Feedback and Time Optimal Control for Quantum Spin Systems

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— Joint Work with Karl Kunisch, K.F. University of Garz and Qin Zhang.

1. MODEL PROBLEMS

Abstract Schrödinger Control System

$$i \frac{\partial}{\partial t} \Psi(x,t) = (\mathcal{H}_0 + \epsilon(t)\mu)\Psi(x,t) + \gamma\,|\Psi|^2\Psi, \quad \Psi(x,0) = \Psi_0(x).$$

Let the internal Hamiltonian $\mathcal{H}_0$ is a positive closed self-adjoint operator on a Hilbert space X. In the presence of an external interaction taken as an electric field modelled by a coupling operator with amplitude $\epsilon(t) \in R$ and a time independent dipole moment operator μ, the new Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \sum_{j=1}^{m} \epsilon_j(t)\mu_j$ gives rise to the dynamical equations to be controlled, (e.g, $\mu_j \Psi = v_j(x)\,\Psi$)

Control Problem We consider the control problem of driving the state $\Psi(t)$ to an energy equilibrium state of $\bar{\Psi}$ of $\mathcal{H}_0$, i.e. $\bar{\Psi} = e^{i\,\phi}\bar{\psi}$ where $\mathcal{H}_0\bar{\psi} = \lambda\,\bar{\psi}$.

Linear Schrödinger Equations: $\gamma = 0$

$$\frac{\partial}{\partial t}\Psi_1(x,t) = (\mathcal{H}_0 + \epsilon(t)\mu)\Psi_2(x,t) + \gamma\,|\Psi|^2\Psi_2$$

$$\frac{\partial}{\partial t}\Psi_2(x,t) = -(\mathcal{H}_0 + \epsilon(t)\mu)\Psi_1(x,t) - \gamma\,|\Psi|^2\Psi_1,$$

i.e., for $\Psi(x,t) = \Psi_1(x,t) + i\,\Psi_2(x,t)$ and $\Psi = (\Psi_1, \Psi_2)$,

$$\frac{d}{dt}\Psi = A\Psi + \epsilon(t)B\Psi \quad \text{with}$$

$$A = \begin{pmatrix} 0 & \mathcal{H}_0 \\ -\mathcal{H}_0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$$

Multiple controls: $\mu(t) = \sum_{j=1}^{m} \epsilon_j(t)\,\mu_j$.

MIld Solution: $\Psi(t) = S(t)\Psi_0 + \int_0^t S(t-s)\epsilon(s)B\Psi(s)\,ds.$

Orbit Tracking:

$$V = V(\Psi(t), \mathcal{O}(t)) = \tfrac{1}{2}|\Psi(t) - \mathcal{O}(t)|_X^2 = 1 - Re\,(\mathcal{O}, \Psi)_H$$
$$i\frac{\partial}{\partial t}\mathcal{O}(t) = \mathcal{H}_0\mathcal{O}(t), \text{ e.g. } \mathcal{O}(t) = e^{-i(\lambda t - \theta)}\psi, \quad H = \lambda\psi$$

$$\frac{d}{dt}V(\Psi(t), \mathcal{O}(t)) = \epsilon(t)\,\mathrm{Im}\,(\mathcal{O}(t), \mu\Psi(t))$$

$$\epsilon(t) = -\frac{1}{\alpha}\,(u(t) + \beta\, sign(u(t))V^{\gamma}), \quad u(t) = \mathrm{Im}\,(\mathcal{O}(t), \mu\Psi(t))$$

$$\frac{d}{dt}V(\Psi(t), \mathcal{O}(t)) = -\frac{1}{\alpha}\,(|u(t)|^2 + \beta\,|u(t)|V^{\gamma}).$$

[K. Beauchard, J.M. Coron, M. Mirrahimi, and P. Rouchon] Implicit Lyapunov control of finite dimensional Schrödinger equations, preprint.

[M.Mirrahimi, P. Rouchon, and G. Turinici] Lyapunov control of bilinear Schrödinger equations, Automatica, 41(2005), 1987-1994.

