MA 341 - Applied Differential Equations I
Final Exam - December 8, 2005
Instructor: *Dr. Kartik Sivaramakrishnan*

SOLUTIONS FOR FINAL EXAM

PREPARED BY KARTIK

INSTRUCTIONS

1. Please write your name and student number clearly on the front page of the exam.
2. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
3. TIME LIMIT: 3 hours
4. There are 7 pages and 6 questions on the exam. Each question appears on a different page. Read each question carefully.
5. The exam is worth 160 points out of which 10 points are extra credit. The distribution of points is clearly indicated on the exam.
6. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
7. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
8. Sources: Open book and class notes.
9. You may use an electronic calculator on the exam.

1. [20 points] Consider the periodic square wave function given by

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases},$$

where $f(t)$ has period $T = 2$.

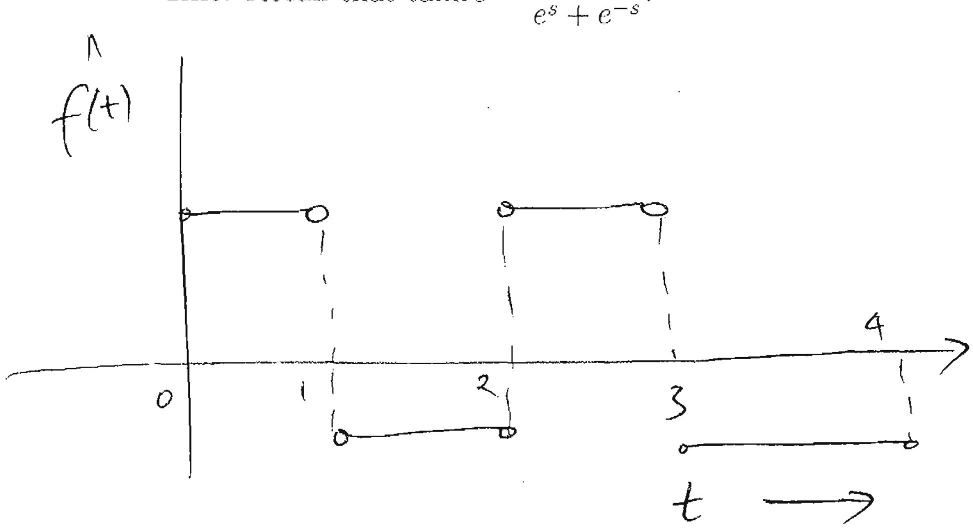
(a) [5 points] Graph $f(t)$ and its *windowed version* $f_T(t)$.

(b) [15 points] Show that

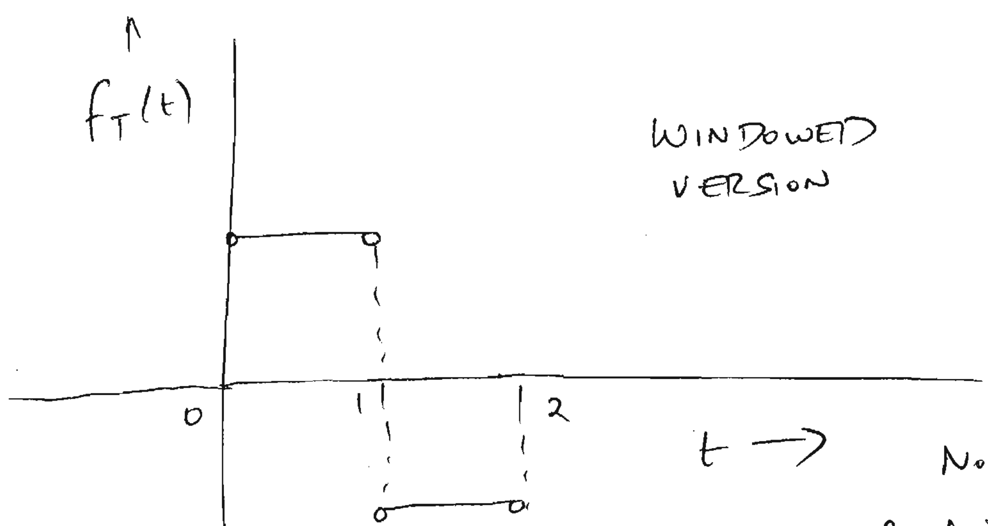
$$\mathcal{L}(f(t)) = \frac{1}{s}\tanh\frac{s}{2}.$$

