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**Department of Mathematics**

**NC STATE** UNIVERSITY

MA 305 Intro Elem Lin Algebra, first mid-semester examination, Feb 13, 2001

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*Your Name:* SOLUTION
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 3 problems, which are subdivided into 11 questions, where each question counts for the explicitly given number of points, adding to a total of **46 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **one** 8.5in $\times$ 11in sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

| | |
|---|---|
| Problem 1 | ____ |
| 2 | ____ |
| 3 | ____ |
| Total | ____ |

If you are taking the exam later, please sign the following statement:

*I, ____________________, affirm that I have no knowledge of the contents of this exam.*

____________________________
Signature

**Problem 1** (19 points)

(a, 3 pts) Please name a well-known mathematician who was born in the USA.

*John Nash, Julia Robinson, Stephen Smale, Benjamin Banneker, Josiah Gibbs*

(b, 4 pts) In how many multipications of pairs of integers can one compute $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{256}$? Please explain.

$256 = 2^8$: *there are 8 matrix multiplications, namely squarings, of* $2 \times 2$ *matrices, each of which needs* $2 \cdot 2 \cdot 2 = 8$ *integer multiplications. A total of* $8 \cdot 8 = 64$ *integer multiplications.*

*Without the doubling trick, there are 255 matrix multiplications, a total of* $255 \cdot 8 = 2040$ *integer multiplicaions.*

(c, 4 pts) By laws that the transposition and inversion operations for matrices satisfy please prove that for an invertible matrix $A$ we have $(A^T)^{-1} = (A^{-1})^T$.

$$(A^T) \cdot (A^{-1})^T \underbrace{=}_{\text{because } (A \cdot B)^T = B^T \cdot A^T} (A^{-1} \cdot A)^T = I^T = I, \text{ and}$$

$$(A^{-1})^T \cdot (A^T) = (A \cdot A^{-1})^T = I^T = I.$$

(d, 4 pts) Suppose that in a Maple session the variable `A` has been assigned a $4 \times 4$ invertible matrix. If you then execute the Maple commands `with(linalg): rref(A);` what answer to you expect?

$$\text{A } 4 \times 4 \text{ identity matrix } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(e, 4 pts) A binary operation on an arbitrary set $S$ may or may not satisfy the mathematical laws of (i) associativity, (ii) $S$ has a unit element, (iii) each element in $S$ has an inverse element, and (iv) commutativity. For the concrete set $S = \mathbb{R}^{n \times n}$ and matrix multiplication as the binary operation, which of these four laws are satisfied?

*(i) and (ii).*

**Problem 2** (15 points): Consider the following augmented matrix of a system of linear equations, where the first 3 columns correspond to the variables $x, y, z$ and where $a$, $b$, and $c$ are real parameters.

$$\begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ -a & 1 & -1 & \vdots & (1+a)c \\ 0 & b & -b & \vdots & c \end{bmatrix} \tag{1}$$

(a, 4pts) For which values of $a$, $b$, and $c$ is the matrix (1) in row echelon form? Please give all the conditions.

*$a = 0$ and $b = 0$.*

(b, 6pts) By performing Gaussian elimination on the cases $a = -1$ and $a \neq -1$ separately, determine for which values of $a$, $b$, and $c$ the linear system corresponding to the augmented matrix (1) is consistent. For each condition please state the row-echelon form.

*Case $a = -1$:*

$$\begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & b & -b & \vdots & c \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & b & -b & \vdots & c \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ 0 & b & -b & \vdots & c \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

*consistent if $b \neq 0$ or if $b = c = 0$.*

*Case $a \neq -1$:*

$$\begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ -a & 1 & -1 & \vdots & (1+a)c \\ 0 & b & -b & \vdots & c \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ 0 & 1+a & -1-a & \vdots & (1+a)c \\ 0 & b & -b & \vdots & c \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & c \\ 0 & b & -b & \vdots & c \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & c \\ 0 & 0 & 0 & \vdots & c - bc \end{bmatrix}.$$

*consistent if $c(1-b) = 0$.*

(c, 5pts) For each of the conditions discovered in part b, please perform the back-substitution to solve the system.

*Case $a = -1, b \neq 0$: $z = z$, $y = (1/b) \cdot (c + bz) = c/b + z$, $x = -y + z = -c/b$.*

*Case $a = -1, b = c = 0$: $z = z$, $y = y$, $x = -y + z$.*

*Case $a \neq -1, c(1-b) = 0$: $z = z$, $y = c + z$, $x = -y + z = -c$.*

**Problem 3** (12 points): Consider the following matrix $A$ together with its factorization into elementary matrices. Here $\alpha$ is a non-zero real parameter, and $\beta$ is a real parameter.

$$\begin{bmatrix} 1 & 0 & 0 \\ \beta & 0 & 1 \\ 0 & 1/\alpha & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix}}_{E_1} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{E_2} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}}_{E_3}$$

This problem computes $A^{-1}$ as $E_3^{-1} \cdot E_2^{-1} \cdot E_1^{-1}$. Please perform the following steps.

(a, 4 pts) Please write $E_1, E_2, E_3$ as elementary matrices $E_I(\dots)$ or $E_{II}(\dots)$ or $E_{III}(\dots)$.

$$E_1 = E_{II}(3,3,1/\alpha)$$
$$E_2 = E_I(3,2,3) \text{ or } E_I(3,3,2)$$
$$E_3 = E_{III}(3,1,3,\beta)$$

(b, 4 pts) Please write $E_3^{-1}, E_2^{-1}, E_1^{-1}$ as elementary matrices $E_I(\dots)$ or $E_{II}(\dots)$ or $E_{III}(\dots)$.

$$E_3^{-1} = E_{III}(3,1,3,-\beta)$$
$$E_2^{-1} = E_I(3,2,3) \text{ or } E_I(3,3,2)$$
$$E_1^{-1} = E_{II}(3,3,\alpha)$$

(c, 4 pts) Please write out $E_1^{-1}$ as a matrix. Then compute the product $E_2^{-1} \cdot E_1^{-1}$ by performing the elementary row operation for $E_2^{-1}$ determined in part b. Then compute the product $E_3^{-1} \cdot (E_2^{-1}E_1^{-1})$, again by performing the elementary row operation for $E_3^{-1}$ of part b.

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

$$E_2^{-1}E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha \\ -\beta & 1 & 0 \end{bmatrix}$$