Optimality: $V = V(\Psi, \mathcal{O}(t))$ satisfies HJB equation

$$\frac{\partial}{\partial t} V(\Psi, \mathcal{O}(t)) + \min_{\epsilon} \left[(A_0 \Psi + \epsilon B \Psi, V_\Psi) \right.$$
$$\left. + \frac{1}{\alpha} \left(|u(t)|^2 + \beta \, |u(t)| V^\gamma \right) + \alpha \left| \epsilon + \frac{\beta}{\alpha} sign(u(t)) V^\gamma \right|^2) \right] = 0,$$

$$u(t) = \mathrm{Im} \left(\mathcal{O}(t), \mu \Psi \right)$$

Asymptotic Tracking $V(\Psi, \mathcal{O}) \to 0$ as $t \to \infty$ Since

$$V(\Psi(t), \mathcal{O}(t)) + \int_0^t (\beta \, |u(s)|^2 + |u(s)| V(\Psi(s), \mathcal{O}(s))) \, ds = V(\Psi(0), \mathcal{O}(0)).$$

$V(\Psi(t), \mathcal{O}(t))$ is monotonically decreasing in t. Using the compactness of the orbit we prove the following invariance principle.

Invariance Principle Suppose

$$\int_0^T \alpha \, |\mathrm{Im}\,(\mathcal{O}_\infty(t), \mu\Psi_\infty(t))|^2 \, dt = 0.$$

$$i \frac{d}{dt} \Psi_\infty = \mathcal{H}_0 \Psi_\infty, \quad \Psi_\infty(t) = \omega - Limit = \sum_{k=1}^{\infty} A_k e^{i(-\lambda_k t + \theta_k)} \psi_k.$$

where for $\mu_{k_0}^k = (\psi_{k_0}, \mu\psi_k)_H$ and $\mathcal{O}_\infty(\tau) = e^{i(\lambda_{k_0}\tau - \tilde{\theta}_{k_0})}\, \psi_{k_0}$,

$$\mathrm{Im}\,(\mathcal{O}(t), \mu\Psi(t)) = Im\,(\sum_{k=1}^{\infty} A_k e^{i((\lambda_k - \lambda_{k_0})\tau - \theta_k + \tilde{\theta}_{k_0}))} \mu_{k_0}^k).$$

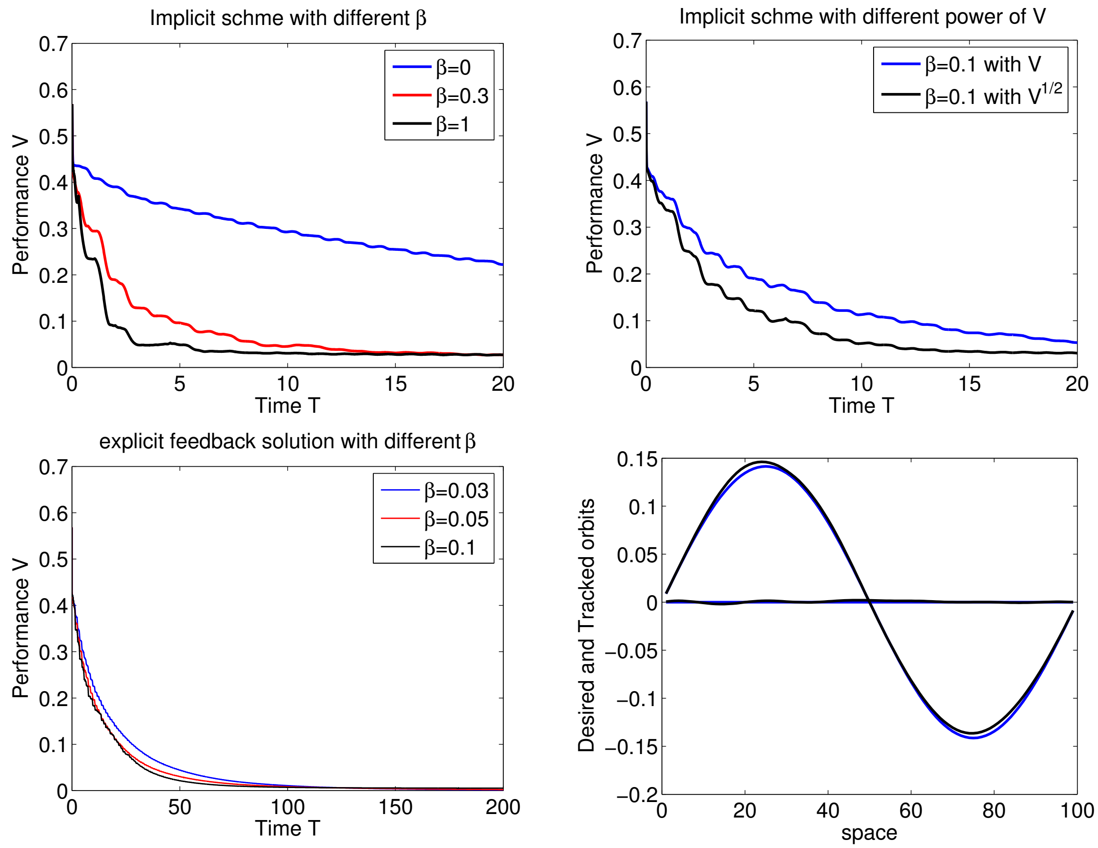
Ingham's Lemma Assume $|\mu_m - \mu_\ell| \geq \delta, \;\; m \neq \ell$ for

