Your Name: __________________
For purpose of anonymous grading, please do not write your name on the subsequent pages.

This examination consists of 3 problems, which are subdivided into 9 questions, where each question counts for the explicitly given number of points, adding to a total of 42 points. Please write your answers in the spaces indicated, or below the questions using the back of the sheets for completing the answers and for all scratch work, if necessary. You are allowed to consult two 8.5in × 11in sheet with notes, but not your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 60 minutes to do this test.

Good luck!

Problem 1 ______

2 ______

3 ______

Total ______

If you are taking the exam later, please sign the following statement:

I, ___________________, affirm that I have no knowledge of the contents of this exam.

__________________________
Signature
Problem 1 (18 points)

(a, 3 pts) True or false: \( \forall A \in \mathbb{R}^{n \times n}: \det(A) = 0 \implies \) the row echelon form of \( A \) contains a row of all zero entries. Please explain.

(b, 3 pts) Is it possible that \( n + 1 \) vectors in \( \mathbb{R}^n \) are linearly independent? Please explain.

(c, 4 pts) Consider the set of \( 2 \times 2 \) Toeplitz matrices over the reals, i.e., \( S = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \subset \mathbb{R}^{2 \times 2} \). Is \( S \) a subspace of \( \mathbb{R}^{2 \times 2} \), and if so, what is its dimension? Please justify your answer.

(d, 4 pts) Let \( V \subseteq \mathbb{R}^n \) be a vector space. Please define the orthogonal complement space \( V^\perp \).

(e, 4 pts) Let \( A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ 2 & -1 & 2 \end{bmatrix} \) and \( b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). Please find a vector \( y \in \mathbb{R}^3 \) such that \( y^T A = 0 \in \mathbb{R}^3 \) and \( y^T b \neq 0 \), which proves that \( Ax = b \) is inconsistent.
Problem 2 (12 points): Please consider

\[
A = \begin{bmatrix}
\alpha & -1 & 0 & 0 \\
1 & \beta & -2 & 0 \\
0 & 2 & \gamma & 0 \\
0 & 0 & 3 & \delta
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}.
\]

where \( \alpha, \beta, \gamma, \delta \) are real-valued parameters.

(a, 5pts) By Cramer's rule, please write down two matrices so that the entry \( x_3 \) in the solution vector \( x \) of \( Ax = b \) is the quotient of the determinants of these two matrices. In addition, \( x_3 \) is an entry of \( A^{-1} \), but in which row and column?

(b, 7pts) By minor expansion, please compute the values of the two determinants of Part (a) as polynomials in \( \alpha, \beta, \gamma, \delta \). Please show all your work.
Problem 3 (12 points): Consider the following vectors in \( \mathbb{R}^4 \):

\[
\begin{align*}
u_1 &= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \\
u_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \\
v_1 &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\end{align*}
\]

(a, 6 pts) Please implicitize the vector spaces \( U = \text{Span}_\mathbb{R}(u_1, u_2) \) and \( V = \text{Span}_\mathbb{R}(v_1, v_2) \) by computing matrices \( A, B \in \mathbb{R}^{1 \times 3} \) such that \( U \) is equal to the right nullspace of \( A \) and \( V \) is equal to the right nullspace of \( B \). Please explain all your work.

(b, 6 pts) Please compute a vector space basis for \( W = U \cap V \) as a vector space basis for the right nullspace of the block matrix \( \begin{bmatrix} A \\ B \end{bmatrix} \in \mathbb{R}^{2 \times 3} \). Please explain all your work.