MA 305 Intro Elem Lin Algebra, first mid-semester examination, Feb 12, 2002
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www.math.ncsu.edu/~kaltofen/courses/LinAlgebra/Spring02/ (URL)
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Your Name: SOLUTION
For purpose of anonymous grading, please do not write your name on the subsequent pages.

This examination consists of 3 problems, which are subdivided into 11 questions, where each question counts for the explicitly given number of points, adding to a total of 44 points. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult one 8.5in \times 11in sheet with notes, but not your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 75 minutes to do this test.

Problem 1

2

3

Total

If you are taking the exam later, please sign the following statement:

I, ____________________, affirm that I have no knowledge of the contents of this exam.

________________________
Signature
**Problem 1** (17 points)

(a, 3 pts) In what country was the famous mathematician Norbert Wiener born? Did Wiener live before, during or after Carl Friedrich Gauss’s lifetime?

*Wiener was born in the United States, and lived after Gauss’s lifetime.*

(b, 4 pts) Consider the following variant of the Fibonacci’s rabbits problem. Instead of having a single pair of offspring each month, after maturity each pair of rabbits gives birth to two pairs (2 male and 2 female rabbits) each month. Please give the $2 \times 2$ transition matrix for this recursion.

$$
\begin{bmatrix}
0 & 1 \\
2 & 1
\end{bmatrix}
$$

(c, 4 pts) Please write down a non-singular matrix $A \in \mathbb{R}^{2\times 2}$ such that $A \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = A^{-1}$.

$$
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
$$

(d, 3 pts) Suppose that in a Maple session you have performed the command `package with(LinearAlgebra)`. Please give the Maple command to assign in the variable $A$ the matrix $A = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \end{bmatrix}$.

$$A := Matrix(2,3,[[a,b,c],[1,2,3]]);$$

(e, 3 pts) True or false: for any matrix $A \in \mathbb{R}^{m\times n}$ there exists a matrix $B \in \mathbb{R}^{m\times n}$ such that $A + B = B + A = 0^{m\times n}$, where $0^{m\times n}$ is an $m \times n$ matrix all of whose entries are zero. Please explain.

*True, if $A \in \mathbb{R}^{m\times n}$, then $B = (-1)A \in \mathbb{R}^{m\times n}$ and $A + B = B + A = 0^{m\times n}$.***
Problem 2 (14 points): Consider the following augmented matrix of a system of linear equations, where the first 4 columns correspond to the variables $x$, $y$, $z$, $w$ and where $a$, $b$, $c$ and $d$ are real parameters.

\[
\begin{bmatrix}
1 & 2 & -3 & 4 & : & 5 \\
0 & 1 & -6 & : & 7 \\
0 & 0 & 0 & b & : & c \\
0 & 0 & 0 & 0 & : & d
\end{bmatrix}
\]  \hfill (1)

(a, 4pts) For which values of $a$, $b$, $c$ and $d$ is the matrix (1) in row echelon form? Please give all the conditions.

\((b \neq 0) \text{ or } (b = 0 \text{ and } d = 0)\)

(b, 4pts) For which values of $a$, $b$, $c$ and $d$ is the system of linear equations corresponding to the matrix (1) consistent? Please give all the conditions.

\((d = 0) \text{ and } (b \neq 0 \text{ or } b = 0 \text{ and } c = 0)\)

(c, 6pts) For the conditions described under (b), please solve the linear system. You need to consider the four cases $a = 0$, $a \neq 0$ and $b = 0$, $b \neq 0$ separately.

There are four cases to consider:

\begin{align*}
\text{a=0 and b=0:} & \quad z = 7 + 6w, \quad x = 26 - 2y + 14w, \quad y = y, \quad w = w \\
\text{a=0 and b\neq0:} & \quad y = y, \quad w = \frac{c}{b}, \quad z = \frac{7b + 6c}{b}, \quad x = -2 \left( \frac{by - 13b - 7c}{b} \right) \\
\text{a\neq0 and b=0:} & \quad z = z, \quad w = w, \quad y = \frac{-z + 6w + 7}{a}, \quad x = \frac{5a + 2z - 12w - 14 + 3az - 4aw}{a} \\
\text{a\neq0 and b\neq0:} & \quad w = \frac{c}{b}, \quad z = z, \quad y = \frac{7b + 6c - 6z}{ab}, \quad x = \frac{3abz + 2bz + 5ab - 14b - 4ca - 12c}{ab}
\end{align*}
Problem 3 (13 points): Consider the following matrix \( A \)

\[
\begin{bmatrix}
1/2 & 0 & -5/2 \\
0 & 0 & 1 \\
0 & 1 & 2
\end{bmatrix}
\]

In this problem you are to compute elementary matrices \( E_1, E_2, E_3, E_4 \) such that \( A = E_1 \cdot E_2 \cdot E_3 \cdot E_4 \). Please perform the following steps.

(a, 4 pts) Compute the reduced row echelon form of \( A \) by Gauss-Jordan elimination.

\[
\begin{bmatrix}
1/2 & 0 & -5/2 \\
0 & 0 & 1 \\
0 & 1 & 2
\end{bmatrix} \xrightarrow{2 \cdot R_1} \begin{bmatrix}
1 & 0 & -5 \\
0 & 0 & 1 \\
0 & 1 & 2
\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}
1 & 0 & -5 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix} \xrightarrow{-2 \cdot R_3 + R_2} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(b, 5 pts) For each row operation performed in (a), give the corresponding elementary matrix in the form of \( E_I(\ldots) \) or \( E_{II}(\ldots) \) or \( E_{III}(\ldots) \). Please give the matrices in the order the operations were done in (a).

\( E_1 = E_{III}(3,1,2), E_2 = E_I(3,2,3), E_3 = E_{III}(3,3,2,-2), E_4 = E_{III}(3,3,1,5) \)

(c, 4 pts) For each elementary matrix in (b), give its inverse as another elementary matrix in the form of \( E_I(\ldots) \) or \( E_{II}(\ldots) \) or \( E_{III}(\ldots) \). Then write \( A \) as the corresponding product of \( 3 \times 3 \) matrices.

\( E_1^{-1} = E_{II}(3,1,1/2), E_2^{-1} = E_I(3,2,3), E_3^{-1} = E_{III}(3,3,2,2), E_4^{-1} = E_{III}(3,3,1,-5) \)

\[
A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]