MA 242-005: Calculus III

Test # 2 - October 9, 2006

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SOLUTIONS TO TEST # 2

PREPARED BY KARTIK

INSTRUCTIONS

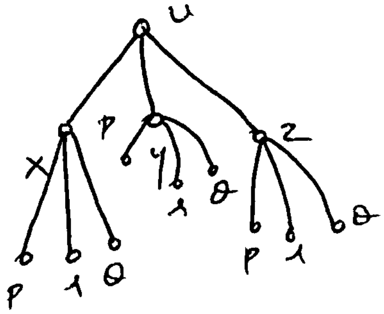
1. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
2. TIME LIMIT: 50 minutes
3. There are 5 pages and 4 questions on the exam. Each question appears on a different page. Read each question carefully.
4. The exam is worth 105 points including 5 extra credit points. The distribution of these points is clearly indicated on the exam.
5. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
6. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
7. This is a closed book exam. You may use one crib sheet of formulas on the exam.
8. You can also use an electronic calculator on the exam.

1. [30 points] Use the chain rule to find the indicated partial derivatives.
$u = x^2 + yz, \quad x = pr\cos\theta, \quad y = pr\sin\theta, \quad z = p + r;$

(a) [10 points] $\dfrac{\partial u}{\partial p}$ when $p = 2$, $r = 3$, $\theta = 0$.

(b) [10 points] $\dfrac{\partial u}{\partial r}$ when $p = 2$, $r = 3$, $\theta = 0$.

(c) [10 points] $\dfrac{\partial u}{\partial \theta}$ when $p = 2$, $r = 3$, $\theta = 0$.



(a) $$\frac{\partial u}{\partial p} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial p}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial p}\right) + \left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial z}{\partial p}\right)$$

$$= (2x)(r\cos\theta) + (z)(r\sin\theta) + (y)(1)$$

$$= 2xr\cos\theta + zr\sin\theta + y$$

When $p=2, r=3, \theta=0$

$$x = (2)(3)\cos(0) = 6, \quad y = (2)(3)\sin(0) = 0$$

$$z = p + r = 5$$

$$\therefore \frac{\partial u}{\partial p} = 2(6)(3)\cos(0) + (5)(3)\sin(0) + 0$$

$$= 36 + 0 + 0 = \boxed{36}$$

(b) $$\frac{\partial u}{\partial r} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial r}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial r}\right) + \left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial z}{\partial r}\right)$$

$$= (2x)(p\cos\theta) + (z)(p\sin\theta) + (y)(1)$$

$$= 2xp\cos\theta + pz\sin\theta + y$$

$$\therefore \frac{\partial u}{\partial r} = 2(6)(2)\cos(0) + (2)(5)\sin(0) + 0$$

$$= \boxed{24}$$

(c) $$\frac{\partial u}{\partial \theta} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial z}{\partial \theta}\right)$$

$$\therefore \left(\frac{\partial u}{\partial \theta}\right) = (2x)(-\rho r\sin\theta) + (z)(\rho r\cos\theta) + (y)(0)$$

$$= -2x\rho r\sin\theta + \rho r z\cos\theta$$

$$\therefore \left(\frac{\partial u}{\partial \theta}\right) = -2(6)(2)\sin(0) + (2)(3)(5)\cos(0)$$

$$= \boxed{30}$$

2. [25 points] The electrical potential V over a certain region of space is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.

(a) [15 points] Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $v = i + j - k$.

(b) [5 points] In what direction does V change most rapidly at P (enter your answer as a unit vector!)?

(c) [5 points] What is the maximum rate of change at P?

2(a) $V(x,y,z) = 5x^2 - 3xy + xyz$

$$\therefore \vec{\nabla} V(x,y,z) = \begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{bmatrix}$$

$\frac{\partial V}{\partial x} = (10x - 3y + yz)$

$\frac{\partial V}{\partial y} = (-3x + xz)$

$\frac{\partial V}{\partial z} = xy$

$\therefore \vec{\nabla} V(3,4,5) = \begin{bmatrix} 10(3) - 3(4) + 20 \\ -3(3) + (3)(5) \\ (3)(4) \end{bmatrix}$ (AT THE POINT P)

$$= \begin{bmatrix} 38 \\ 6 \\ 12 \end{bmatrix}$$

THE RATE OF THE CHANGE OF THE POTENTIAL AT $P(3,4,5)$ in the direction of $\vec{v} = i + j - k$ IS

$$D_{\vec{v}} V(3,4,5) = \vec{\nabla} V(3,4,5) \cdot \left(\vec{v}/|\vec{v}|\right) \rightarrow \text{UNIT VECTOR}$$

$$= (38)\left(\frac{1}{\sqrt{3}}\right) + 6\left(\frac{1}{\sqrt{3}}\right) - 12\left(\frac{1}{\sqrt{3}}\right) = (38 + 6 - 12)/\sqrt{3} = \boxed{\frac{32}{\sqrt{3}}}$$

