MA 341-002 - Applied Differential Equations I
Summer II session
Test 3 - August 2, 2007
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SOLUTIONS TO TEST 3

PREPARED BY KARTIK

INSTRUCTIONS

1. Please write your name and student number clearly on the front page of the exam.
2. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
3. TIME LIMIT: 90 minutes
4. There are 5 pages and 4 questions on the exam. Each question appears on a different page. Read each question carefully.
5. The exam is worth 105 points with 5 extra credit points. The distribution of points is clearly indicated on the exam.
6. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
7. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
8. You are allowed two crib sheets of formulas on the exam.
9. You may also use an electronic calculator on the exam.

1. [30 points] Find the inverse Laplace transform of

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

using the following techniques

(a) [20 points] Convolution Theorem.

(b) [10 points] Partial fractions.

You should get the same answer in either case.

(1)
(a)

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{(s+1)} \times \frac{1}{(s+2)}\right\}$$

$$= \int_0^t e^{-v} e^{-2(t-v)} dv = \int_0^t e^{-2t} e^{v} dv$$

$$= e^{-2t}\left[e^{v}\right]_{v=0}^{v=t} = e^{-2t}\left(e^{t}-1\right) = \boxed{e^{-t}-e^{-2t}}$$

1 (b)

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{A}{(s+1)} + \frac{B}{(s+2)}\right\}$$

$$= L^{-1}\left\{\frac{A(s+2)+B(s+1)}{(s+1)(s+2)}\right\}$$

$\therefore \; A(s+2) + B(s+1) = 1$

$\Rightarrow \; (A+B)s + (2A+B) = 1$

$A+B=0$ $\qquad$ $2A+B=1$

$A=-B$ $\qquad$ $-2B+B=1$

$\qquad\qquad\quad B=-1$

$\therefore \; A=1$

$$\Rightarrow \; \therefore \; L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)} - \frac{1}{(s+2)}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)}\right\} - L^{-1}\left\{\frac{1}{(s+2)}\right\}$$

$$= e^{-t} - e^{-2t}$$

2. [20 points] Find the solution $y(t)$ to the following initial value problem

$$\frac{d^2y}{dt^2} + y = u(t-3); \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1;$$

using Laplace transforms.

Let $Y(s) = L\{y(t)\}$

Taking Laplace transforms on both sides of ODE ques

$$\left(s^2 Y(s) - sy(0) - y'(0)\right) + Y(s) = \frac{e^{-3s}}{s}$$

$$\therefore \left(s^2 Y(s) - 1\right) + Y(s) = \frac{e^{-3s}}{s}$$

$$\therefore (s^2+1)Y(s) = 1 + \frac{e^{-3s}}{s}$$

$$\therefore \boxed{Y(s) = \frac{1}{(s^2+1)} + \frac{e^{-3s}}{s(s^2+1)}}$$

Taking inverse Laplace transforms ques

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{(s^2+1)}\right\} + L^{-1}\left\{\frac{e^{-3s}}{s(s^2+1)}\right\}$$

Now $L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = L^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{(s^2+1)}\right\}$

$$= L^{-1}\left\{\frac{A(s^2+1) + Bs^2 + Cs}{s(s^2+1)}\right\}$$

$\therefore A(s^2+1) + Bs^2 + Cs = 1$

$\Rightarrow (A+B)s^2 + Cs + A = 1$

$\Rightarrow A = -B$ $\quad \therefore B = -1$

$C = 0$

$A = 1$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{s}{(s^2+1)}\right\}$$

$$= 1 - \cos t$$

$$\therefore\ y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2+1)}\right\}$$

$$= \boxed{\sin t + \left(1-\cos(t-3)\right)u(t-3)}$$

(USING

$\mathcal{L}^{-1}\{e^{-as}F(s)\}$

$= f(t-a)\,u(t-a)$

where $\mathcal{L}^{-1}\{F(s)\} = f(t)$)

3. [35 points] Consider the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 8y = 0; \;\; y(0) = 3, \;\; \frac{dy}{dt}(0) = -12.$$

(a) [10 points] Write the initial value problem as a system of two first order ODEs in matrix notation.

(b) [25 points] Solve the initial value problem in matrix notation.