$$\mu_k = \lambda_k - \lambda_{k_0}, \;\; k \geq 1, \quad \mu_{-k} = -(\lambda_k - \lambda_{k_0}) \;\; k \neq k_0.$$

If $T > \frac{2\pi}{\delta}$, there exits a constant c, depending on T and $\delta > 0$ such that

$$c \sum_{m \in I} |a_m|^2 \leq \int_0^T f(\tau)|^2 \, d\tau, \;\; f(\tau) = \sum_{m \in I} a_m e^{i\mu_m \tau}.$$

Thus, $A_k = 0, \; k \neq k_0$ and $\tilde{\theta}_{k_0} = \theta_{k_0}$ if $\mu_{k_0}^k \neq 0$.



$\beta = 1/500$, $\bar{\Psi} = \sin(2\pi\, x)$, $\Psi_0 = e^{-100(x-.8)^2}$, $\mu_i \Psi = v_i(x)\Psi(x)$ with $v_1(x) = x - 0.5 - 1.75(x-0.5)^2$, and $v_2(x) = x - 0.5 - 1.75(x-0.5)^2 + 2.5(x-0.5)^3 - 2.5(x-0.5)^4$.

Feedback Control of Quantum Spin Systems

Observable process: $X_t = U_t^* X U_t$, where the unitary operator satisfies the Schrodinger type quantum stochastic differential equation (QSDE)

$$dU_t = (-iHdt - \frac{1}{2}L^*Ldt + LdA_t^* - L^*dA_t)U_t, \quad U_0 = I \qquad (1)$$

where A_t is the annihilation and A_t^* is the creation operator and L is an atomic operator. The homodyne detection process:

$$dY_t = U_t^*(L + L^*)U_t^+ dA_t + dA_t^*$$

By using Quantum Ito's rule, we get the QSDE for $X(t)$

$$dX_t = j_t(\mathcal{L}_{L,H}(X))dt + j_t([L^*, X])dA_t + j_t([X, L])dA_t^*,$$

where $j_t(X) = U_t^* X U_t$ and the (Lindblad) generator given by

$$\mathcal{L}_{L,H}(X) = i[H, X] + L^*XL - \frac{1}{2}(XL^*L + L^*LX).$$

K. R. Parthasarathy, An Introduction to Quantum Stochastic Calculus, Birkhauser, Basel, 1992

Luc. Bouten, Ramon van Handel, Matthew R James: An Introduction to Quantum Filtering, arXiv:math. OC/-601741v1, 2006.

M. Mirrahimi, R.A.Handel, Stabilizing Feedback Controls for Quantum Systems, arXiv:math.ph/0510066v2, 2005.

K. Tsumura, Global Stabilization of N-dimensional Quantum Spin Systems via Continuous Feedback.

Follow the quantum filtering theory, the conditional expectation of the observable X, $\mathbb{P}(X_t|\mathcal{Y}_t) = \pi_t(X) = Tr(\rho_t X)$ is given by

$$d\rho_t = (-i[H, \rho_t] + L\rho_t L^* - \frac{1}{2}\rho_t L^* L - \frac{1}{2}L^* L\rho_t)dt \\ + (L\rho_t + \rho_t L^* - Tr[\rho_t(L + L^*)]\rho_t)dW_t$$

where The Brownian process $dW_t = dY_t - Tr[(L + L^*)\rho_t)]dt$ is the innovation process.