Hint: Recall that $\tanh s = \dfrac{e^s - e^{-s}}{e^s + e^{-s}}$.

(a)



(b)



WINDOWED VERSION

$f_T(t) = 1, \quad 0<t<1$
$= -1, \quad 1<t<2$
$= 0, \quad$ OTHERWISE

Note

$f_T(t) = 1 - 2u(t-1) + u(t-2)$

$\therefore\ f_T(t) = 1 - 2u(t-1) + u(t-2)$

$$L\{f_T(t)\} = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$= \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{1}{s}\left(1 - e^{-s}\right)^2$$

$$L\{f(t)\} = \frac{L\{f_T(t)\}}{1 - e^{-sT}} = \frac{L\{f_T(t)\}}{1 - e^{-2s}}$$

Since $T = 2$

$$= \frac{1}{s} \frac{\left(1 - e^{-s}\right)^2}{\left(1 - e^{-2s}\right)}$$

$$= \frac{1}{s} \frac{\left(1 - e^{-s}\right)}{\left(1 + e^{-s}\right)}$$

$$= \frac{1}{s} \frac{\left(e^{s/2} - e^{-s/2}\right)}{\left(e^{s/2} + e^{-s/2}\right)}$$

$$= \frac{1}{s} \operatorname{Tanh}\left(\frac{s}{2}\right)$$

2. [25 points] You will be required to find inverse Laplace transforms in this question.

(a) [5 points] Find $\mathcal{L}^{-1}\left(\frac{1-e^{-2\pi s}}{s^2+1}\right)$.

(b) [10 points] Use the convolution theorem to find $\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+9)}\right)$.

(c) [10 points] Use partial fractions to find $\mathcal{L}^{-1}\left(\frac{1}{s^3-5s^2}\right)$.

Note: You can use the table of integrals in the front of the book if necessary.

2(a) $$L^{-1}\left(\frac{1-e^{-2\pi s}}{s^2+1}\right) = L^{-1}\left(\frac{1}{s^2+1}\right) - L^{-1}\left(\frac{e^{-2\pi s}}{s^2+1}\right)$$

$$= \sin t - \sin(t-2\pi)\, u(t-2\pi)$$

$$= \sin t - \sin t\, u(t-2\pi)$$ since $\sin(t-2\pi) = \sin t$

$$= \boxed{\sin t\,(1-u(t-2\pi))}$$

2(b) $$L^{-1}\left(\frac{1}{s^2(s^2+9)}\right) = L^{-1}\left(\left(\frac{1}{s^2}\right)\frac{1}{3}\left(\frac{3}{s^2+9}\right)\right) = t * \left(\frac{1}{3}\sin 3t\right)$$

$$= \int_0^t (t-v)\left(\frac{1}{3}\sin 3v\right) dv$$

$$= \frac{t}{3}\left[-\frac{\cos 3v}{3}\right]_0^t - \frac{1}{3}\int_0^t v\sin 3v\, dv$$ ↳ INTEGRATE BY PARTS

$$= \frac{t}{3}\left[\frac{1}{3}(1-\cos 3t)\right] - \frac{1}{3}\left[v\left(-\frac{\cos 3v}{3}\right)\Big|_0^t - \int_0^t \left(-\frac{\cos 3v}{3}\right) dv\right]$$

