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Department of Mathematics

NC STATE UNIVERSITY

MA 305 Intro Elem Lin Algebra, final examination, May 13, 2003
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Your Name: SOLUTION
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 problems, which are subdivided into 12 questions, where each question counts for the explicitly given number of points, adding to a total of **49 points**. Please write your answers in the spaces indicated, or below the questions, using the back of the sheets for scratch work and for completing the answers, if necessary. You are allowed to consult **three** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **120 minutes** to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
5	____
Total	____

Problem 1 (17 points)

(a, 3 pts) True or false: for all positive integers m,n and all $A \in \mathbb{R}^{m\times n}$ the dimension of the column space of A plus the dimension of the nullspace of A is equal n. Please explain.

True. The dimension of the nullspace of A is equal to n-rank(A). The dimension of the column space of A is the rank(A). So the dimension of the column space of A plus the dimension of the nullspace of A is equal to n.

(b, 4 pts) Please give a matrix $A \in \mathbb{R}^{m\times 3}$, where m is to be determined by you, and a vector $b \in \mathbb{R}^2$ such that the solution set $\{x \in \mathbb{R}^3 \mid Ax = b\}$ is equal to the parametric set $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix} + t\cdot \begin{bmatrix}-1\\0\\1\end{bmatrix} \,\middle|\, t \in \mathbb{R}\right\}$.

$$m = 2,\ A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\end{bmatrix},\ b = \begin{bmatrix}4\\2\end{bmatrix}$$

(c, 3 pts) Consider the weighted inner product function from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}$: $\left\langle \begin{bmatrix}x_1\\y_1\end{bmatrix}, \begin{bmatrix}x_2\\y_2\end{bmatrix} \right\rangle_{\langle 2,3\rangle} = 2x_1x_2 + 3y_1y_2$. Please give two vectors $u, v \in \mathbb{R}^2$ that are orthogonal with respect to that weighted inner product.

$$\left\{u = \begin{bmatrix}3\\2\end{bmatrix}, v = \begin{bmatrix}1\\-1\end{bmatrix}\right\}, \left\{u = \begin{bmatrix}1\\0\end{bmatrix}, v = \begin{bmatrix}0\\1\end{bmatrix}\right\}, \left\{u = \begin{bmatrix}-3\\2\end{bmatrix}, v = \begin{bmatrix}2\\2\end{bmatrix}\right\}$$

(d, 4 pts) Consider the infinity norm function from $\mathbb{R}^2$ to $\mathbb{R}_{\geq 0}$: $\left\| \begin{bmatrix}x\\y\end{bmatrix} \right\|_\infty = \max\{|x|,|y|\}$. Please prove that for the vectors $u = \begin{bmatrix}-3\\-2\end{bmatrix}$ and $v = \begin{bmatrix}4\\-5\end{bmatrix}$ the triangle inequality is satisfied.

$$\left\| \begin{bmatrix}-3\\-2\end{bmatrix} + \begin{bmatrix}4\\-5\end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix}1\\-7\end{bmatrix} \right\|_\infty = 7$$
$$\left\| \begin{bmatrix}-3\\-2\end{bmatrix} \right\|_\infty + \left\| \begin{bmatrix}4\\-5\end{bmatrix} \right\|_\infty = 3 + 5 = 8$$
$$7 \leq 8$$

(e, 3 pts) Let L be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ and let $u, v \in \mathbb{R}^3$. If $L(u) = \begin{bmatrix}1\\-3\end{bmatrix}$ and $L(v) = \begin{bmatrix}2\\-1\end{bmatrix}$, please compute $L(2u - 5v)$.

$$L(2u-5v) = L(2u) - L(5v) = 2L(u) - 5L(v) = 2 * \begin{bmatrix}1\\-3\end{bmatrix} - 5 * \begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}-8\\-1\end{bmatrix}$$

Problem 2: (5 pts) Let R be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ corresponding to the rotation of any three dimensional vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ around a specific axis by a specifice degree. Suppose

$$R\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\sqrt{3}/2 \\ 0 \\ -1/2 \end{bmatrix}, R\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, R\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ 0 \\ -\sqrt{3}/2 \end{bmatrix}.$$

