 17$ m and do two iterations of Newton's method in all.

We have

$$S = \pi r\sqrt{r^2+h^2} \quad \text{where } S = 1200, \; h = 20$$

$$\therefore 1200 = \pi r\sqrt{r^2+400}$$

$$\therefore f(r) = \pi r\sqrt{r^2+400} - 1200 = 0 \rightarrow \text{I}$$

We have to solve $f(r) = 0$ for r

$$f'(r) = \pi\left[\sqrt{r^2+400} + r \cdot \frac{1}{2\sqrt{r^2+400}}(2r)\right]$$

$$= \pi\left[\sqrt{r^2+400} + \frac{r^2}{\sqrt{r^2+400}}\right] \rightarrow \text{II}$$

NEWTON ITERATION

$$r_{i+1} = r_i - \frac{f(r_i)}{f'(r_i)} \quad i = 0, 1, 2, \ldots \text{ where } r_0 = 17$$

$$r_1 = 17 - \frac{f(17)}{f'(17)} = \boxed{15.27536}$$

$$r_2 = -\frac{f(r_1)}{f'(r_1)} + r_1 = \boxed{15.204}$$

3. [15 points] For a function f, the forward divided differences are given by

$$\begin{array}{llll} x_0 = 0.0 & f[x_0] & & \\ & & f[x_0, x_1] & \\ x_1 = 0.4 & f[x_1] & & f[x_0, x_1, x_2] = \frac{50}{7} \\ & & f[x_1, x_2] = 10 & \\ x_2 = 0.7 & f[x_2] = 6 & & \end{array} \tag{3}$$

(a) [10 points] Determine the missing entries in the table (3).

(b) [5 points] Use the divided difference coefficients to find the interpolating polynomial $P_2(x)$.

We have:

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = 10$$

$$\therefore \frac{6 - f[x_1]}{0.3} = 10 \qquad \therefore \boxed{f[x_1] = 3}$$

We also have:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{50}{7}$$

$$\frac{10 - f[x_0, x_1]}{0.7} = \frac{50}{7} \qquad \therefore \boxed{f[x_0, x_1] = 5}$$

Finally $f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = 5$

$$\frac{3 - f[x_0]}{0.4} = 5 \qquad \therefore \boxed{f[x_0] = 1}$$

The interpolating
polynomial

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= 1 + 5(x - 0)$$
$$+ \frac{50}{7}(x)(x - 0.4)$$

$$= \boxed{1 + 5x + \frac{50}{7}x(x - 0.4)}$$

4. [25 points] The motion of a swinging pendulum is described by the following initial value problem

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \quad \theta(0) = \frac{\pi}{6}, \quad \frac{d\theta}{dt}(t=0) = 0 \tag{4}$$

where $L = 2$ ft and $g = 32.17$ ft/s². Using $h = 0.2$ and the 4th order Runge Kutta method, determine the approximate value of $\theta(t)$ at $t = 0.2$. Show all the steps in your calculations.

Let $y_1 = \theta$ and $y_2 = \left(\frac{d\theta}{dt}\right)$

We have $y_1' = y_2$

and $y_2' + \frac{g}{L}\sin y_1 = 0 \quad \therefore \quad y_2' = -\frac{g}{L}\sin y_1$

$\therefore$ OUR INITIAL value problem is

$$\underbrace{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}}_{y'} = \underbrace{\begin{bmatrix} y_2 \\ -\frac{g}{L}\sin y_1 \end{bmatrix}}_{f(t,y)} \quad \text{and} \quad y_1(0) = \pi/6 \quad \text{and} \quad y_2(0) = 0$$

where $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

ONE ITERATION OF THE 4th order Runge Kutta Method

$t_0 = 0$ and $w_0 = \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix}$

$k_1 = h f(t_0, w_0)$

$= 0.2 \times \begin{bmatrix} 0 \\ -\frac{g}{L}\sin(\pi/6) \end{bmatrix} = 0.2 \begin{bmatrix} 0 \\ -\frac{32.17}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ -1.6085 \end{bmatrix}$