$$= \frac{t}{9}(1-\cos 3t) - \frac{1}{3}\left[-\frac{t\cos 3t}{3} + \frac{1}{3}\left[\frac{\sin 3v}{3}\right]_0^t\right]$$

$$= \frac{t}{9}(1-\cos 3t) + \frac{t}{9}\cos 3t - \frac{1}{27}\sin 3t = \frac{t}{9} - \frac{1}{27}\sin 3t$$

$$= \boxed{\frac{1}{9}\left(t - \frac{1}{3}\sin 3t\right)}$$

2(c) $\mathcal{L}^{-1}\left(\frac{1}{s^3-5s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2(s-5)}\right)$

Now $\frac{1}{s^2(s-5)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s-5)} = \frac{A(s-5)+Bs(s-5)+Cs^2}{s^2(s-5)}$

$= \frac{(B+C)s^2 + (A-5B)s - 5A}{s^2(s-5)}$

$\therefore -5A = 1$ $\quad A - 5B = 0$ $\quad B + C = 0$

$A = -1/5$ $\quad 5B = A$ $\quad C = -B$

$B = A/5 = -1/25$ $\quad = 1/25$

$\therefore \frac{1}{s^2(s-5)} = \frac{-1}{5s^2} - \frac{1}{25s} + \frac{1}{25(s-5)}$

$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^3-5s^2}\right) = \mathcal{L}^{-1}\left(\frac{-1}{5s^2} - \frac{1}{25s} + \frac{1}{25(s-5)}\right)$

$= \boxed{\frac{-1}{5}t - \frac{1}{25} + \frac{1}{25}e^{5t}}$

3. [30 points] Consider the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 8y = 0; \;\; y(0) = 3, \;\; \frac{dy}{dt}(0) = -12. \qquad (1)$$

(a) [10 points] Using state variables $x_1 = y$ and $x_2 = \frac{dy}{dt}$, write (1) as an initial value problem in matrix notation.

(b) [20 points] Solve the initial value problem in matrix notation using the techniques in Chapter 9. (**Note:** Do not use the techniques in Chapter 4 to solve (1) directly!)

(a) Consider

$y'' + 2y' - 8y = 0$; $y(0) = 3$, $y'(0) = -12$

Let $x_1 = y$, $x_2 = y'$

Also $y'' + 2y' - 8y$
$= x_2' + 2x_2 - 8x_1$
$= 0$

$\therefore x_2 = x_1'$

$\therefore x_1' = x_2$

$x_2' = 8x_1 - 2x_2$

$\therefore \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix}$

$\hookrightarrow A$

(b) Consider $A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$

Eigenvalues of A are solutions to

$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 8 & -2-\lambda \end{bmatrix} = 0$

i.e. $-\lambda(-2-\lambda) - 8 = 0$

$\lambda^2 + 2\lambda - 8 = 0$

$\therefore \lambda_1 = -4$ and $\lambda_2 = 2$

(c) Find eigenvectors $u_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$ and $u_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$ corresponding to $\lambda_1 = -4$ and $\lambda_2 = 2$.

We have

$(A - \lambda_1 I) u_1 = 0$

$$\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore 4u_{11} + u_{12} = 0$

$u_{12} = -4u_{11} = -4$ when $u_{11} = 1$

$\therefore u_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\therefore X_1(t) = e^{\lambda_1 t} u_1$

$= \begin{bmatrix} e^{-4t} \\ -4e^{-4t} \end{bmatrix}$

We have

$(A - \lambda_2 I) u_2 = 0$

$$\begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -2u_{21} + u_{22} = 0$

$u_{22} = 2u_{21}$

$= 2$

when $u_{21} = 1$

$\therefore u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$X_2(t) = e^{\lambda_2 t} u_2$

$= \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$

$\therefore X(t) = c_1 X_1(t) + c_2 X_2(t)$

$$= \begin{bmatrix} e^{-4t} & e^{2t} \\ -4e^{-4t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Now $X(0) = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix}$

