Problem 1

2

3

Total

Good luck!

If you are taking the exam later, please sign the following statement:

I, ____________________, affirm that I have no knowledge of the contents of this exam.

__________________________
Signature
Problem 1 (19 points)

(a, 3 pts) Please name a well-known mathematician who was born in the USA.

John Nash, Julia Robinson, Stephen Smale, Benjamin Banneker, Josiah Gibbs

(b, 4 pts) In how many multiplications of pairs of integers can one compute \[
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{256}
\] ? Please explain.

\[256 = 2^8: \text{there are 8 matrix multiplications, namely squarings, of } 2 \times 2 \text{ matrices, each of which needs } 2 \cdot 2 \cdot 2 = 8 \text{ integer multiplications. A total of } 8 \cdot 8 = 64 \text{ integer multiplications. Without the doubling trick, there are 255 matrix multiplications, a total of } 255 \cdot 8 = 2040 \text{ integer multiplications.}\]

(c, 4 pts) By laws that the transposition and inversion operations for matrices satisfy please prove that for an invertible matrix \(A\) we have \((A^T)^{-1} = (A^{-1})^T\).

\[
(A^T) \cdot (A^{-1})^T \quad \quad \quad \quad (A^{-1} \cdot A)^T = I^T = I, \text{ and because } (A-B)^T = B^T \cdot A^T
\]

\[
(A^{-1})^T \cdot (A^T) = (A \cdot A^{-1})^T = I^T = I.
\]

(d, 4 pts) Suppose that in a Maple session the variable \(A\) has been assigned a \(4 \times 4\) invertible matrix. If you then execute the Maple commands \(\text{with(linalg): rref(A);}\) what answer to you expect?

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(e, 4 pts) A binary operation on an arbitrary set \(S\) may or may not satisfy the mathematical laws of (i) associativity, (ii) \(S\) has a unit element, (iii) each element in \(S\) has an inverse element, and (iv) commutativity. For the concrete set \(S = \mathbb{R}^{n \times n}\) and matrix multiplication as the binary operation, which of these four laws are satisfied?

(i) and (ii).
Problem 2 (15 points): Consider the following augmented matrix of a system of linear equations, where the first 3 columns correspond to the variables $x, y, z$ and where $a, b,$ and $c$ are real parameters.

$$\begin{bmatrix} 1 & 1 & -1 & : & 0 \\ -a & 1 & -1 & : & (1+a)c \\ 0 & b & -b & : & c \end{bmatrix}$$

(a, 4pts) For which values of $a, b,$ and $c$ is the matrix (1) in row echelon form? Please give all the conditions.

$a = 0 \text{ and } b = 0$.

(b, 6pts) By performing Gaussian elimination on the cases $a = -1$ and $a \neq -1$ separately, determine for which values of $a, b,$ and $c$ the linear system corresponding to the augmented matrix (1) is consistent. For each condition please state the row-echelon form.

*Case $a = -1$:*

$$\begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 1 & 1 & -1 & : & 0 \\ 0 & b & -b & : & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

consistent if $b \neq 0$ or if $b = c = 0$.

*Case $a \neq -1$:*

$$\begin{bmatrix} 1 & 1 & -1 & : & 0 \\ -a & 1 & -1 & : & (1+a)c \\ 0 & b & -b & : & c \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1+a & -1-a & : & (1+a)c \\ 0 & b & -b & : & c \\ 1 & 1 & -1 & : & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1+a & -1-a & : & (1+a)c \\ 0 & b & -b & : & c \\ 0 & 1 & -1 & : & c \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & : & c-bc \end{bmatrix}$$

consistent if $c(1-b) = 0$.

(c, 5pts) For each of the conditions discovered in part b, please perform the back-substitution to solve the system.

*Case $a = -1, b \neq 0$:*

$z = z, y = (1/b) \cdot (c + bz) = c/b + z, x = -y + z = -c/b$.

*Case $a = -1, b = c = 0$:*

$z = z, y = y, x = -y + z$.

*Case $a \neq -1, c(1-b) = 0$:*

$z = z, y = c + z, x = -y + z = -c$. 


Problem 3 (12 points): Consider the following matrix $A$ together with its factorization into elementary matrices. Here $\alpha$ is a non-zero real parameter, and $\beta$ is a real parameter.

\[
\begin{bmatrix}
1 & 0 & 0 \\
\beta & 0 & 1 \\
0 & 1/\alpha & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/\alpha
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\beta & 0 & 1
\end{bmatrix}
\]

This problem computes $A^{-1}$ as $E_3^{-1} \cdot E_2^{-1} \cdot E_1^{-1}$. Please perform the following steps.

(a, 4 pts) Please write $E_1, E_2, E_3$ as elementary matrices $E_I(\ldots)$ or $E_{II}(\ldots)$ or $E_{III}(\ldots)$.

\[
E_1 = E_{II}(3, 3, 1/\alpha) \\
E_2 = E_I(3, 2, 3) \text{ or } E_I(3, 3, 2) \\
E_3 = E_{III}(3, 1, 3, \beta)
\]

(b, 4 pts) Please write $E_3^{-1}, E_2^{-1}, E_1^{-1}$ as elementary matrices $E_I(\ldots)$ or $E_{II}(\ldots)$ or $E_{III}(\ldots)$.

\[
E_3^{-1} = E_{III}(3, 1, 3, -\beta) \\
E_2^{-1} = E_I(3, 2, 3) \text{ or } E_I(3, 3, 2) \\
E_1^{-1} = E_{II}(3, 3, \alpha)
\]

(c, 4 pts) Please write out $E_1^{-1}$ as a matrix. Then compute the product $E_2^{-1} \cdot E_1^{-1}$ by performing the elementary row operation for $E_2^{-1}$ determined in part b. Then compute the product $E_3^{-1} \cdot (E_2^{-1}E_1^{-1})$, again by performing the elementary row operation for $E_3^{-1}$ of part b.

\[
E_1^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \alpha
\end{bmatrix}
\]

\[
E_2^{-1}E_1^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & \alpha \\
0 & 1 & 0
\end{bmatrix}
\]

\[
E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & \alpha \\
-\beta & 1 & 0
\end{bmatrix}
\]