$$k_2 = h f\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2} k_1\right)$$

$$w_0 + \frac{1}{2} k_1 = \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ -1.6085 \end{bmatrix} = \begin{bmatrix} \pi/6 \\ -0.8042 \end{bmatrix}$$

$$= 0.2 \times \begin{bmatrix} -0.8042 \\ -\frac{g}{L} \sin(\pi/6) \end{bmatrix}$$

$$= \begin{bmatrix} -0.16084 \\ -1.6085 \end{bmatrix}$$

$$k_3 = h f\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2} k_2\right)$$

$$w_0 + \frac{1}{2} k_2 = \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -0.1684 \\ -1.6085 \end{bmatrix} = \begin{bmatrix} 0.44317 \\ -0.8042 \end{bmatrix}$$

$$= 0.2 \times \begin{bmatrix} -0.8042 \\ -\frac{g}{L} \sin(0.44317) \end{bmatrix}$$

$$= \begin{bmatrix} -0.16084 \\ -1.37949 \end{bmatrix}$$

$$k_4 = h f(t_0 + h, w_0 + k_3)$$

$$w_0 + k_3 = \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.16084 \\ -1.37949 \end{bmatrix} = \begin{bmatrix} 0.36275 \\ -1.37949 \end{bmatrix}$$

$$= 0.2 \times \begin{bmatrix} -1.37949 \\ -\frac{g}{L} \sin(0.36275) \end{bmatrix}$$

$$= \begin{bmatrix} -0.2759 \\ -1.14154 \end{bmatrix}$$

$$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix} + \frac{0.2}{6}\left(\quad \right)$$

$$= \begin{bmatrix} 0.3703 \\ -1.4543 \end{bmatrix}$$

$$\boxed{\therefore y(0.2) = 0.3703}$$

5. [25 points] Consider the definite integral

$$\int_1^2 x \ln x \; dx. \tag{5}$$

(a) [15 points] Determine the values of n and h required to approximate the integral (5) to within 10^{-4} using the composite Simpson's rule.

(b) [10 points] Compute the value of the integral using composite Simpson's rule.

5(a) Let $f(x) = x \ln x$

$f'(x) = \ln x + 1$

$f''(x) = 1/x$

$f'''(x) = -1/x^2$

and $f^{(4)}(x) = 2/x^3$

ERROR $= \left| \left(\frac{b-a}{180}\right) h^4 f^{(4)}(\mu) \right|$

The largest value of $f^{(4)}(x)$ over $[1,2]$ is 2 (when $x=1$)

$\therefore \left(\frac{1}{180}\right) h^4 (2) \leq 10^{-4}$

$\therefore h^4 \leq \frac{180 \times 10^{-4}}{2}$

$\therefore h \leq 0.308$

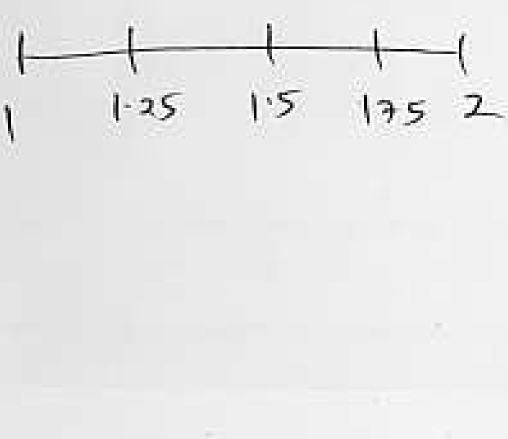
$\therefore h = \left(\frac{b-a}{n}\right) \leq 0.308$

$\therefore n \geq 3.2466$

$\therefore n = 4 \rightarrow$ EVEN

and $h = \left(\frac{b-a}{4}\right)$

$= 1/4$



$$\therefore \int_1^2 x \ln x$$

$$= \frac{h}{3}\left[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)\right]$$

$$= \frac{0.25}{3}\left[0 + 4(1.25 \times \ln(1.25)) + 2(1.5 \times \ln(1.5)) + 4(1.75 \times \ln(1.75)) + 2 \times \ln 2\right]$$

$$= \boxed{0.636309}$$