$c_1 + c_2 = 3$

$c_2 = 3 - c_1$

$\therefore -4c_1 + 2(3 - c_1) = -12$

$-6c_1 = -18$

$c_1 = 3$

$c_2 = 3 - c_1 = 0$

$$\therefore X(t) = \begin{bmatrix} e^{-4t} & e^{2t} \\ -4e^{-4t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3e^{-4t} \\ -12e^{-4t} \end{bmatrix}$$

4. [30 points] Find a solution to (2) using the steps outlined below:

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = 0; \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 3. \tag{2}$$

(a) [10 points] Show that taking Laplace transforms reduces (2) to the linear first order differential equation

$$\frac{dY(s)}{ds} + (\frac{2}{s} - s)Y(s) = -\frac{3}{s}, \tag{3}$$

where $Y(s) = \mathcal{L}(y(t))$.

(b) [15 points] Solve (3) for $Y(s)$. Why can you set the constant of integration to zero?

(c) [5 points] Taking inverse Laplace transforms show that the solution to (2) is $y(t) = 3t$.

(a) Taking Laplace transforms we have:

$$\left(s^2Y(s) - 3\right) + (-1)\frac{d}{ds}\left(sY(s)\right) - Y(s) = 0$$

$$\left(s^2Y(s) - 3\right) - \left(Y(s) + s\frac{dY(s)}{ds}\right) - Y(s) = 0$$

$$\therefore -s\frac{dY(s)}{ds} + (s^2-2)\,Y(s) = 3$$

$$\therefore \frac{dY(s)}{ds} + \left(\frac{2}{s} - s\right)Y(s) = -3/s$$

(b) $$\text{I.F} = e^{\int(\frac{2}{s}-s)\,ds} = e^{\left[\ln s^2 - \frac{s^2}{2}\right]} = s^2e^{-s^2/2}$$

$$\therefore Y(s) = \frac{1}{\text{I.F}}\left[\int (\text{I.F})\left(\frac{-3}{s}\right)ds + c\right] = \frac{e^{s^2/2}}{s^2}\left[\int s^2e^{-s^2/2}\left(\frac{-3}{s}\right)ds + c\right]$$

$$\therefore Y(s) = \frac{e^{s^2/2}}{s^2}\left[+3\int e^{-s^2/2}(-s\,ds) + C\right]$$

$$= \frac{e^{s^2/2}}{s^2}\left[3e^{-s^2/2} + C\right]$$

$$= \frac{3}{s^2} + \frac{Ce^{s^2/2}}{s^2}$$

Since $y(t)$ is of exponential order

$\lim_{s \to \infty} Y(s) = 0$

$\therefore C = 0$

$$\therefore Y(s) = \frac{3}{s^2}$$

(c) Taking the inverse Laplace transform gives

$y(t) = 3t$ (since $L(t) = 1/s^2$)

5. [30 points] Determine the equation of motion for an undamped spring mass system at resonance governed by

$$\frac{d^2y}{dt^2} + y = 5\cos t; \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 1. \tag{4}$$

