North Carolina State University is a land-grant university and a constituent institution of The University of North Carolina

Department of Mathematics

NC STATE UNIVERSITY

MA 305 Intro Elem Lin Algebra, second mid-semester examination, Mar 27, 2003
919.515.8785 (phone)
kaltofen@math.ncsu.edu (email)
919.515.3798 (fax)
http://www.math.ncsu.edu/ kaltofen/courses/LinAlgebra/Spring03/ (URL)

Your Name: SOLUTION
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 3 problems, which are subdivided into 9 questions, where each question counts for the explicitly given number of points, adding to a total of **44 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **two** 8.5in × 11in sheets with notes and your calculators, but **not** your book or your class notes or laptops. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

Problem 1	_____
2	_____
3	_____
Total	_____

Problem 1 (20 points)

(a, 4 pts) Let $A \in \mathbb{R}^{n\times n}$ and suppose $\det(A) = 0$. Is A singular or non-singular? Please explain.

A is singular because by definition if $\det(A)$*=0 then A is singular.*

(b, 4 pts) Compute rank(A) if $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 0 & 4 & 0 \end{bmatrix}$

rank(A) = *3*

(c, 4 pts) Consider $\mathbb{R}^2$ with $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ for addition and $\alpha \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \alpha^2 x_1 \\ \alpha y_1 \end{bmatrix}$ for scalar multiplication. Is $\mathbb{R}^2$ with these operations a vector space? Please justify your answer.

No, the scalar multiplication fails axiom 6.

$$(a+b) \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} (a+b)^2 x_1 \\ (a+b) y_1 \end{bmatrix} \neq \begin{bmatrix} (a^2+b^2) x_1 \\ (a+b) y_1 \end{bmatrix} = a \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus b \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

(d, 4 pts) Consider the following subset $S \subset \mathbb{R}^{2\times 2}$: $S = \{\begin{bmatrix} 0 & a \\ b & a-b \end{bmatrix} \mid a, b \in \mathbb{R}\}$. With the usual matrix addition and multiplication of matrices by scalars, does S form a subspace of $\mathbb{R}^{2\times 2}$? Please justify your answer.

Yes, S is a subspace of $\mathbb{R}^{2\times 2}$.

$$\begin{bmatrix} 0 & a \\ b & a-b \end{bmatrix} + \begin{bmatrix} 0 & c \\ d & c-d \end{bmatrix} = \begin{bmatrix} 0 & a+c \\ b+d & (a+c)-(b+d) \end{bmatrix} \in S$$

$$r \cdot \begin{bmatrix} 0 & a \\ b & a-b \end{bmatrix} = \begin{bmatrix} 0 & ra \\ rb & ra-rb \end{bmatrix} \in S$$

(e, 4 pts) Let $A \in \mathbb{R}^{2\times 3}$. Please describe all the possible dimensions of the right null space of A as a subspace of $\mathbb{R}^3$. Please explain.

Since A has rank at most 2, then the right null space of A has dimension at least 1. So the possible dimensions of the null space are 1, 2, or 3.

Problem 2 (12 points): Consider the following 4×4 matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & s & 3 & -2 \\ -1 & t & 4 & 1 \end{bmatrix} \tag{1}$$

(a, 6pts) By Cramer's rule, please write down two matrices so that the 3,2 entry of A^{-1} is $(-1)^i$ times the quotient of their determinants. Also please determine if $i = 0$ or 1.

$$A^{-1}_{3,2} = \frac{(-1)^{3+2}\det(A\downarrow_{2,3})}{\det(A)} = \frac{(-1)\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & s & -2 \\ -1 & t & 1 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & s & 3 & -2 \\ -1 & t & 4 & 1 \end{bmatrix}\right)}$$

$$i = 1$$

(b, 6pts) Please compute the values of these two matrix determinants by minor (also known as cofactor) expansion. Using these determine the 3,2 entry of A^{-1}.

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & s & -2 \\ -1 & t & 1 \end{bmatrix}\right) = 2 \cdot \det\left(\begin{bmatrix} s & -2 \\ t & 1 \end{bmatrix}\right) = 2(s + 2t)$$

$$\det\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & s & 3 & -2 \\ -1 & t & 4 & 1 \end{bmatrix}\right) = 2{\cdot}\det\left(\begin{bmatrix} 5 & 0 & 0 \\ s & 3 & -2 \\ t & 4 & 1 \end{bmatrix}\right) = 2{\cdot}5{\cdot}\det\left(\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}\right) = 10(3{+}8) = 110$$

$$A^{-1}_{3,2} = \frac{-2(s+2t)}{110} = \frac{-(s+2t)}{55}$$

Problem 3 (12 points): Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 3 \\ 6 \\ -8 \end{bmatrix}.$$

(a, 6 pts) Please pare down the list v_1, v_2, v_3, v_4 to form a basis of $\text{Span}_{\mathbb{R}}(v_1, v_2, v_3, v_4)$.

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 3 & 3 \\ 0 & 0 & 0 & 6 \\ 3 & 6 & 2 & -8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & -1 & -5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are pivots in columns 1,3 and 4, then $\{v_1, v_3, v_4\}$ form a basis for the span.

(b, 6 pts) Consider all vectors $w = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ in $\mathbb{R}^4$. Please write down a condition on the real numbers x, y, z, u such that $\text{Span}_{\mathbb{R}}(v_1, v_2, v_3, v_4, w) = \mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & -1 & x \\ 2 & 3 & 3 & y \\ 0 & 0 & 6 & z \\ 3 & 2 & -8 & u \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & -1 & x \\ 0 & 1 & 5 & y-2x \\ 0 & 0 & 6 & z \\ 0 & -1 & -5 & u-3x \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & -1 & x \\ 0 & 1 & 5 & y-2x \\ 0 & 0 & 6 & z \\ 0 & 0 & 0 & u+y-5x \end{bmatrix}$$

If $\{v_1, v_3, v_4, w\}$ are to form a basis of $\mathbb{R}^4$ then there must be a pivot in each column.
So $u + y - 5x \neq 0$ must be true.