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Department of Mathematics

NC STATE UNIVERSITY

MA 305 Intro Elem Lin Algebra, second mid-semester examination, Mar 30, 2000

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Your Name: SOLUTION
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 3 problems, which are subdivided into 10 questions, where each question counts for the explicitly given number of points, adding to a total of **43 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **two** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

Problem 1	____
2	____
3	____
Total	____

If you are taking the exam later, please sign the following statement:

I, ____________________, affirm that I have no knowledge of the contents of this exam.

____
Signature

Problem 1 (18 points) Please answer the following questions and **give a brief explanation for your answer**.

(a, 4 pts) What is the advantage to solving linear systems when the coefficient matrix is (already) factored into LU-form?

The cost is lower: One forward and one backward substitution $O(n^2)$ vs. Gaussian elimination $O(n^3)$.

(b, 4 pts) True or false: the number of multiplications performed when computing the determinant of an $n \times n$ matrix by minor expansion grows exponentially in n, in the worst case. Please explain.

True: $n! \geq 2^{n-1}$ without option remember; $n \cdot 2^n$ with option remember.

(c, 4 pts) Consider the subset of 2-dimensional vectors, $S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \cdot y = 0\} \subset \mathbb{R}^2$. Does this subset form a subspace of $\mathbb{R}^2$? Please explain.

NO. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in S$, *but* $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin S$.

(d, 3 pts) Is it possible that a vector space over $\mathbb{R}$ has a basis that has infinitely many elements? Please explain.

YES. E.g., $\mathbb{R}[x]$.

(e, 3 pts) True or false: the linear system $Ax = b$ is consistent if and only if the vector b is in the column space of the matrix A. Please explain.

TRUE. $b = x_1 \cdot A_{*,1} + \cdots + x_n A_{*,n} \in$ *col-space*(A).

Problem 2 (10 points): Consider the following linear system given as a matrix equation:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}. \tag{1}$$

Here a, b, c, d are parameters.

(a, 4pts) By Cramer's rule, please write down two matrices so that the solution x_1 of (1) is the quotient of their determinants.

$$\begin{bmatrix} a & -2 & 0 & 0 \\ b & 1 & -2 & 0 \\ c & 0 & 1 & -2 \\ d & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b, 6pts) Please compute the values of these two matrix determinants by minor expansion. From them, determine the value of x_1 (as a function in a, b, c, d).

Minor expansion shows that the determinant of a triangular matrix is the product of its diagonal elements. Hence, by minor expansion along the first column we have

$$\det \begin{bmatrix} a & -2 & 0 & 0 \\ b & 1 & -2 & 0 \\ c & 0 & 1 & -2 \\ d & 0 & 0 & 1 \end{bmatrix} = a \cdot \underbrace{\det \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{=1} - b \cdot \underbrace{\det \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{=-2}$$
$$+ c \cdot \underbrace{\det \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=4} - d \cdot \underbrace{\det \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}}_{=-8} = a + 2b + 4c + 8d.$$

and we have

$$\det\left(\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$$

Therefore, $x_1 = a + 2b + 4c + 8d$.

Problem 3: Consider the list of four 4-dimensional vectors, $S = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-2\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\3\end{bmatrix}$.
(15 points)

(a, 5 pts) Please pare down the list S to a subset of linearly independent vectors that spans the same vector space.

Performing row reduction to ref on the corresponding matrix yields

$$\begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\1 & -2 & 0 & 3\end{bmatrix} \underset{r_4 \leftarrow r_4 - r_1}{\Longrightarrow} \begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\0 & -2 & -2 & 3\end{bmatrix}$$

$$\underset{r_4 \leftarrow r_4 + 2r_2}{\Longrightarrow} \begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 3\end{bmatrix} \underset{r_4 \leftarrow r_4 - 3r_3}{\Longrightarrow} \begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

There are pivot elements in columns 1, 2, and 4, hence the first, second and fourth vector form a basis for Span(S).

(b, 6 pts) Please find a matrix $A \in \mathbb{R}^{m\times 4}$ such that $\text{Span}_{\mathbb{R}}(S)$ is the null space of A.

$m = 1$, *because dim(Span*$(S)) = 3$, *so the space is a hyperplane in* $\mathbb{R}^4$. *The hyperplane satisfies*

$$ax + by + cz + dw = 0,$$

where a, b, c, d *are to be determined. Plugging in vectors in the basis from part a, we get:*

$$\textit{For } \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} : a + d = 0; \quad \textit{for } \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}0\\1\\0\\-2\end{bmatrix} : b - 2d = 0; \quad \textit{for } \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}0\\0\\1\\3\end{bmatrix} : c + 3d = 0.$$

This is a linear system for a, b, c, d. *The variable* d *is free, hence we may choose* $d = 1$. *Then* $c = -3d = -3$, $b = 2d = 2$, $a = -d = -1$. *An answer is*

$$A = \begin{bmatrix}-1 & 2 & -3 & 1\end{bmatrix}.$$

(c, 4 pts) Please extend the basis for $\text{Span}_{\mathbb{R}}(S)$, which you have determined in part a above, to a basis for the full space $\mathbb{R}^4$. [Hint: add vectors that do not belong to the null space of A computed in part b.]

The vector $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ *is linearly independent from the other 3, because it does not lie on the hyperplane (in the null space of A):* $Ax = (-1)\cdot 1 + 2\cdot 0 + (-3)\cdot 0 + 1\cdot 0 = -1 \neq 0$. *Therefore the four vectors* $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ *are linearly independent, and thus form a basis for* $\mathbb{R}^4$.