(a) Find the homogeneous solution $y_h(t)$

Char equation is $r^2 + 1 = 0$

$\therefore r_1 = i$ and $r_2 = -i$

$\therefore y_h(t) = c_1 \cos t + c_2 \sin t$

(b) Find the particular solution $y_p(t)$

$y_p(t) = t(A\cos t + B\sin t)$

FIND A & B using method of undetermined coefficients

$\therefore y_p'(t) = (A\cos t + B\sin t) + t(-A\sin t + B\cos t)$

$y_p''(t) = (-A\sin t + B\cos t) + (-A\sin t + B\cos t) + t(-A\cos t - B\sin t)$

$= -2A\sin t + 2B\cos t - t(A\cos t + B\sin t)$

$\therefore y_p''(t) + y_p(t) = -2A\sin t + 2B\cos t = 5\cos t$

$\therefore B = 5/2 \quad A = 0$

$\therefore y_p(t) = \frac{5}{2} t \sin t$

(c) general solution

$$y(t) = y_h(t) + y_p(t)$$
$$= c_1 \cos t + c_2 \sin t + \frac{5}{2} t \sin t$$

$$y(0) = c_1 = 0 \qquad \therefore c_1 = 0$$

$$y'(t) = -c_1 \sin t + c_2 \cos t + \frac{5}{2} \sin t + \frac{5}{2} t \cos t$$

$$\therefore y'(0) = c_2 = 1 \qquad \therefore c_2 = 1$$

$$\therefore y(t) = \sin t + \frac{5}{2} t \sin t$$
$$= \boxed{\left(1 + \frac{5}{2} t\right) \sin t}$$

6. [25 points] Consider a body that moves horizontally through a medium whose resistance is proportional to the *square* of the velocity v, so that

$$\frac{dv}{dt} = -kv^2.$$

Show the following

(a) [10 points] Assuming $v(0) = v_0$ show that

$$v(t) = \frac{v_0}{1 + v_0 kt}. \quad (5)$$

(b) [10 points] Using $v = \frac{dx}{dt}$ in (5) and $x(0) = x_0$ show that

$$x(t) = x_0 + \frac{1}{k}\ln(1 + v_0 kt).$$

(c) [5 points] What is $x(t)$ as $t \to \infty$?

6 (a)

$$\frac{dv}{dt} = -kv^2$$

$$\therefore \frac{dv}{v^2} = -k\,dt$$

$$\int \frac{dv}{v^2} = -k\int dt + c$$

$$\therefore \left[\frac{v^{-1}}{-1}\right] = -kt + c$$

when $t = 0$ $\quad v = v_0 \quad \therefore c = -\frac{1}{v_0}$

$$\therefore -\frac{1}{v} = -kt - \frac{1}{v_0}$$

$$\frac{1}{v} = kt + \frac{1}{v_0} \qquad \therefore \frac{1}{v} = \frac{kt\,v_0 + 1}{v_0}$$

$$\boxed{v = \frac{v_0}{1 + kt\,v_0}}$$

6(b)

$$v = \frac{v_0}{1+ktv_0}$$

$$\left(\frac{dx}{dt}\right) = \frac{v_0}{1+ktv_0}$$

$$\therefore dx = \frac{v_0 \, dt}{kv_0\left(\frac{1}{kv_0}+t\right)}$$

$$\therefore dx = \frac{dt}{k\left(\frac{1}{kv_0}+t\right)}$$

$$\therefore x = \frac{1}{k}\ln\left|\frac{1}{kv_0}+t\right| + c$$

$$\therefore \ln\left|\frac{1}{kv_0}+t\right| = (kx - c)$$

$$\therefore \left|\frac{1}{kv_0}+t\right| = e^{-c}e^{kx}$$

$$\therefore \left|\frac{1}{kv_0}+t\right| = Ce^{kx} \quad \text{where } C>0$$

$$\therefore \left(\frac{1}{kv_0}+t\right) = Ke^{kx} \quad K \text{ unrestricted}$$

At $t=0$ $x = x_0$

$$\therefore \frac{1}{kv_0} = Ke^{kx_0}$$

$$\therefore K = \frac{1}{(kv_0e^{kx_0})}$$

$$\therefore \left(\frac{1}{kv_0}+t\right) = \frac{1}{kv_0e^{kx_0}}e^{kx}$$

$$(1+kv_0t) = e^{k(x-x_0)}$$

$$\therefore k(x-x_0) = \ln(1+kv_0t)$$

$$\boxed{x = x_0 + \frac{1}{k}\ln(1+kv_0t)}$$

6(c)

As $t \to \infty$

$x(t) \to \infty$