

MA 341 - Applied Differential Equations I  
Test # 3 - November 22, 2005  
Instructor: *Dr. Kartik Sivaramakrishnan*

## INSTRUCTIONS

1. Please write your name and student number clearly on the front page of the exam.
2. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
3. TIME LIMIT: 75 minutes
4. There are 6 pages and 5 questions on the exam. Each question appears on a different page. Read each question carefully.
5. The exam is worth 100 points. All problems are worth 20 points.
6. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
7. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
8. Sources: Open book and class notes.
9. You may use an electronic calculator on the exam.

1. [20 points] Apply the convolution theorem to find the following inverse Laplace transform

$$\mathcal{L}^{-1} \left\{ \frac{s}{(s-3)(s^2+1)} \right\}.$$

**Hint:** Use the *table of integrals* in the front of the book to evaluate the integral.

$$\mathcal{L}^{-1} \left\{ \frac{s}{(s-3)(s^2+1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+1)} \times \frac{1}{(s-3)} \right\}$$

$$= \cos t * e^{3t}$$

$$= \int_0^t e^{3(t-u)} \cos u \, du$$

$$= e^{3t} \int_0^t e^{-3u} \cos u \, du$$

$$= e^{3t} \int_0^t e^{-3u} \left[ \frac{-3 \cos u + \sin u}{10} \right] \Big|_{u=0}^{u=t}$$

$$= e^{3t} \left[ e^{-3t} \left( \frac{-3 \cos t + \sin t}{10} \right) + \frac{3}{10} \right]$$

$$= \frac{3}{10} e^{3t} - \frac{3}{10} \cos t + \frac{\sin t}{10}$$

2. [20 points] Find the solution  $y(t)$  to the following initial value problem

$$y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1;$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq t < \infty \end{cases}$$

Hint: You may assume that

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+1)} \right\} = 1 - \cos(t).$$

Now  $f(t) = 1 - u(t - \pi/2)$

where  $u(t-a)$  is the unit step function

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-\pi/2 s}}{s} = \frac{1}{s} (1 - e^{-\pi/2 s})$$

Taking Laplace transforms we have

$$s^2 Y(s) - 1 + Y(s) = \frac{1}{s} (1 - e^{-\pi/2 s})$$

$$\therefore Y(s) = \frac{1}{(s^2+1)} + \frac{1}{s(s^2+1)} (1 - e^{-\pi/2 s})$$

$$\therefore y(t) = \sin t + (1 - \cos t) - (1 - \cos(t - \pi/2)) u(t - \pi/2)$$

(Using the HINT!)

$$y(t) = (1 + \sin t - \cos t) - (1 - \sin t) u(t - \pi/2)$$

Since  $\cos(t - \pi/2) = \sin t$

3. [20 points] Consider the system of linear equations

$$\begin{aligned}2x_1 - x_2 &= 2, \\ -6x_1 + 3x_2 &= k.\end{aligned}$$

where  $k$  is a given scalar.

- Write down the system of equations in the matrix form  $Ax = b$ . Demonstrate that the matrix  $A$  is singular by showing that  $\det(A) = 0$ .
- If  $k = 4$ , show that the system does not have a solution.
- If  $k = -6$ , show that the system has infinitely many solutions. What is the general solution in this case?

(a) we have  $Ax = b$

with  $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$  and  $b = \begin{bmatrix} 2 \\ k \end{bmatrix}$

Now  $\det A = \det \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} = 6 - 6 = 0$

$\therefore A$  is singular

(b) Consider

Let  $k = 4$

$$2x_1 - x_2 = 2$$

$$-6x_1 + 3x_2 = k$$

$$\therefore 2x_1 - x_2 = 2$$

$$-6x_1 + 3x_2 = 4$$

Multiplying the first equation by 3  
and adding them together  
gives

$$0 = 10$$

$\therefore$  The system of equations  
is inconsistent, i.e., there is  
no solution

(c) If  $k = -6$

In this case there is only one equation

$$2x_1 - x_2 = 2$$

(since the 2nd equation is obtained by multiplying the first equation by  $-3$ )

$$\therefore x_2 = 2x_1 - 2$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 - 2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

is a solution for all values of  $x_1$ .

$\therefore$  In this case, we have infinitely many solutions.

4. [20 points] Consider the periodic half-rectified sine wave function  $f(t)$  given by

$$f(t) = \begin{cases} \sin(t), & 0 \leq t \leq \pi, \\ 0, & \pi \leq t \leq 2\pi. \end{cases}$$

where  $f(t)$  has period  $T = 2\pi$ .

(a) [5 points] Sketch the functions  $f(t)$  and  $f_T(t)$ , where  $f_T(t)$  is the *windowed* version of  $f(t)$ ? Derive an expression for  $f_T(t)$  in terms of unit step functions.

