GIVEN $\int_0^{\pi} \sin^3(x)\,dx = 4/3 \qquad \int_0^{2\pi} \sin^2\theta\,d\theta = \pi$

4. [30 points] Evaluate the surface integral $\int\int_S \boldsymbol{F}\cdot d\boldsymbol{S}$ for the given vector field $\boldsymbol{F}(x,y,z)$ and the closed surface S. In other words, find the flux of $\boldsymbol{F}$ across S. For the closed surface S, use the positive (outward) orientation.
$\boldsymbol{F} = z\boldsymbol{i} + y\boldsymbol{j} + x\boldsymbol{k}$, S is the unit sphere $x^2+y^2+z^2=1$.

The parametric equation for the unit sphere is

$$r(\Phi,\theta) = (\sin\Phi\cos\theta)\,i + (\sin\Phi\sin\theta)\,j + (\cos\Phi)\,k \qquad 0\le\Phi\le\pi,\ 0\le\theta\le 2\pi$$

$$\therefore F(r(\Phi,\theta)) = (\cos\Phi)\,i + (\sin\Phi\sin\theta)\,j + (\sin\Phi\cos\theta)\,k$$

We know that

$$r_\Phi \times r_\theta = (\sin^2\Phi\cos\theta)\,i + (\sin^2\Phi\sin\theta)\,j + (\sin\Phi\cos\Phi)\,k$$

(FORMULA ALLOWED ON CRIB SHEET)

$$\therefore F(r(\Phi,\theta))\cdot(r_\Phi\times r_\theta) = (\cos\Phi\sin^2\Phi\cos\theta) + (\sin^3\Phi\sin^2\theta) + (\sin^2\Phi\cos\Phi\cos\theta)$$

Using the formula

$$\iint_S F\cdot dS = \iint_D F\cdot(r_\Phi\times r_\theta)\,dA$$

$$= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\Phi\cos\Phi\cos\theta + \sin^3\Phi\sin^2\theta)\,d\Phi\,d\theta$$

$$= 2\left(\int_0^{\pi}\sin^2\Phi\cos\Phi\,d\Phi\right)\underbrace{\left(\int_0^{2\pi}\cos\theta\,d\theta\right)}_{\to 0} + \left(\int_0^{\pi}\sin^3\Phi\,d\Phi\right)\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)$$

$$= 0 + \int_0^{\pi}\sin^3\Phi\,d\Phi\int_0^{2\pi}\sin^2\theta\,d\theta$$

$$= \left(\frac{4}{3}\right)(\pi) = \boxed{\frac{4\pi}{3}}$$