North Carolina State University

Department of Mathematics

MA-305 Elem Linear Algebra
Second Midsemester Quiz

Spring 1997

Your Name: SOLUTION
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 questions, each question counting for the given number of points, adding to a total of **20 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **two** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

Problem 1	_____
2	_____
3	_____
4	_____
5	_____
Total	_____

If you are taking the exam later, please sign the following statement:

I, ____________________*, affirm that I have no knowledge of the contents of this exam.*

____
Signature

Problem 1 (5 points, 2 points for (a) and (b), 1 point for (c)): Please answer the following questions about vector spaces and matrices. Please, also **justify your answers** briefly.

(a) Suppose that you are given the LU decomposition of a square matrix, $A = L \cdot U$. Is there an easy way of computing the determinant of A? If so, how would you do it?

$\det(L) =$ product of diagonal elements $(= 1)$
$\det(U) =$ product of diagonal elements
$\det(A) = \det(L) \cdot \det(U)$

(b) Consider the subset of 2×2 real matrices:

$$S = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 0 \} \subset \mathbb{R}^{2\times 2}.$$

(The book uses the notation M_{22} for $\mathbb{R}^{2\times 2}$.) Does S form a subspace of $\mathbb{R}^{2\times 2}$ with matrix addition and multiplication of a matrix by a scalar as the addition and scalar multiplication operations? Please explain.

NO. Not closed under $+$: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, i.e., the sum of two singular matrices can be non-singular.

(c) Suppose $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$ with $n > m$. Then $v_1, v_2, \ldots, v_n$ must be linearly dependent in $\mathbb{R}^m$. Please explain why.

Take $[v_1 \vdots \ldots \vdots v_n]$ and convert to ref. Then not every column can have a pivot element, as there are more columns than rows and each pivot occupies its own row. Therefore, all the columns that have no pivot elements correspond to free variables and therefore to linearly dependent vectors.

Problem 2 (3 points): Consider the following matrix:

$$A = \begin{bmatrix} a & c & 0 & 0 \\ b & a & c & 0 \\ 0 & b & a & c \\ 0 & 0 & b & a \end{bmatrix}.$$

As an expression in a, b, and c, please compute by co-factor expansion (also called "minor expansion") the determinant of A.

Minor expansion along the first row yields

$$a \cdot \underbrace{\det \begin{bmatrix} a & c & 0 \\ b & a & c \\ 0 & b & a \end{bmatrix}}_{a \cdot \underbrace{\det \begin{bmatrix} a & c \\ b & a \end{bmatrix}}_{a^2 - bc} - c \cdot \underbrace{\det \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}}_{ab}} - c \cdot \underbrace{\det \begin{bmatrix} b & c & 0 \\ 0 & a & c \\ 0 & b & a \end{bmatrix}}_{b \cdot \underbrace{\det \begin{bmatrix} a & c \\ b & a \end{bmatrix}}_{a^2 - bc}} = a^4 - 3a^2bc + b^2c^2$$

Problem 3 (3 points, 1.5 points each part):
(a) Please solve the linear system

$$\begin{array}{rrrcl} x & +2y & & = & 7 \\ 3x & & +4z & = & 8 \\ & 5y & +6z & = & 9 \end{array} \tag{3.1}$$

by Cramer's rule. Please do not compute rational number values for x, y, and z, but instead give each value as a quotient of determinants of matrices.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 5 & 6 \end{bmatrix}, \Delta = \det(A)$$

$$x = \frac{\det \begin{bmatrix} 7 & 2 & 0 \\ 8 & 0 & 4 \\ 9 & 5 & 6 \end{bmatrix}}{\Delta}, y = \frac{\det \begin{bmatrix} 1 & 7 & 0 \\ 3 & 8 & 4 \\ 0 & 9 & 6 \end{bmatrix}}{\Delta}, z = \frac{\det \begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 8 \\ 0 & 5 & 9 \end{bmatrix}}{\Delta},$$

(b) Please give the entry in row 1 and column 2 of the inverse matrix of the coefficient matrix of equation (3.1) above, again as a quotient of two determinants of matrices.

$$\frac{(-1)^{1+2} \cdot \det \begin{bmatrix} 2 & 0 \\ 5 & 6 \end{bmatrix}}{\Delta} = \frac{-\det \begin{bmatrix} 2 & 0 \\ 5 & 6 \end{bmatrix}}{\Delta}$$

Problem 4 (5 points): Consider the four vectors in $\mathbb{R}^4$:

$$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}a\\b\\c\\d\end{bmatrix}.$$

Under what **necessary and sufficient** condition for the parameters a, b, c, and d are these vectors linearly dependent? Please show your work.

Convert the matrix whose columns are the vectors into ref:

$$\begin{bmatrix}1 & 0 & 1 & a\\0 & 1 & 1 & b\\0 & 1 & 0 & c\\1 & 0 & 0 & d\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 1 & a\\0 & 1 & 1 & b\\0 & 1 & 0 & c\\0 & 0 & -1 & d-a\end{bmatrix} \text{(subtract row 1 from row 4)}$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 1 & a\\0 & 1 & 1 & b\\0 & 0 & -1 & c-b\\0 & 0 & -1 & d-a\end{bmatrix} \text{(subtract row 2 from row 3)}$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 1 & a\\0 & 1 & 1 & b\\0 & 0 & -1 & c-b\\0 & 0 & 0 & (d-a)-(c-b)\end{bmatrix} \text{(subtract row 3 from row 4)}$$

The last column is linearly dependent if and only if there is no pivot element in it. Hence $(d-a)-(c-b)=0$, or $a+c=b+d$.

Problem 5 (4 points): Consider the following matrix:

$$A = \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Please compute a basis and the dimension of (right) null space of A.

Solve

$$A = \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix is already in ref, so only back substibution must be done: The free variables are $w = w$ and $y = y$. Now $z = 4w$ and $x = 2y - 2z - 3w = 2y - 11w$. Therefore

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = y \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -11 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

hence a basis for the right null space is $\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ 0 \\ 4 \\ 1 \end{bmatrix} \}$ and its dimension is 2.

Alternative derivation based on **column** echelon form (note that T keeps track of the elementary column operations, starting with the identity matrix):

$$\begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\text{add 2 times column 1 to column 2: } T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\text{subtract 2 times column 1 from column 3: } T = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\text{subtract 3 times column 1 from column 4: } T = \begin{bmatrix} 1 & 2 & -2 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\text{exchange column 2 and 3: } T = \begin{bmatrix} 1 & -2 & 2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\text{add 4 times column 2 to column 4: } T = \begin{bmatrix} 1 & 2 & -2 & -11 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

We therefore have

$$\begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -2 & -11 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the columns of T that do not produce a pivot element, namely the last 2 columns, form a basis for the right nullspace.