HOMEWORK # 6
SOLUTIONS BY KARTIK

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Section 9.5
PROBLEM 35(a)

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$$

(a) Eigenvalues of A are the solutions λ to

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-3-\lambda) + 4 = 0$$

$$-3 - \lambda + 3\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0$$

$\therefore$ The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -1$.

(a) The eigenvector u_1 associated with $\lambda_1 = -1$ is the solution to

$(A - \lambda_1 I)\, u_1 = 0$ Let $u_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = 0$$

$\therefore\ 2u_{11} - u_{12} = 0$

$u_{12} = 2u_{11}$ Let $u_{11} = 1$, $u_{12} = 2$

$\therefore\ u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector.

(b) A nontrivial soln $x_1(t)$ to $x' = Ax$ is $x_1(t) = e^{\lambda_1 t} u_1$

$$x_1(t) = e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(c) Let $x_2(t) = te^{-t} u_1 + e^{-t} u_2$

$$\therefore\ x_2'(t) = \left(e^{-t} u_1 - te^{-t} u_1\right) - e^{-t} u_2$$

$$= e^{-t}(u_1 - u_2) - te^{-t} u_1$$

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$\therefore X_2'(t) = A X_2(t)$ gives

$e^{-t}(u_1-u_2) - te^{-t}u_1 = A\left(te^{-t}u_1 + e^{-t}u_2\right)$

$\therefore \quad -u_1 = Au_1 \quad$ OR $\quad (A+I)u_1 = 0$
(comparing the coeffs of te^{-t} on each side)

Also

$Au_2 = (u_1 - u_2)$

$(A+I)u_2 = u_1$

Now u_1 is the eigenvector $\begin{pmatrix}1\\2\end{pmatrix}$ that we obtained in step (a)

$\therefore \quad (A+I)u_2 = \begin{pmatrix}1\\2\end{pmatrix}$ Let $u_2 = \begin{pmatrix}u_{21}\\u_{22}\end{pmatrix}$

$\begin{pmatrix}2 & -1\\4 & -2\end{pmatrix}\begin{pmatrix}u_{21}\\u_{22}\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$\therefore \quad 2u_{21} - u_{22} = 1$

$\therefore \quad u_{22} = -1 + 2u_{21}$ $\quad \therefore \quad u_2 = \begin{bmatrix}u_{21}\\u_{22}\end{bmatrix}$

If $u_{21} = 1$
then $u_{22} = 1$ $\quad = \begin{bmatrix}1\\1\end{bmatrix}$.

(d)

$$(A+I)^2 u_2$$

$$A+I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

$$(A+I)^2 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \boxed{(A+I)^2 u_2 = 0}$$

(2)

Sec 9.6
Problem 41

Consider
$$ay'' + by' + cy = 0$$

Aux equation
is $ar^2 + br + c = 0$
(from Chapter 4)

Let $x_1 = y$
$x_2 = y'$

$\therefore$ We have:

$$x_1' = x_2$$

and

$$a x_2' + b x_2 + c x_1 = 0$$

$$x_2' = -\frac{c}{a} x_1 - \frac{b}{a} x_2$$

$$\therefore \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Consider the char equation of A