Consider a quantum spin system with fixed angular momentum J. Let $H_t = u_t F_y$ be the control Hamiltonian and $L = F_z$ be the coupling operator. Then the corresponding quantum filtering equation is

$$d\rho_t = -iu_t[F_y, \rho_t]dt - \frac{1}{2}[F_z, [F_z, \rho_t]]dt \\ + \sqrt{\eta}(F_z\rho_t + \rho_t F_z - 2Tr[F_z\rho_t]\rho_t)dW_t$$

where F_y and F_z are self adjoint angular moment operators along axis y and z.

$$F_y = \frac{1}{2i}\begin{pmatrix} 0 & -c_1 & & & \\ c_1 & 0 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{2J-1} & 0 & -c_{2J} \\ & & & c_{2J} & 0 \end{pmatrix} \quad , c_m = \sqrt{(2J+1-m)m}$$

$$F_z = \begin{pmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{pmatrix}$$

For any $\epsilon > 0$, let

$$\rho_{\tilde{f}} = (1-\epsilon)\rho_f + \frac{\epsilon}{2J}\sum_{m\neq f}\rho_{\Psi_m}$$

where $\rho_f,\ f \in [-J\ J]$ is an equilibrium solution $(\frac{d\rho_t}{dt}|_{\rho_t=\rho_f} = 0)$. For $\alpha,\ \beta > 0$ consider the control law:

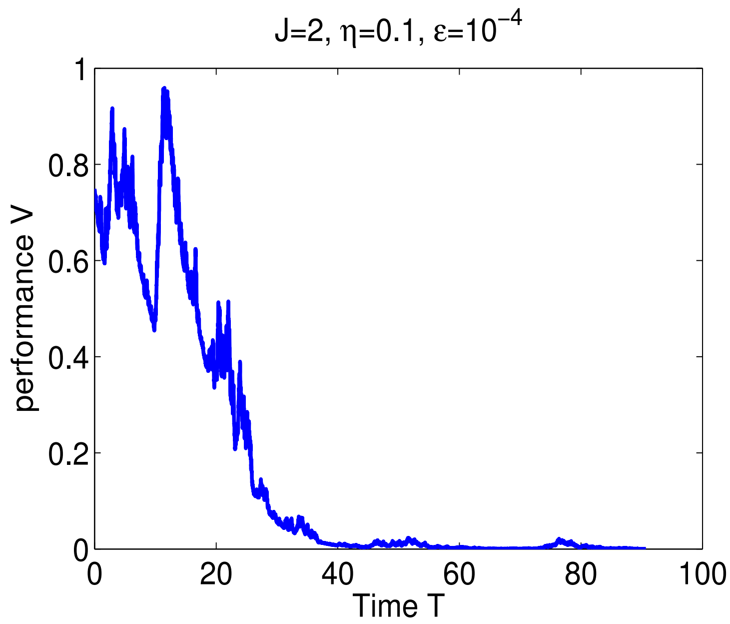
$$u_t = -(\alpha\,\tilde{u}_t + \beta \mathrm{sign}(\tilde{u}_t)\,V^I_{\rho_f}(\rho_t)), \quad \tilde{u}_t = Tr(i[F_y,\rho_t]\rho_{\tilde{f}}).$$

