CAAM 353: Computational Numerical Analysis
Midterm Exam - March 3, 2003
Instructor: *Dr. Kartik Krishnan*

INSTRUCTIONS

1. TIME LIMIT: 2 hours.
2. There are 4 questions in all. Read each question carefully.
3. 100 points in all. 5 points are available as extra credit in question 4.
4. Solve each problem in detail. Define variables carefully. Always explain your answer, unless otherwise stated.
5. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
6. Sources: Open book, notes, homeworks, and solutions to the homework assignments.
7. This has to be all your own work. Do not consult anyone else. If you have any questions, my email address is kartik@caam.rice.edu, and phone number 348-2649. I'll have my regular office hours on Monday [3rd], and Thursday [5th] between 11 - 12 pm. I will also have additional office hours on Tuesday [4th] between 12 - 1 pm.
8. Due in class Thursday, March 6. No late exams will be accepted.

On my honor I have neither given nor received any aid on this examination.

_____

1. [25 points] Here is a variant of the Cholesky factorization of a $n \times n$ symmetric positive definite matrix $A = R^T R$, where R is a upper triangular matrix. The pseudocode is given below. The input A is overwritten by the upper triangular matrix R at the output.

Cholesky factorization
$R = A$
for $k = 1$ **to** n
  for $j = k + 1$ **to** n
    for $i = j$ to n
      $R(j,i) = R(j,i) - \frac{R(k,i)R(k,j)}{R(k,k)}$
    end
  end
  for $j = k$ to n
    $R(k,j) = \frac{R(k,j)}{\sqrt{R(k,k)}}$
  end
end

   (a) [15 points] What is the complexity of the Cholesky factorization routine?. I want a detailed complexity analysis as we discussed in class for the LU factorization routine. What is the asymptotic flop complexity?. Is this the most efficient way to store the matrix R?. Why?.

   (b) [5 points] How would you use the Cholesky factorization $A = R^T R$ in solving $Ax = b$?. What is the overall complexity of the procedure?.

   (c) [5 points] How does this compare with the LU factorization in solving $Ax = b$?.

2. [25 points] Approximating a derivative by a difference quotient :

   (a) [10 points] Use my program *cancellation.m* in the *http://www.caam.rice.edu/∼kartik/caam353/programs* directory to compute an approximate value for the derivative of a function using the finite-difference formula.

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

   My program is for $f(x) = \sin(x)$ at $x = 1$, but you should test your program using the function $f(x) = \tan(x)$ for $x = 1$. Determine the error by using $f'(x) = \sec^2(x)$. Plot the magnitude of the error as a function of h, for $h = 10^{-k}$, $k = 0, \ldots, 16$. You should use a log scale for h, and for the magnitude of the error. Is there a minimum value for the magnitude of the error?. (I have already incorporated this in mu code, so you need not make nay changes). How does the corresponding value of h compare with the rule of thumb $h = \sqrt{\epsilon_{mach}}$ derived in Exampled 1.3, Chapter 1, page 8 of Heath?.

(b) [10 points] Repeat the exercise using the centered difference approximation.

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

(c) [5 points] Tabulate your observations clearly. Can you provide a brief explanation for your results?. Please attach the plots with your exam.

3. [25 points] Let $p = 5$. What is the value of the following continued fraction?.

$$x = \cfrac{1}{p+\cfrac{1}{p+\cfrac{1}{p+\ldots}}}$$

(a) [10 points] Can you generate a suitable fixed point iteration function to evaluate the value of x?. This is the function that comes immediately to mind by examining the expression for x?.

(b) [15 points] Prove that the sequence of values of x converges using the fixed point iteration theorem?.

4. [30 points] Given an $n \times n$ nonsingular matrix A, how do you efficiently solve the following problems, using Gaussian elimination with partial pivoting, i.e. given $PA = LU$?.

(a) [10 points] Solve the linear system $A^k x = b$, where k is a positive integer.

(b) [10 points] Compute $\alpha = c^T A^{-1} b$.

(c) [10 points] Solve the matrix equation $AX = B$, where B is $n \times m$.

You should (1) describe your algorithms clearly (making use of the factorization $PA = LU$), and (2) give the required flops.