North Carolina State University
Department of Mathematics
College of Physical and Mathematical Sciences

MA 305 Elem Linear Algebra
First mid-semester examination
Spring 1997

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Professor

Your Name: ______________________
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 4 questions, each question counting for the given number of points, adding to a total of 20 points. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult a single $8.5' \times 11'$ sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 75 minutes to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
Total	____

If you are taking the exam later, please sign the following statement:

I, ________________ *affirm that I have not contacted my class mates about the contents of this exam.*

____
Signature

Problem 1 (7 points, 1 point for part (a), 1.5 points for each other part): Please answer the following questions.

(a) Please name two well-known mathematicians who have made contributions to the field of linear algebra.

Carl Friedrich Gauss, (M.-E.) Camille Jordan, Leonardo of Pisa Fibonacci

(b) Is the following identity true for all pairs of invertible matrices $A, B \in \mathbb{R}^{n \times n}$?

$$\left((AB)^{-1}\right)^T = (A^{-1})^T (B^{-1})^T$$

Please explain your answer.

Yes. $((AB)^{-1})^T = (B^{-1}A^{-1})^T = (A^{-1})^T(B^{-1})^T$

(c) Please compute the explicit value of the vector $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^4 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(d) The set of all $n \times n$ matrices over $\mathbb{R}$ forms a commutative group with matrix addition as its binary operation. What does this mean?

For all $A, B, C \in \mathbb{R}^{n \times n}$ we have

$$\begin{aligned} &A + (B + C) = (A + B) + C \quad (+ \text{ is associative}) \\ &A + 0^{n \times n} = 0^{n \times n} + A \quad (0^{n \times n} \text{ is an additive unit element}) \\ &A + (-A) = (-A) + A = 0^{n \times n} \quad (-A \text{ is an additive inverse}) \\ &A + B = B + A \quad (+ \text{ is commutative}) \end{aligned}$$

(e) The name of the Maple procedure `rref` in the `linalg` package is an acronym of which sequence of words?

reduced row echelon form

Problem 2 (4 points): The Gaussian elimination process has produced the following augmented matrix

$$\begin{bmatrix} -3 & 2 & 1 & 4 & | & 6 \\ 0 & 2 & 1 & 2 & | & -4 \\ 0 & 0 & 2 & 1 & | & 2 \end{bmatrix}$$

where the first four columns correspond to the variable x, y, z, and w. Please give the solution to this system in terms of linear forms in the free variables.

$$\begin{aligned} w &= w \\ z &= 1 - \frac{1}{2}w \\ y &= -2 - \frac{1}{2}z - w = -2 - \frac{1}{2} + \frac{1}{4}w - w \\ &= -\frac{5}{2} - \frac{3}{4}w \\ x &= -2 + \frac{2}{3}y + \frac{1}{3}z + \frac{4}{3}w = -2 + (-\frac{5}{3} - \frac{1}{2}w) + (\frac{1}{3} - \frac{1}{6}w) + \frac{4}{3}w \\ &= -\frac{10}{3} + \frac{2}{3}w \end{aligned}$$

Problem 3 (5 points, 1 point for part (a) and (b), 1.5 points for part (c) and (d)): Consider the following matrix with symbolic real parameters α, β, and γ:

$$A = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \alpha & \beta & 0 \\ 0 & \gamma & \alpha & \beta \\ 0 & 0 & \gamma & \alpha \end{bmatrix}.$$

Give necessary **and sufficient** conditions that the parameters α, β, and γ must satisfy (simultaneous equalities and inequalities—there may be more than one) such that

(a) A is upper triangular

$\gamma = 0$

(b) A is diagonal **and invertible**

$\gamma = 0$ and $\beta = 0$ and $\alpha \neq 0$

(c) A is in row echelon form

$\gamma = 0$

(d) A is in reduced row echelon form

$\gamma = 0$ and $[(\alpha = 1 \text{ and } \beta = 0) \text{ or } (\alpha = 0 \text{ and } \beta = 1) \text{ or } (\alpha = 0 \text{ and } \beta = 0)]$

Problem 4 (4 points): The following is product of 4 matrices with 3 rows and 3 columns. The matrices contain the variables a, b, c, d, e, and f.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & db-e & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -d & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & a \\ 1 & b & c \\ d & e & f \end{bmatrix}.$$

Give a 3 by 3 matrix that is the product multiplied out; some entries in the resulting matrix depend on the variables. Hint: perform the product from right to left using properties of elementary matrices.

Using properties of elementary matrices, we get

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & a \\ 1 & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ d & e & f \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -d & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & e-db & f-dc \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & db-e & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & e-db & f-dc \end{bmatrix} = \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & f-dc+a(db-e) \end{bmatrix}$$

Bonus question (1 point): Give a sequence of Maple expressions that compute the above product automatically.

```
with(refpkg);
A := matrix(3,3,[0,1,a, 1,b,c, d,e,f]);
E1 := E_I(3,1,3); E2:=E_III(3,1,3,-d); E3:=E_III(3,2,3,d*b-e);
evalm( E3 \&* E2 \&* E1 \&* A);
```