This examination consists of 3 problems, which are subdivided into 11 questions, where each question counts for the explicitly given number of points, adding to a total of 44 points. Please write your answers in the spaces indicated, or below the questions using the back of the sheets for completing the answers and for all scratch work, if necessary. You are allowed to consult two 8.5in × 11in sheet with notes, but not your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 60 minutes to do this test.

Good luck!

Problem 1  _____

2  _____

3  _____

Total  _____
Problem 1 (20 points)

(a, 4 pts) True or false: \( \forall A \in \text{GL}_n(\mathbb{R}): \det(A^{-1}) = \det(A)^{-1} \). Please explain.

(b, 4 pts) What is a two-fold? Please define.

(c, 4 pts) Consider \( \mathbb{R}^2 \) with \( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \) for addition and \( \alpha \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix} \) for scalar multiplication. Is \( \mathbb{R}^2 \) with these operations a vector space? Please justify your answer.

(d, 4 pts) Consider the following subset \( S \subset \mathbb{R}[x]: S = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R} \text{ with } a - 2b + 4c = 0\} \). With the usual addition and multiplication by scalars of real polynomials in \( x \) does \( S \) form a subspace of \( \mathbb{R}[x] \)? Please justify your answer.

(e, 4 pts) Please give a matrix \( A \in \mathbb{R}^{2 \times 4} \) such that the rank of \( A \) is equal to the dimension of the right null space of \( A \).
Problem 2 (12 points): Consider the $4 \times 4$ matrix $A = \begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -a & -b & -c & \lambda - d \end{bmatrix}$ where $a, b, c, d, \lambda$ are parameters (for real numbers).

(a, 5pts) By Cramer’s rule, please write down two matrices so that the entry in row 3 and column 3 of $A^{-1}$ is $(-1)^i$ times the quotient of the determinants of these two matrices. In addition, please determine whether $i = 0$ or $i = 1$.

(b, 7pts) By minor expansion, please compute the value of the two determinants of part a as polynomials in $a, b, c, d, \lambda$. 
Problem 3 (12 points): Consider the following vectors in $\mathbb{R}^4$:

\[
\begin{align*}
v_1 &= \begin{bmatrix}
1 \\
-1 \\
1 \\
-1
\end{bmatrix}, \\
v_2 &= \begin{bmatrix}
2 \\
-2 \\
2 \\
-2
\end{bmatrix}, \\
v_3 &= \begin{bmatrix}
1 \\
1 \\
0 \\
1
\end{bmatrix}, \\
v_4 &= \begin{bmatrix}
2 \\
0 \\
1 \\
0
\end{bmatrix}.
\end{align*}
\]

(a, 6 pts) Please pare down the list $v_1, v_2, v_3, v_4$ to form a basis of $\text{Span}_{\mathbb{R}}(v_1, v_2, v_3, v_4)$.

(b, 6 pts) Please write down a matrix $\mathbb{R}^{m \times 4}$, where the minimum $m$ is determined in part a, such that right nullspace$(A) = \text{Span}_{\mathbb{R}}(v_1, v_2, v_3, v_4)$. 