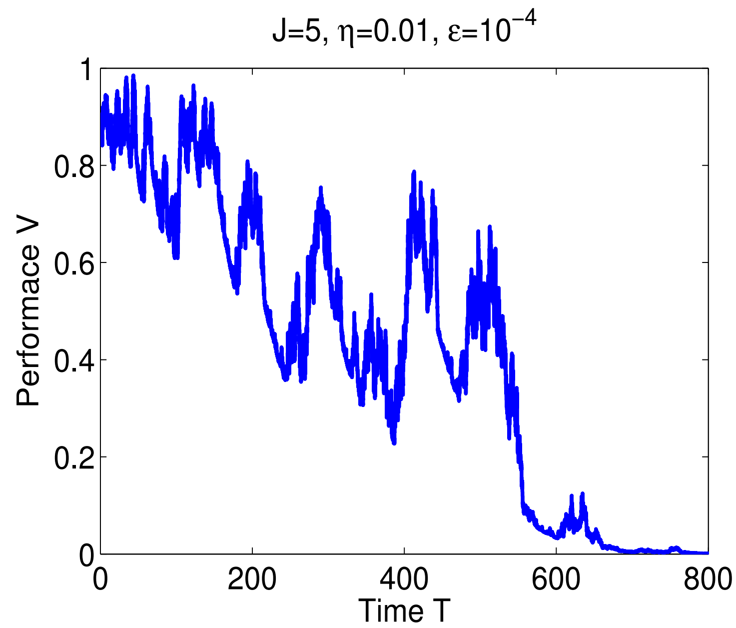
Claim: $\rho_t \to \rho_f$ a.s. as $t \to \infty$. Let

$$V^I_{\rho_f} = 1 - Tr(\rho\rho_f), \quad V^{II}_{\rho_f} = 1 - (Tr(\rho\rho_f))^2$$

$$\mathcal{A}V^I = -u_t\,Tr(i[F_y,\rho_t]\rho_{\tilde{f}}) = -(\alpha\,|\tilde{u}_t|^2 + \beta|\tilde{u}_t|V^I(\rho_t))$$

$$\mathcal{A}V^{II} = -2Tr(\rho_t\rho_{\tilde{f}})(\alpha\,|\tilde{u}_t|^2 + \beta\,|\tilde{u}_t|V^I(\rho_t)). + 4\eta(f - tr(F_z\rho_t))^2 tr(\rho_t\rho_f)^2.$$





— Time Integration: Splitting of Deterministic and Stochastic terms. Implicit Euler scheme is $\mathcal{S}$-invariant.

M. Mirrahimi, R.A.Handel, Stabilizing Feedback Controls for Quantum Systems, arXiv:math.ph/0510066v2, 2005.

Quantum Control System

$$\frac{d}{dt}\Psi(t) = -iH(t)\Psi(t)$$

with controlled Hamiltonian: $\quad H(t) = H_0 + \sum_{i=1}^{m} \epsilon_i(t)\, H_i.$

Spin-$\frac{1}{2}$ system

$$\frac{d}{dt}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 0 & I_z \\ -I_z & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} + \epsilon(t)\begin{pmatrix} 0 & I_x \\ -I_x & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

where the Pauli spin matrices are given by

$$I_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\Psi(0) = \frac{1}{\sqrt{2}}(1\ 0\ 0\ 1), \quad \frac{1}{2}|\Psi(\tau) - \mathcal{O}|^2 \leq \delta, \;\; \mathcal{O} = (1\ 0\ 0\ 0).$$

Time Optimal control $x = (\Psi_1, \Psi_2)$. $u(t) = \epsilon(t)$

$$\min \quad \int_0^\tau (1 + \epsilon\,|u(s)|^2)\,ds \quad \text{subject to}$$

$$\frac{d}{dt}x(t) = (A{+}u(t)B)x(t), \;\; |u(t)| \le \gamma \quad |x(\tau){-}x_1|^2 \le \delta, \; x(0) = x_0 \in X.$$

Necessary optimality condition

$$\frac{d}{dt}x(t) = f(x,u), \quad u(t) = -\mathrm{sign}_\epsilon(\lambda(t)\cdot Bx(t)). \quad x(0) = x_0, \;\; x(\tau) = x_1$$
$$-\frac{d}{dt}\lambda(t) = (A + uB)^t\lambda(t)$$
$$1 + \lambda(\tau)\cdot(Ax(1) + u(1)Bx(1)) = 0$$

$$\mathrm{sign}_\epsilon(s) = \begin{cases} 1 & s \ge \epsilon \\ \frac{s}{\epsilon} & |s| \le \epsilon \\ -1 & s \le -\epsilon \end{cases}$$

