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Department of Mathematics

NC STATE UNIVERSITY

MA 305 Intro Elem Lin Algebra, final examination, May 6, 2004

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Your Name: ____________________

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 problems, which are subdivided into 11 questions, where each question counts for the explicitly given number of points, adding to a total of **49 points**. Please write your answers in the spaces indicated, or below the questions, using the **back of the sheets** for completing the answers and **for all scratch work,** if necessary. You are allowed to consult **three** 8.5in $\times$ 11in sheets with notes and your calculators, but **not** your book or your class notes or laptops. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **120 minutes** to do this test.

Good luck!

Problem 1	_____
2	_____
3	_____
4	_____
5	_____
Total	_____

If you are taking the exam early, please sign the following statement:

I, ________________, *affirm that I will not disclose the contents of this exam to anyone.*

____
Signature

Problem 1 (18 points)

(a, 3 pts) Consider a system of linear equations in n variables. Suppose that its solution set is parameterized by $k \leq n$ free variables. What is the minimum number of equations that system must have? Please explain.

(b, 4 pts) Please give a basis for the following orthogonal complement space: $\mathrm{Span}_{\mathbb{R}}(\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T, \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^T)^{\perp}$.

(c, 4 pts) We have discussed orthogonal projection and least squares fitting with respect to weighted inner products. Please give conditions under which a model obtained with varying weights is more accurate than the standard one.

(d, 4 pts) Consider the weighted Manhattan norm function N from $\mathbb{R}^2$ to $\mathbb{R}_{\geq 0}$: $N\left(\begin{bmatrix}x \\ y\end{bmatrix}\right) = 2|x| + |y|/3$. Please prove that for the vectors $u = \begin{bmatrix}-1 \\ -3\end{bmatrix}$ and $v = \begin{bmatrix}5 \\ -6\end{bmatrix}$ the triangle inequality is satisfied.

(e, 3 pts) Consider the following linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$: $L\left(\begin{bmatrix}x \\ y\end{bmatrix}\right) = \begin{bmatrix}y \\ x \\ -2x+3y\end{bmatrix}$. Please give a matrix $A \in \mathbb{R}^{3\times 2}$ such that $L(v) = Av$ for $v \in \mathbb{R}^2$.

Problem 2: (5 pts) Consider the 3-dimensional rotation around the y-axis (parallel to the x-z plane) by 315 degrees ($= \frac{7}{4}\pi = -\frac{1}{4}\pi$) counter-clockwise. Please give the 3×3 matrix which when multiplied by any vector in $\mathbb{R}^3$ effects this rotation.

Problem 3: (6 pts) Consider the set of points in $\mathbb{R}^3$ to the right. You want to best fit the points to the surface

x	1	-1	2	-2	3	-3
y	1	2	3	-1	-2	-3
z	4	5	6	7	-6	-5

$$z = c_1 \cdot x^2 + c_2 \cdot y^2 + c_3 \cdot x \cdot y + c_4 \cdot y + c_5 \tag{1}$$

for some constants c_1, c_2, c_3, c_4, c_5. Please write down a 6×5 matrix A and a 6-dimensional vector b such that the solution to the corresponding least squares problem $A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} \approx b$ yields a best fit to the model (1). You **do not** need to compute the solution.

Problem 4: (14 pts)
Consider
$$A = \begin{bmatrix} 1 & -1 & -4 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -5 & 10 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ -4 \\ 2 \\ -31 \end{bmatrix}.$$

(a, 6 pts) Using the Gram-Schmidt orthogonalization algorithm (as done in class, **without normalizing** the vectors to unit length) please compute the QR-factorization of A, that is compute a matrix Q whose columns are orthogonal and an upper triangular matrix R such that $A = Q \cdot R$. Please show your work.

(b, 4 pts) Using the columns of Q and the vector b, please compute the orthogonal projection, denoted by $\hat{b}$, of b onto the column space of A.

Problem 4 continued:

(c, 4 pts) Solving the normal equations via the QR-factorization of part (a), please find $\hat{x} \in \mathbb{R}^3$ such that $\hat{b} = A \cdot \hat{x}$, where $\hat{b}$ is defined in part (b).

Problem 5: (6 pts) Let $A = \begin{bmatrix} 10 & 12 \\ -6 & -8 \end{bmatrix}$. **Without** the use of a **computer or a calculator**, please compute the eigenvalues and corresponding eigenvectors for A. Please show your work.