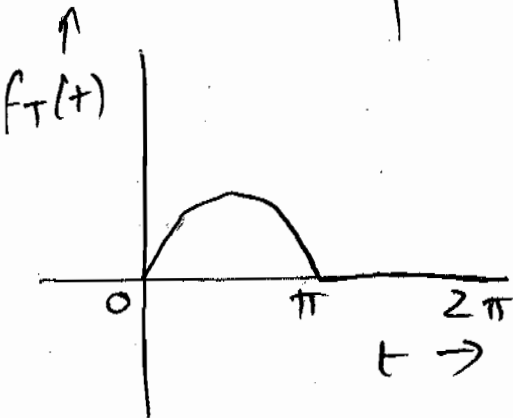
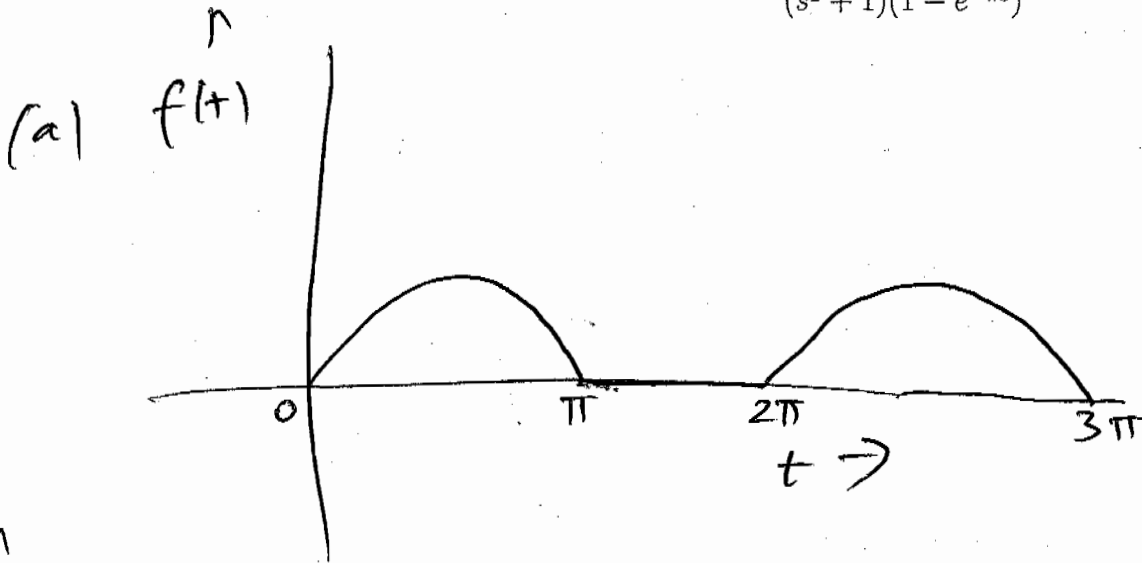
(b) [10 points] Show that

$$\mathcal{L}(f_T(t)) = \frac{(1 + e^{-\pi s})}{(s^2 + 1)}.$$

**Hint:** Use formula (8) on page 387 of the book to compute the Laplace transform. You may also assume that  $\sin(t + \pi) = -\sin(t)$ .

(c) [5 points] Show that

$$\mathcal{L}(f(t)) = \frac{1}{(s^2 + 1)(1 - e^{-\pi s})}.$$



(WINDOWED  
VERSION OF  
 $f(t)$ )

Now

$$\begin{aligned} f_T(t) &= \sin t \\ &\quad - \sin t u(t - \pi) \\ &= \sin t (1 - u(t - \pi)) \end{aligned}$$

$$\begin{aligned}
(b) \quad & \mathcal{L} \{ f_T(t) \} \\
&= \mathcal{L} \{ \sin t (1 - u(t - \pi)) \} \\
&= \mathcal{L} \{ \sin t \} - \mathcal{L} \{ \sin t u(t - \pi) \} \\
&= \frac{1}{s^2 + 1} - e^{-\pi s} \mathcal{L} \{ \sin(t + \pi) \} \\
&= \frac{1}{s^2 + 1} + e^{-\pi s} \cdot \frac{1}{s^2 + 1} \\
&= \frac{1}{s^2 + 1} (1 + e^{-\pi s})
\end{aligned}$$

$$\begin{aligned}
(c) \quad \mathcal{L} \{ f(t) \} &= \frac{\mathcal{L} \{ f_T(t) \}}{1 - e^{-sT}} \\
&= \frac{1}{s^2 + 1} \times \frac{(1 + e^{-\pi s})}{(1 - e^{-2\pi s})} \\
&= \frac{1}{(s^2 + 1)(1 - e^{-\pi s})}
\end{aligned}$$

5. [20 points] Find a general solution  $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$  for the following homogeneous system of linear differential equations

$$x'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x(t).$$

we have

$$x'(t) = A x(t)$$

where  $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

- (a) Eigenvalues of  $A$  are solutions to  $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

i.e.  $\lambda_1 = 3$  and  $\lambda_2 = -1$

- (b) Eigenvector  $u_1$  for  $\lambda_1 = 3$  is

$$(A - \lambda_1 I) u_1 = 0 \quad \text{where } u_1 = \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -2u_{11} + u_{12} = 0 \Rightarrow u_{12} = 2u_{11}$$

$$\therefore u_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} u_{11} \\ 2u_{11} \end{bmatrix} = u_{11} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

when  $u_{11} = 1$

Similarly  $u_2$  corresponds to  $\lambda_2 = -1$

$$(A - \lambda_2 I) u_2 = 0 \quad \text{where } u_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2u_{21} + u_{22} = 0$$

$$u_{22} = -2u_{21}$$

$$\therefore u_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = u_{21} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{when } u_{21} = 1$$

(c)  $\therefore x_1(t) = e^{\lambda_1 t} u_1$  (REAL AND DISTINCT CASE)

$$= e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$$

$$x_2(t) = e^{\lambda_2 t} u_2 = e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$$

$$\therefore x(t) = c_1 x_1(t) + c_2 x_2(t)$$

$$= c_1 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$$

$\therefore$  is the general solution to this system of diff equations.