i.e. $\det (A - \lambda I) = 0$

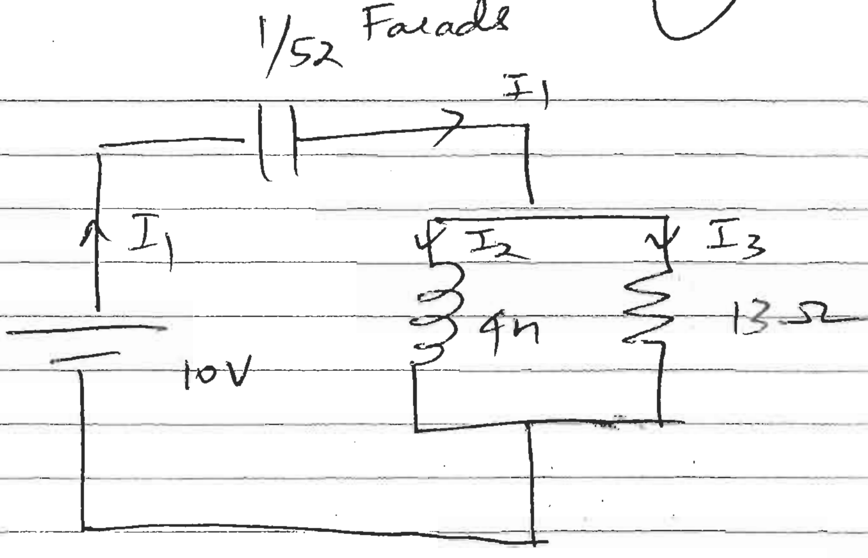
$$\det \begin{bmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{bmatrix} = 0$$

i.e. $-\lambda \left(-\frac{b}{a} - \lambda\right) + \frac{c}{a} = 0$

OR $a\lambda^2 + b\lambda + c = 0$
which is our char equation too.

(3)

PROBLEM 21
EXERCISE 9.6



The set of equations describing the electrical circuit are

$$4 I_2'(t) + 52 q_1(t) = 10$$

$$13 I_3(t) + 52 q_1(t) = 10$$

$$I_1(t) = I_2(t) + I_3(t)$$

Diff the first 2 equations and substitute the value of I_1 in these equations gives

$$4 I_2''(t) + 52 (I_2(t) + I_3(t)) = 0$$

$$13 I_3'(t) + 52 (I_2(t) + I_3(t)) = 0$$

Since $I_1(t) = \frac{dq_1(t)}{dt}$ etc.

Let $x_1 = I_2$, $x_2 = I_2'$

and $x_3 = I_3$

∴ The system of diff equations can be written as the system of the following three first order differential equations

$$4x_2'(t) + 52x_1(t) + 52x_3(t) = 0$$

$$13x_3'(t) + 52x_1(t) + 52x_3(t) = 0$$

$$x_1'(t) = x_2(t)$$

∴ We have

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -13 & 0 & -13 \\ -4 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The initial conditions are

$$4I_2'(0) + 52q_1(0) = 10$$

$$\therefore I_2'(0) = 5/2 \quad \text{since } q_1(0) = 0$$

Also

$$13 I_3(0) + 52 q_1(0) = 10$$

$$\therefore I_3(0) = 10/13$$

Also $I_1(0) = I_2(0) + I_3(0) = 0$

$$\therefore I_2(0) = -I_3(0) = -10/13$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} I_2(0) \\ I_2'(0) \\ I_3(0) \end{bmatrix}$$

$$= \begin{bmatrix} -10/13 \\ 5/2 \\ 10/13 \end{bmatrix}$$

(a) Compute eigenvalues and eigenvectors of A

$$[V, D] = eig(A)$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ -13 & 0 & -13 \\ -4 & 0 & -4 \end{bmatrix}$

The eigenvalues are

$l_1 = -2+3i$, $l_2 = -2-3i$
and $l_3 = 0$

with eigenvectors

$$u_1 = \begin{bmatrix} 0.1421 \\ -0.9239 \\ -0.2843 \end{bmatrix} + i \begin{bmatrix} 0.2132 \\ 0 \\ 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0.1421 \\ -0.9239 \\ -0.2843 \end{bmatrix} - i \begin{bmatrix} 0.2132 \\ 0 \\ 0 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix}$$

Construct 3 li solutions
$x_1(t)$, $x_2(t)$, and $x_3(t)$
from the eigenvalue - eigenvector
information