With coordinate change $t = \tau s$, $F(z) = 0$ for $z = (x, \lambda, u, \tau) \in H^1(0,1;R^n) \times H^1(0,1,R^n) \times L^2(0,1,R) \times R$:

$$\frac{d}{dt}x(t) = \tau\,(Ax + uBx)$$

$$u(t) = -\mathsf{sign}_\epsilon(\lambda(t) \cdot Bx)$$

$$x(0) = x_0, \;\; x(1) = x_1,$$

$$-\frac{d}{dt}\lambda(t) = \tau\,(A + uB)^t\lambda,$$

$$1 + \lambda(1) \cdot (Ax(1) + u(1)Bx(1)) = 0.$$

Semismmoth Newton algorithm

$$z_{k+1} = z_k - J^{-1} F(z_k)$$

where $J\Delta z + F = 0$ is written as

$$\frac{d}{dt}\Delta x = \tau(A\Delta x + B\Delta u) + \Delta\tau(Ax + Bu) + f = 0 \quad \text{in } L^2(0,1,R^n)$$

$$-\frac{d}{dt}\Delta\lambda = \tau A^*\Delta\lambda + \Delta\tau A^*\lambda + g = 0 \quad \text{in } L^2(0,1,R^n)$$

$$\Delta u + G_\epsilon(B^*\lambda)B^*\Delta\lambda + e = 0 \quad \text{in } L^2(0,1,R^n)$$

$$\Delta x(0) = 0, \quad \Delta x(1) + \Delta_1 = 0 \quad \text{in } R^n$$

$$\Delta\lambda(1)\cdot(Ax(1) + Bu(1)) + \lambda(1)\cdot(A\Delta x + B\Delta u) + \Delta_2 = 0 \text{in } R.$$

$$\text{with } G_\epsilon(s) = \begin{cases} 0 & |s| \geq \epsilon \\ \frac{1}{\epsilon} & |s| < \epsilon \end{cases}$$

Semismooth Function: $F : D \subset X \to Z$ is called N-differentiable, if there exists a family of mappings $G : U \to \mathcal{L}(X, Z)$ such that

$$\lim_{|h|\to 0} \frac{|F(x+h) - F(x) - G(x+h)h|_Z}{|h|_X} = 0$$

for all $x \in U$. Moreover, F is semismooth at x if

$$\lim_{t\to 0^+} G(x + t\,h)h \quad \text{exists unifomly in } |h| = 1.$$

Theorem (Local) Suppose F is semismooth at x^* and $|F'(x,h)| \geq \beta\,|h|$ for $\beta > 0$ and all $h \in X$. and assume for a N-derivative G in a neighborhood of x $|G(y)^{-1}| \leq 2\beta$ for all $y \in N(x^*)$. Then the Newton iterates:

$$x^{k+1} = x^k - G(x^k)^{-1}F(x^k)$$

are well-defined and converges to x^* superlineraly in a neighborhood $N(x^*)$.

$$|x^{k+1} - x^*| \leq |G(x^k)^{-1}(F(x^k) - F(x^*) - G(x^k)(x^k - x^*)| \leq o(|x^k - x^*|).$$

Solvability (Linear Case $\frac{d}{dt}x = Ax + Bu$): Let $(\Delta\lambda(1), \Delta\tau)$ be unknown.

$$\Delta\lambda(t) = e^{\tau A^*(1-t)}\Delta\lambda(1) + \int_t^1 e^{\tau A^*(s-t)}(A^*\lambda(s)\Delta\tau + g(s))\, ds$$

$$\Delta x(t) = \int_0^t e^{\tau A(t-s)}(BG_\epsilon(b^*\lambda(s))B^*\Delta\lambda(s) + (Ax + Bu)\Delta\tau + be(s) + g(s))\, ds$$

Thus, we obtain the system of linear equations for $(\Delta\lambda(1), \Delta\tau)$:

$$\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} \Delta\lambda(1) \\ \Delta\tau \end{pmatrix} + q = 0 \qquad (2)$$

where $q \in R^{n+1,1}$ depends on $F = (f, g, e, \Delta_1, \Delta_2)$ linearly. That is, if the matrix $\Phi \in R^{(n+1)\times(n+1)}$ is invertible , then J is bounded invertible.