2(b) $\vec{V}$ changes most rapidly at P in the direction $\vec{u} = \vec{\nabla} V(3,4,5) = \begin{bmatrix} 38 \\ 6 \\ 12 \end{bmatrix}$

The direction as a unit vector is

$$= \boxed{\frac{1}{\sqrt{(38)^2+6^2+(12)^2}} \begin{bmatrix} 38 \\ 6 \\ 12 \end{bmatrix}} = \boxed{\begin{bmatrix} 0.9430 \\ 0.1489 \\ 0.2978 \end{bmatrix}}$$

2(c) The maximum rate of change at P

$$= \left| \vec{\nabla} V(3,4,5) \right| = \boxed{\sqrt{(38)^2+(6)^2+(12)^2}}$$

$$= \boxed{40.2989}$$

3. [25 points]

(a) [20 points] Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2+z^2}$ at the point $(2, 3, 4)$.

(b) [5 points] Use the linear approximation to estimate the number $(1.98)^3\sqrt{(3.01)^2+(3.97)^2}$.

3(a) We have $f(x,y,z) = x^3\sqrt{y^2+z^2}$

$$\frac{\partial f}{\partial x} = 3x^2\sqrt{y^2+z^2} \qquad \frac{\partial f}{\partial z} = \frac{x^3 z}{\sqrt{y^2+z^2}}$$

$$\frac{\partial f}{\partial y} = x^3 \cdot \frac{1}{2\sqrt{y^2+z^2}}(2y) = \frac{x^3 y}{\sqrt{y^2+z^2}}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(2,3,4)} = 3(2)^2\sqrt{(3)^2+(4)^2} = 3\times 4\times 5 = 60$$

$$\left.\frac{\partial f}{\partial y}\right|_{(2,3,4)} = \frac{8\times 3}{5} = \frac{24}{5}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(2,3,4)} = \frac{8\times 4}{5} = \frac{32}{5}$$

ALSO $f(2,3,4) = 8(5) = 40$

The LINEAR approximation $L(x,y,z)$ to the function $f(x,y,z)$ at the point $(2,3,4)$ is given by:

$$L(x,y,z) = f(2,3,4) + \left.\frac{\partial f}{\partial x}\right|_{(2,3,4)}(x-2) + \left.\frac{\partial f}{\partial y}\right|_{(2,3,4)}(y-3) + \left.\frac{\partial f}{\partial z}\right|_{(2,3,4)}(z-4)$$

$$= \boxed{40 + 60(x-2) + \frac{24}{5}(y-3) + \frac{32}{5}(z-4)}$$

$\therefore L(x,y,z) = \boxed{60x + \frac{24}{5}y + \frac{32}{5}z - 120}$

3(b) $L(1.98, 3.01, 3.97) = 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120$

$= \boxed{38.6560}$

4. [25 points]

(a) [20 points] Find three positive numbers whose sum is 9 and whose product is a maximum.

(b) [5 points] What is the value of this maximum product?

4 (a) Let x, y, z be the three positive numbers

We have $x + y + z = 9 \longrightarrow (1)$

We want to maximize xyz

We can use (1) to eliminate z, ie,

$$z = (9 - x - y)$$

$\therefore$ $xyz = xy(9-x-y)$

$\therefore$ We want to maximize

$$f(x,y) = xy(9-x-y)$$

$$\frac{\partial f}{\partial x} = y\left((9-x-y) + x(-1)\right) = y(9-2x-y)$$

$$\frac{\partial f}{\partial y} = x\left((9-x-y) + y(-1)\right) = x(9-x-2y)$$

AT A LOCAL MAX

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

which gives

$$y(9-2x-y) = 0$$
$$x(9-x-2y) = 0$$

$y(9-2x-y)=0$
$x(9-x-2y)=0$

(Since x, y are required to be positive)

we must have

$$9-2x-y=0 \qquad 2x+y=9$$
$$9-x-2y=0 \qquad x+2y=9$$

This gives $x=3$ and $y=3$

$\therefore z=(9-x-y)$
$=(9-3-3)=6$

$\therefore$ The three positive numbers are
$\boxed{x=3,\ y=3,\ \text{and } z=3}$

9(b) The value of the maximum product is $= xyz = (3)(3)(3) = \boxed{27}$