Consider $x_1(t)$ and $x_2(t)$ constructed
from the complex eigenvalue

Consider

$$l_1 = -2+3i \qquad \alpha = -2, \quad \beta = 3$$

$$u_1 = \underbrace{\begin{bmatrix} 0.1421 \\ -0.9239 \\ -0.2843 \end{bmatrix}}_{a} + i \underbrace{\begin{bmatrix} 0.2132 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$$\therefore X_1(t) = (e^{\alpha t}\cos\beta t)\, a - (e^{\alpha t}\sin\beta t)\, b$$

$$= e^{-2t}\begin{bmatrix} 0.1421\cos 3t - 0.2132\sin 3t \\ -0.9239\cos 3t \\ -0.2843\cos 3t \end{bmatrix}$$

$$X_2(t) = (e^{\alpha t}\sin\beta t)\, a + (e^{\alpha t}\cos\beta t)\, b$$

$$= e^{-2t}\begin{bmatrix} 0.1421\sin 3t + 0.2132\cos 3t \\ -0.9239\sin 3t \\ -0.2843\sin 3t \end{bmatrix}$$

Also

$$X_3(t) = e^{l_3 t} u_3$$

$$= \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix}$$

(4)

∴ The general solution

$$X(t) = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ X_1(t) & X_2(t) & X_3(t) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\therefore X(t) = \begin{bmatrix} e^{-2t}(0.1432\cos 3t - 0.2132\sin 3t) & e^{-2t}(0.1432\sin 3t + 0.2132\cos 3t) & -0.7071 \\ (-0.9239\cos 3t)\,e^{-2t} & (-0.9239\sin 3t)\,e^{-2t} & 0\,e^{2t} \\ (-0.2843\cos 3t)\,e^{-2t} & (-0.2843\sin 3t)\,e^{-2t} & 0.7071 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\therefore X(0) = \begin{bmatrix} 0.1432 & 0.2132 & -0.7071 \\ -0.9239 & 0 & 0 \\ -0.2843 & 0 & 0.7071 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} -10/13 \\ 5/2 \\ 10/13 \end{bmatrix}$$

$$C = \begin{bmatrix} -2.7059 \\ -1.7908 \\ -0.001 \end{bmatrix}$$ Using MATLAB

$C = A \backslash b$ where $A = \begin{bmatrix} 0.1432 & 0.2132 & -0.7071 \\ -0.9239 & 0 & 0 \\ -0.2843 & 0 & 0.7071 \end{bmatrix}$

and $b = \begin{bmatrix} -10/13 \\ 5/2 \\ 10/13 \end{bmatrix}$

$$\therefore X(t) = \begin{bmatrix} e^{-2t}(0.1432 \cos 3t - 0.2132 \sin 3t) & e^{-2t}(0.1432 \sin 3t + 0.2132 \cos 3t) & -0.7071 \\ (-0.9239 \cos 3t)e^{-2t} & (-0.9239 \sin 3t)e^{-2t} & 0 \\ (-0.2843 \cos 3t)e^{-2t} & (-0.2843 \sin 3t)e^{-2t} & 0.7071 \end{bmatrix}$$

$$\begin{bmatrix} -2.7059 \\ -1.7908 \\ -0.001 \end{bmatrix}$$

$$\therefore X(t) = \begin{bmatrix} e^{-2t}(-0.3875 \cos 3t + 0.3869 \sin 3t) \\ e^{-2t}(-0.2569 \sin 3t \; [illegible] \; 0.3818 \cos 3t) \\ e^{-2t}(2.5 \cos 3t + 1.6545 \sin 3t) \\ (0.7693 \cos 3t + 0.5091 \sin 3t)e^{-2t} \end{bmatrix}$$

$$\therefore \; x(t) = \begin{bmatrix} (-0.7693\cos 3t + 0.3205 \sin 3t)\, e^{-2t} \\ e^{-2t}\,(2.5\cos 3t + 1.6595 \sin 3t) \\ e^{-2t}\,(0.7693 \cos 3t + 0.5091 \sin 3t) \end{bmatrix}$$

$\therefore \; I_2(t)$

$$= e^{-2t}\,(-0.7693\cos 3t + 0.3205 \sin 3t)$$

$$I_3(t) = e^{-2t}\,(0.7693 \cos 3t + 0.5091 \sin 3t)$$

$$I_1(t) = I_2(t) + I_3(t)$$

$$= 0.8296\, e^{-2t} \sin 3t$$

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