Let $\{t : |B^*\lambda(t)| < \epsilon\} = \cup_i(t_i, t_{i+1})$ then

$$\Phi_{11} = \frac{1}{\epsilon}\sum_i e^{\tau A(1-t_i)}\mathcal{G}(\Delta t_i)e^{\tau A^*(1-t_i)}$$

$$\mathcal{G}(\Delta) = \int_0^\Delta e^{\tau A s}BB^* e^{\tau A^* s}\,ds$$

Reduced Equations and Backward Shooting method

Linear Case: $\lambda(t)$ is linear equation (independent of x) and u can eliminated by $u = -\text{sign}_\epsilon(B^*\lambda)$. Thus, $x(\cdot)$ is a function of $(\lambda(1), \tau)$ by solving the state equation with $x(0) = x_0$. One can define adjoint-forward shooting method: Find $(\lambda(1), \tau)$ that satisfies

$$\Phi(\lambda(1), \tau) = (x(1) - x_1, 1 + \lambda(1)\cdot(Ax(1) + Bu(1)) = 0$$

Backward Shooting Method We use the scaled transversality condition

$$1 + \tau\lambda(1) \cdot (Ax(1) + u(\tau)Bx(1)) = 0$$

In this way one can eliminate $\tau > 0$ by this. So, we may employ the backward shooting method: find $\lambda(1)$ such that $\Phi(p(1)) = x(0) = x_0 = 0$ in which we solve the coupled system with the terminal condition $(x(1), \lambda(1))$ is given.

Theorem u_ϵ converges to u^* with minimum norm. For $0 \leq \epsilon \leq \hat{\epsilon}$

$$\tau_\epsilon \leq \tau_{\hat{\epsilon}}$$

$$\tau_\epsilon \int_0^1 |u_\epsilon|^2 \, dt \geq \tau_{\hat{\epsilon}} \int_0^1 |u_{\hat{\epsilon}}|^2 \, dt$$

$$\tau_0 \leq \tau_\epsilon \leq \tau_0 (1 + \frac{\epsilon}{2})$$

Necessary Optimality

With coordinate change $t = \tau s$,

$$\min \quad J_\epsilon(u,\tau) = \tau \int_0^1 (1 + \frac{\epsilon}{2}|u(t)|^2)\, dt$$

subject to $\frac{d}{dt}x = \tau\,(Ax + uBx),\;\; x(0) = x_0$ and

$$u \in U_{ad} = \{u \in L^2(0,1;R^m) : |u(t)| \leq \gamma,\ \text{a.e. }\}.$$

Let $g_1(u,\tau) = Cx(1) - y = 0$ (Finite Rank), $g_2(u,\tau) = |x(1) - \bar{x}|^2 \leq \delta$

$$\min \quad J_\epsilon(u,\tau) \quad \text{subject to } g_1(u,\tau) = 0,\ g_2(u,\tau) \leq 0 \text{ and } u \in U_{ad}.$$

Regular Point Condition: $0 \in \text{int}\{g_u(U_{ad} - u_\epsilon) + g_\tau(R^+ - \tau_\epsilon)\} \quad (3)$

$$\int_0^1 (\epsilon\, u_\epsilon, u - u_\epsilon)\, dt + (g_u(u - u_\epsilon), \mu_\epsilon) \geq 0, \;\; (g_\tau, \mu_\epsilon) = 0 \text{ for all } u \in U_{ad}.$$

$$\color{blue}{Adjoint Equation:} \quad -\frac{d}{dt}\lambda_\epsilon(t) = (A + uB)\lambda_\epsilon$$

$$(g_\tau, \mu_\epsilon) = \int_0^1 ((A + u_\epsilon B)x_\epsilon, \lambda_\epsilon)\, dt.$$

$$(g_u(v), \mu_\epsilon) = \tau \int_0^1 (\lambda_\epsilon \cdot Bx_\epsilon, v)\, ds$$

Thus, the necessary optimality is written as

$$\int_0^1 (\epsilon u_\epsilon + \lambda_\epsilon \cdot Bx_\epsilon, u - u_\epsilon)\, dt \geq 0.$$

for all $u \in U_{ad}$ and

$$\int_0^1 (1 + \frac{\epsilon}{2}|u_\epsilon|^2 + (A + u_\epsilon B)x_\epsilon, p_\epsilon))\, dt = 0.$$

