This examination consists of 5 problems, which are subdivided into 11 questions, where each question counts for the explicitly given number of points, adding to a total of 49 points. Please write your answers in the spaces indicated, or below the questions, using the back of the sheets for completing the answers and for all scratch work, if necessary. You are allowed to consult three 8.5in × 11in sheets with notes and your calculators, but not your book or your class notes or laptops. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 120 minutes to do this test.

Problem 1 ______

2 ______

3 ______

4 ______

5 ______

Total ______

If you are taking the exam early, please sign the following statement:

I, __________________, affirm that I will not disclose the contents of this exam to anyone.

________________________
Signature
Problem 1 (18 points)

(a, 3 pts) Consider a system of linear equations in \( n \) variables. Suppose that its solution set is parameterized by \( k \leq n \) free variables. What is the minimum number of equations that system must have? Please explain.

(b, 4 pts) Please give a basis for the following orthogonal complement space: \( \text{Span}_{\mathbb{R}}([1 \ 1 \ 1 \ 1]^T, [1 \ -1 \ -1 \ 1]^T)^\perp \).

(c, 4 pts) We have discussed orthogonal projection and least squares fitting with respect to weighted inner products. Please give conditions under which a model obtained with varying weights is more accurate than the standard one.

(d, 4 pts) Consider the weighted Manhattan norm function \( N \) from \( \mathbb{R}^2 \) to \( \mathbb{R}_{\geq 0} \): \( N \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = 2|x| + |y|/3 \). Please prove that for the vectors \( u = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \) and \( v = \begin{bmatrix} 5 \\ -6 \end{bmatrix} \) the triangle inequality is satisfied.

(e, 3 pts) Consider the following linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \): \( L \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} y \\ x \\ -2x + 3y \end{bmatrix} \).
Please give a matrix \( A \in \mathbb{R}^{3\times 2} \) such that \( L(v) = Av \) for \( v \in \mathbb{R}^2 \).
Problem 2: (5 pts) Consider the 3-dimensional rotation around the y-axis (parallel to the x-z plane) by 315 degrees (\(= \frac{7}{4}\pi = -\frac{1}{4}\pi\)) counter-clockwise. Please give the \(3 \times 3\) matrix which when multiplied by any vector in \(\mathbb{R}^3\) effects this rotation.

Problem 3: (6 pts) Consider the set of points in \(\mathbb{R}^3\) to the right. You want to best fit the points to the surface

\[ z = c_1 \cdot x^2 + c_2 \cdot y^2 + c_3 \cdot x \cdot y + c_4 \cdot y + c_5 \]  

(1)

for some constants \(c_1, c_2, c_3, c_4, c_5\). Please write down a \(6 \times 5\) matrix \(A\) and a 6-dimensional vector \(b\) such that the solution to the corresponding least squares problem \(A \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \\ \hat{c}_5 \end{bmatrix} \approx b\) yields a best fit to the model (1). You do not need to compute the solution.
Problem 4: (14 pts)
Consider 
\[
A = \begin{bmatrix}
1 & -1 & -4 \\
1 & -1 & -1 \\
1 & -1 & -1 \\
1 & -5 & 10 \\
\end{bmatrix}
\quad \text{and} \quad 
\begin{bmatrix}
5 \\
-4 \\
2 \\
-31 \\
\end{bmatrix}
\]

(a, 6 pts) Using the Gram-Schmidt orthogonalization algorithm (as done in class, \textbf{without normalizing} the vectors to unit length) please compute the QR-factorization of \( A \), that is compute a matrix \( Q \) whose columns are orthogonal and an upper triangular matrix \( R \) such that \( A = Q \cdot R \). Please show your work.

(b, 4 pts) Using the columns of \( Q \) and the vector \( b \), please compute the orthogonal projection, denoted by \( \hat{b} \), of \( b \) onto the column space of \( A \).
Problem 4 continued:

(c, 4 pts) Solving the normal equations via the QR-factorization of part (a), please find \( \hat{x} \in \mathbb{R}^3 \) such that \( \hat{b} = A \cdot \hat{x} \), where \( \hat{b} \) is defined in part (b).

Problem 5: (6 pts) Let \( A = \begin{bmatrix} 10 & 12 \\ -6 & -8 \end{bmatrix} \). Without the use of a computer or a calculator, please compute the eigenvalues and corresponding eigenvectors for \( A \). Please show your work.