3(a) Let $x_1 = y$ and $x_2 = y' = \left(\frac{dy}{dt}\right)$

$\therefore \dot{x}_1 = x_2$ OR $\boxed{\left(\frac{dx_1}{dt}\right) = x_2}$

we also have

$\frac{dx_2}{dt} + 2x_2 - 8x_1 = 0 \implies \boxed{\left(\frac{dx_2}{dt}\right) = 8x_1 - 2x_2}$

$$\therefore \boxed{\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

3(b) Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$

Characteristic equation is

$\det(A - \lambda I) = 0$

$\implies \det \begin{bmatrix} -\lambda & 1 \\ 8 & -2-\lambda \end{bmatrix} = 0 \implies -\lambda(-2-\lambda) - 8 = 0$

$\implies \lambda^2 + 2\lambda - 8 = 0$

$(\lambda + 4)(\lambda - 2) = 0$

$\implies \lambda_1 = 2$ and $\lambda_2 = -4$

Compute eigenvector u corresponding to $\lambda_1 = 2$

$(A - \lambda_1 I)u = 0$

$$\begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Augmented system

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 8 & -4 & 0 \end{array}\right]$$

$\downarrow \quad R_1 = \frac{-1}{2} R_1$

$$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 8 & -4 & 0 \end{array}\right]$$

$\downarrow \quad R_2 = R_2 - 8R_1$

$$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This gives

$u_1 - 1/2\, u_2 = 0$

Choosing $u_2 = 2$

gives $u_1 = 1$

$\therefore$ Eigenvector

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Compute eigenvector u corresponding to $\lambda_2 = -4$

$(A - \lambda_2 I)u = 0$

$$\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Augmented system

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 8 & 2 & 0 \end{array}\right]$$

$\downarrow \quad R_1 = 1/4\, R_1$

$$\left[\begin{array}{cc|c} 1 & 1/4 & 0 \\ 8 & 2 & 0 \end{array}\right]$$

$\downarrow \quad R_2 = R_2 - 8R_1$

$$\left[\begin{array}{cc|c} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This gives

$u_1 + 1/4\, u_2 = 0$

$u_1 = -1/4\, u_2$

Taking $u_2 = 4$

gives $u_1 = -1$.

$\therefore$ Eigenvector

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$\therefore$ General solution

$x(t) = c_1 x_1(t) + c_2 x_2(t)$

where

$x_1(t) = e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $x_2(t) = e^{-4t}\begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$$\therefore x(t) = \begin{bmatrix} c_1 e^{2t} - c_2 e^{-4t} \\ 2c_1 e^{2t} + 4c_2 e^{-4t} \end{bmatrix}$$

SUBSTITUTING INITIAL CONDITIONS GIVES

$$\begin{bmatrix} 3 \\ -12 \end{bmatrix} = x(0) = \begin{bmatrix} c_1 - c_2 \\ 2c_1 + 4c_2 \end{bmatrix}$$

$\therefore c_1 - c_2 = 3$

$c_1 + 2c_2 = -6$

$\therefore 3c_1 = 0$

$c_1 = 0$

and $c_2 = -3$

$$\therefore x(t) = \begin{bmatrix} +3e^{-4t} \\ -12e^{-4t} \end{bmatrix}$$

$$\boxed{-3e^{-4t}\begin{bmatrix} -1 \\ 4 \end{bmatrix}}$$

4. [20 points] Determine the Laplace transform of the periodic function

$$f(t) = \begin{cases} e^{-t} & 0 < t < 1, \\ 1 & 1 < t < 2, \end{cases}$$

where $f(t)$ has period 2.

The Snapshot function

$$f_T(t) = e^{-t} \qquad 0 < t < 1$$
$$= 1 \qquad 1 < t < 2$$
$$= 0 \qquad t > 2$$

$$f_T(t) = e^{-t} + (1 - e^{-t})\, u(t-1) + (0-1)\, u(t-2)$$

$$F_T(s) = L\{f_T(t)\} = L\{e^{-t} + u(t-1) - e^{-t}u(t-1) - u(t-2)\}$$

$$= L\{e^{-t}\} + L\{u(t-1)\} - L\{e^{-t}u(t-1)\} - L\{u(t-2)\}$$

$$= \frac{1}{(s+1)} + \frac{e^{-s}}{s} - e^{-s}\, L\{e^{-(t+1)}\} - \frac{e^{-2s}}{s}$$

$$= \frac{1}{(s+1)} + \frac{e^{-s}}{s} - e^{-s} e^{-1}\, L\{e^{-t}\} - \frac{e^{-2s}}{s}$$

$$= \frac{1}{(s+1)} + \frac{e^{-s}}{s} - e^{-(s+1)} \left[\frac{1}{s+1}\right] - \frac{e^{-2s}}{s}$$

$$\therefore\ F(s) = L\{f(t)\} = \frac{F_T(s)}{1 - e^{-sT}}$$

$$= \boxed{\frac{\left(1 - e^{-(s+1)}\right)\dfrac{1}{(s+1)} + \dfrac{e^{-s}}{s}\left(1 - e^{-s}\right)}{1 - e^{-2s}}}$$