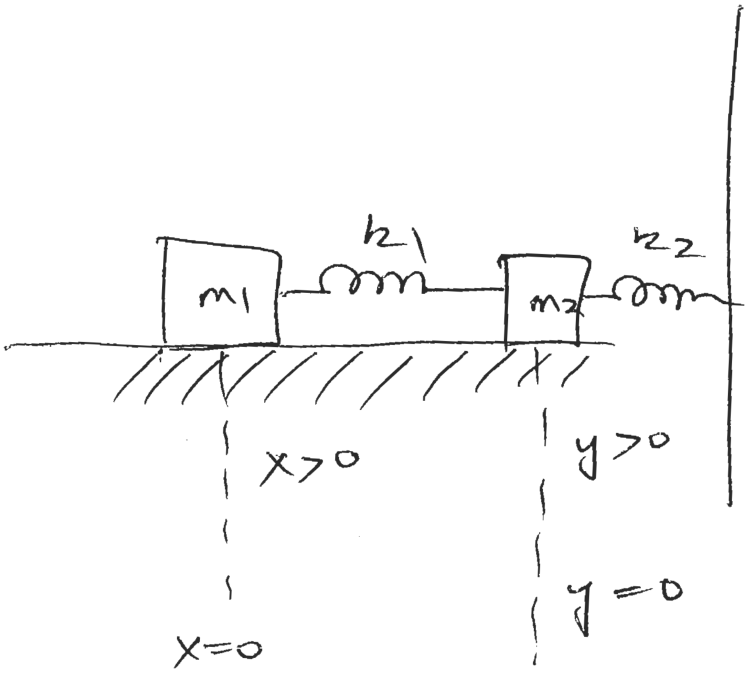
HOMEWORK # 6

SOLUTIONS (PREPARED BY KARTIK)

PROBLEM 4



$m_1 = 1$ kg
$m_2 = 2$ kg
$k_1 = 4$ N/m
$k_2 = 10/3$ N/m

Using the freebody diagrams the system of differential equations describing the spring-mass system can be written as

$$m_1 x'' = k_1 (y - x)$$
$$m_2 y'' = -k_1 (y - x) - k_2 y$$

(SIMILAR TO THE DISCUSSION IN CLASS)

The initial conditions are $x(0) = -1$, $x'(0) = 0$, $y(0) = 0$, and $y'(0) = 0$.

Let $x_1 = x$

$x_2 = x' = x_1'$

$x_3 = y$

$x_4 = y' = x_3'$

We have

$$m_1 x_2' = k_1 x_3 - k_1 x_1$$

$$m_2 x_4' = -k_1 x_3 + k_1 x_1 - k_2 x_3$$

$$\therefore x_2' = \frac{k_1}{m_1} x_3 - \frac{k_1}{m_1} x_1$$

$$x_4' = \frac{-k_1}{m_2} x_3 + \frac{k_1}{m_2} x_1 - \frac{k_2}{m_2} x_3$$

$$\therefore \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & 0 & \frac{k_1}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & 0 & -\frac{(k_1+k_2)}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -\frac{11}{3} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$9 + 10/3$

$\frac{22}{3 \times 2}$

ALSO

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(3)

The eigenvalues are

$l_1 = 2.5820\,i \qquad l_2 = -2.5820\,i$

$l_3 = i$ and $l_4 = -i$

with associated eigenvectors

$$u_1 = \begin{bmatrix} -0.3005\,i \\ 0.7759 \\ 0.2003\,i \\ -0.5173 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.3005\,i \\ 0.7759 \\ -0.2003\,i \\ -0.5173 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -0.5657\,i \\ 0.5657 \\ -0.4243\,i \\ 0.4243 \end{bmatrix} \text{ and } u_4 = \begin{bmatrix} 0.5657\,i \\ 0.5657 \\ 0.4243\,i \\ 0.4243 \end{bmatrix}$$

Consider $l_1 = 2.5820\,i$

$\therefore \alpha = 0 \qquad \beta = 2.5820$

and

$$u_1 = \underbrace{\begin{bmatrix} 0 \\ 0.7759 \\ 0 \\ -0.5173 \end{bmatrix}}_{a} + i \underbrace{\begin{bmatrix} -0.3005 \\ 0 \\ 0.2003 \\ 0 \end{bmatrix}}_{b}$$

$$\therefore X_1 = (e^{\alpha t}\cos\beta t)\,a - (e^{\alpha t}\sin\beta t)\,b$$

$$X_2 = (e^{\alpha t}\sin\beta t)\,a + (e^{\alpha t}\cos\beta t)\,b$$

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$$\therefore X_1 = \begin{bmatrix} 0.3005 \sin(2.5820t) \\ 0.7759 \cos(2.5820t) \\ -0.2003 \sin(2.5820t) \\ -0.5173 \cos(2.5820t) \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -0.3005 \cos(2.5820t) \\ 0.7759 \sin(2.5820t) \\ 0.2003 \cos(2.5820t) \\ -0.5173 \sin(2.5820t) \end{bmatrix}$$

SIMILARLY

for $\lambda_3 = i$

$\alpha = 0$

$\beta = 1$

$$u_3 = \underbrace{\begin{bmatrix} 0 \\ 0.5657 \\ 0 \\ 0.4243 \end{bmatrix}}_{\to a} + i \underbrace{\begin{bmatrix} -0.5657 \\ 0 \\ -0.4243 \\ 0 \end{bmatrix}}_{\to b}$$

$$X_3 = \begin{bmatrix} 0.5657 \sin(t) \\ 0.5657 \cos(t) \\ 0.4243 \sin(t) \\ 0.4243 \cos(t) \end{bmatrix} \qquad X_4 = \begin{bmatrix} -0.5657 \cos(t) \\ 0.5657 \sin(t) \\ -0.4243 \cos(t) \\ 0.4243 \sin(t) \end{bmatrix}$$

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$$\therefore x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 0 & -0.3005 & 0 & -0.5657 \\ 0.7759 & 0 & 0.5657 & 0 \\ 0 & 0.2003 & 0 & -0.4243 \\ -0.5173 & 0 & 0.4243 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ +1.7620 \\ 0 \\ +0.8318 \end{bmatrix}$$

$$\therefore x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix} \begin{bmatrix} 0 \\ +1.7620 \\ 0 \\ +0.8318 \end{bmatrix}$$

$$\therefore X(t) = -(0.3005)(1.7620)\cos(2.5820t)$$

$$= x_1(t) \quad + (0.8318)(-0.5657)\cos(t)$$

$$\therefore\ x(t) = -0.5294\cos(2.5820t) - 0.4706\cos(t)$$ 6

Also

$$y(t) = x_3(t) = (0.2003)(1.7620)\cos(2.5820t) + (-0.4243)\cos(t)\,(0.8318)$$

$$= 0.3529\cos(2.5820t) - 0.3529\cos t$$