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Department of Mathematics

NC STATE UNIVERSITY

MA 305 Intro Elem Lin Algebra, final examination, May 16, 2000
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Your Name: SOLUTION
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 4 problems, which are subdivided into 11 questions, where each question counts for the explicitly given number of points, adding to a total of **43 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **three** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **120 minutes** to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
Total	____

Problem 1 (17 points) Please answer the following questions and **give a brief explanation for your answer**.

(a, 3 pts) True or false: for any matrix A with real entries, the number of linearly independent rows is equal to the number of linearly independent columns. Please explain.

True. This number is the rank of A.

(b, 3 pts) Please give a basis for the following orthogonal complement space: $\left(\text{Span}_{\mathbb{R}}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right)^{\perp}$.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$$

(c, 4 pts) Consider the following function on $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$: $\left\langle \begin{bmatrix}x_1\\x_2\end{bmatrix}, \begin{bmatrix}y_1\\y_2\end{bmatrix} \right\rangle = 1000\,x_1y_1 + \frac{1}{1000}x_2y_2$. Is this function an inner product in the vector space $\mathbb{R}^2$? Please explain.

Yes. This function is a weighted inner product with the positive weights 1000 *and* $1/1000$.

(d, 4 pts) Consider the following function on $\mathbb{R}^2 \to \mathbb{R}^2$: $F\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1+x_2\\x_1\cdot x_2\end{bmatrix}$. Is F a linear transformation? Please explain.

No. $F(\begin{bmatrix}2\\2\end{bmatrix}) = \begin{bmatrix}4\\4\end{bmatrix} \neq 2F(\begin{bmatrix}1\\1\end{bmatrix}) = 2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$.
Also $F(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}) = F(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}2\\1\end{bmatrix} \neq F(\begin{bmatrix}1\\0\end{bmatrix}) + F(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$.

(e, 3 pts) True or false: The Gram-Schmidt produces the **same set** of orthogonal basis vectors independently of the order in which the input basis vectors are processed. Please explain.

False. gram_schmidt$(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ *but* gram_schmidt$(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1/2\\-1/2\end{bmatrix}$.

Problem 2: (9 pts)

(a, 4 pts) The matrix

$$\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

defines the rotation of any three-dimensional vector around one of the coordinate axes by a specific degree. Which axis and by what degree? Please explain.

Since $\cos(\pi/4) = \sqrt{2}/2$ *and* $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ *is not affected, the rotation is by a degree of* $\pi/4$ *around the* y*-axis.*

(b, 5 pts) A plane $ax + by + c$ is to be best fitted to the values of $f(x,y)$ observed at (x,y):

x	0	1	−1	2	−2
y	0	0	1	1	2
$f(x,y)$	3	4	5	6	7

Please write down a 5×3 matrix A and a 5-dimensional vector b such that the solution to the corresponding least squares problem $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx b$ gives the coefficients of the optimal plane. You **do not** need to compute the solution.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$

Problem 3: (9 pts) Consider the QR-factorization of the 4×3 matrix A and a 4-dimensional vector b:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{R}, \quad b = \begin{bmatrix} -3 \\ -1 \\ -5 \\ 1 \end{bmatrix}.$$

(a, 6 pts) With the help of the QR-factorization, please solve the system of linear equations $Ax = b$ for the given matrix A and the given vector b. Please show all your work.

We follow the least squares approach.

$$Ax = b$$

$$QRx = b$$

$$\underbrace{(Q^T Q)}_{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \ Rx = \underbrace{Q^T b}_{\begin{bmatrix} -2 \\ -6 \\ 8 \end{bmatrix}}$$

$$Rx = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} -2 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

$$x_3 = 2$$

$$x_2 = -3 + x_3 = -1$$

$$x_1 = -1 - 2x_2 = 1$$

$$x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(b, 3 pts) Is the system $Ax = b$ from part a consistent? Please explain.

Yes. $Ax = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ -5 \\ 1 \end{bmatrix} = b$*, hence the orthogonal projection of* b *is* b *itself.*

Problem 4: (8 pts) Consider the 2×2 matrix A and the corresponding system of linear differential equations.

$$A = \begin{bmatrix} 5 & 6 \\ -3 & -4 \end{bmatrix}, \qquad \begin{aligned} y_1' &= 5y_1 + 6y_2, \\ y_2' &= -3y_1 - 4y_2. \end{aligned}$$

(a) (a, 4 pts) Please compute the eigenvalues and corresponding eigenvectors for A.

$$\det\left(\begin{bmatrix} 5-\lambda & 6 \\ -3 & -4-\lambda \end{bmatrix}\right) = (5-\lambda)(-4-\lambda) + 18 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

$$\lambda_1 = 2\text{: nullspace}\left(\begin{bmatrix} 3 & 6 \\ -3 & -6 \end{bmatrix}\right) = \text{Span}\Big(\underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{x^{[1]}}\Big).$$

$$\lambda_2 = -1\text{: nullspace}\left(\begin{bmatrix} 6 & 6 \\ -3 & -3 \end{bmatrix}\right) = \text{Span}\Big(\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{x^{[2]}}\Big).$$

(b) (b, 4 pts) Using the solution of part a, please give the general solution of the system of differential equations (with parameters that are determined by initial conditions).

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

$$y_1 = 2c_1 e^{2t} + c_2 e^{-t},$$

$$y_2 = -c_1 e^{2t} - c_2 e^{-t}.$$