Compute the matrix A representing the transformation R and compute the axis and angle of the rotation. Please explain.

$$A = \begin{bmatrix} -\sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & -\sqrt{3}/2 \end{bmatrix}$$

This matrix represents rotation around the y axis by an angle of 210° *or* $\frac{7\pi}{6}$ *rad.*

Problem 3: (6 pts) Consider the set of points in $\mathbb{R}^3$ to the right. You want to best fit the points to the surface

x	0	1	2	3	4	5	6
y	0	0	0	1	−1	1	2
z	20	17	−16	14	−11	9	8

$$z = c_0 + c_1 \cdot x \cdot y + c_2 \cdot x^2 + c_3 \cdot y^3 \tag{1}$$

for some constants c_0, c_1, c_2, c_3. Please write down a 7×4 matrix A and a 7-dimensional vector b such that the solution to the corresponding least squares problem $A \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \approx b$ yields a best fit to the model (1). You **do not** need to compute the solution.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 3 & 9 & 1 \\ 1 & -4 & 16 & -1 \\ 1 & 5 & 25 & 1 \\ 1 & 12 & 36 & 8 \end{bmatrix}$$

$$b = \begin{bmatrix} 20 \\ 17 \\ -16 \\ 14 \\ -11 \\ 9 \\ 8 \end{bmatrix}$$

Problem 4: (12 pts) Consider a subspace of $\mathbb{R}^5$ spanned by the three vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, v_3 := \begin{bmatrix} 0 \\ 1 \\ -2 \\ -3 \\ 1 \end{bmatrix}$.

(a, 5 pts) Using the Gram-Schmidt orthogonalization algorithm (as done in class, **without normalizing** the vectors to unit length) please compute by hand the orthogonalization u_1, u_2, u_3 of the basis v_1, v_2, v_3. Please show your work.

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$u_2 = v_2 - \frac{u_1^T v_2}{u_1^T u_1} u_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} - \frac{6}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_3 = v_3 - \frac{u_1^T v_3}{u_1^T u_1} u_1 - \frac{u_2^T v_3}{u_2^T u_2} u_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -3 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{-4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

(b, 2 pts) By use of the results of part (a), please present a matrix $R \in \mathbb{R}^{3\times 3}$ such that $[v_1 \vdots v_2 \vdots v_3] = [u_1 \vdots u_2 \vdots u_3] \cdot R$.

$$R = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(c, 5 pts) Using the QR-factorization of part (b), please find $c_1, c_2, c_3 \in \mathbb{R}$ such that $\hat{b} = c_1 v_1 + c_2 v_2 + c_3 v_3$ is the orthogonal projection of $b = [12 \ \ -2 \ \ 0 \ \ 2 \ \ 0]^T$ onto $\mathrm{Span}_{\mathbb{R}}(v_1, v_2, v_3)$.

$$R \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = (Q^T Q)^{-1} Q^T b$$

$$R \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

$$c_3 = -1, c_2 = 3 + c_3 = 2, c_1 = 4 - 2c_2 - c_1 = 1$$

Problem 5: (9 pts)

(a, 6 pts) Let $A \in \mathbb{R}^{2\times 2}$ be the following:

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

Without the use of a **computer or a calculator**, please compute the eigenvalues and corresponding eigenvectors for A. Please show your work.

$$\det\left(\begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2)$$

So the eigenvalues are $\lambda_1 = 5$ *and* $\lambda_2 = -2$.

The eigenvectors for λ_1 *are given by the following.*

$$Nullspace\left(\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\right) = Span_{\mathbb{R}}(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$$

The eigenvectors for λ_2 *are given by the following.*

$$Nullspace\left(\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\right) = Span_{\mathbb{R}}(\begin{bmatrix} -4/3 \\ 1 \end{bmatrix})$$

(b, 3 pts) Find the general solution to the following system of differential equations.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_1 + 4y_2 \\ 3y_1 + 2y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} -4/3 \\ 1 \end{bmatrix} e^{-2t}$$