Standard Minimum Norm Problems (unconstrained):

$$\min \quad \int_0^\tau |u|^2 \, dt$$

subject to $x(\tau) = x_1$.

$$u^*(t) = B^* e^{A^*(\tau - t)} \mu, \;\; \mu = G(\tau)^{-1}(x_1 - e^{A\tau} x_0).$$

L^∞ Minimum Norm Problem minimize $\sum_{i=1}^m \gamma_i^2$ subject to

$$|u_i(t)| \leq \gamma_i \text{ on } [0, \tau], \quad \text{and} \quad x(\tau) = x_1.$$

Necessary Optimality:

$$u^* = -\text{sign}_\epsilon(\lambda(t) \cdot Bx(t)), \quad \text{and} \quad \gamma_i - \int_0^\tau |(\lambda(t) \cdot Bx(t))_i| \, dt = 0$$

Mixed Problem Min $\tau + \frac{1}{2}|\gamma|^2$

γ	δ	guess:τ	optimal τ	# of switch	control form
19.25	$0.01/2$	0.5	0.7236	0	1
15	$0.01/2$		0.7434	0	-1
11	$0.01/2$	1	1.0098	0	-1
10	$0.01/2$	1	1.0804	0	-1
8	$0.01/2$		1.3529	0	-1
7	$0.01/2$		1.1139	2	-1,1,-1
6	$0.01/2$		1.3406	2	-1,1,-1
5	$0.01/2$		1.6516	2	-1,1,-1
4	$0.01/2$		2.1064	2	-1,1,-1
3	$0.01/2$		2.8334	2	-1,1,-1

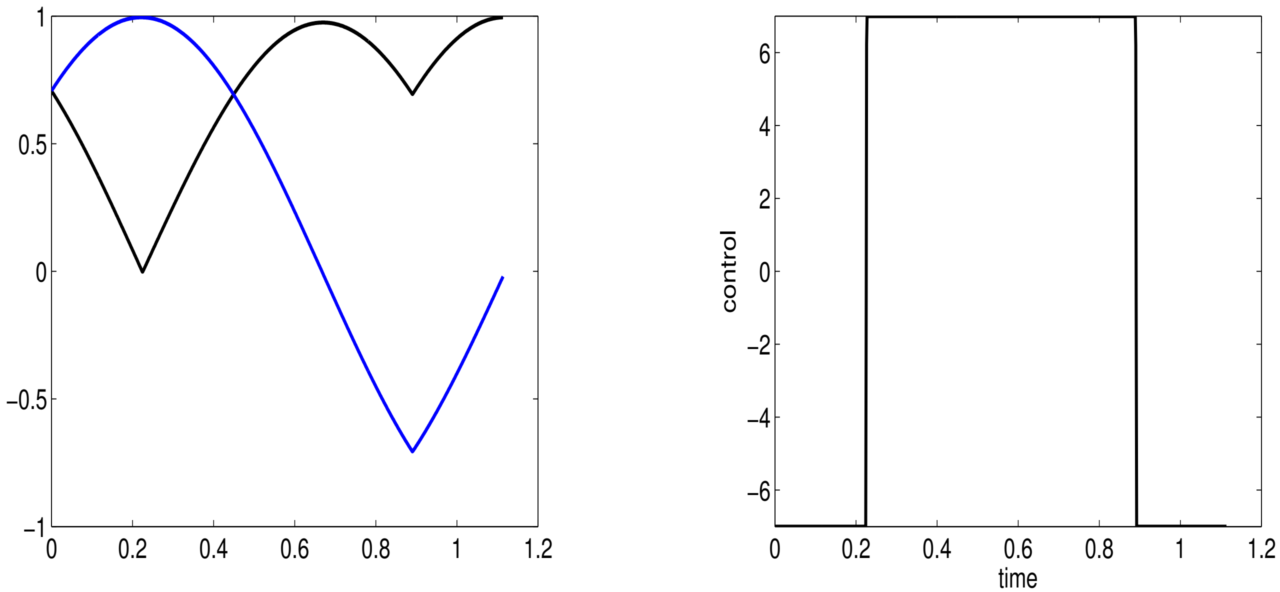


Figure 1: the Trajectory and Control with $\delta = 0.01/2$.