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Department of Mathematics

NC STATE UNIVERSITY

MA 305 Elem. Linear Algebra, Final Examination, December 9, 1997

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Your Name: SOLUTION
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 questions, each question counting for the given number of points, adding to a total of **42 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **three** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later. You will have **two hours** to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
5	____
Total	____

If you are taking the exam later, please sign the following statement:

I, ____________________, affirm that I have no knowledge of the contents of this exam.

____
Signature

Problem 1 (10 points, 3 points for (b) and (d), 2 points for (a) and (c)): Please answer the following questions about rank, inner products, and linear maps. Please, also **justify your answers** briefly.

(a) Is the following situation possible? $A \in \mathbb{R}^{m \times n}$, $m < n$, $b \in \mathbb{R}^m$, and there is a single, unique $x \in \mathbb{R}^n$ such that $Ax = b$.

NO

Since $\operatorname{rank}(A) \leq m < n$, the dimension of nullspace/kernel(A) is equal to $n - \operatorname{rank}(A) > 0$, hence the nullspace/kernel of A must contain a non-zero vector y. Thus, if $Ax = b$, then

$$A(x + y) = Ax + Ay = Ax + 0 = b.$$

So both x and $x + y$ are solutions, and because $y \neq 0$ we have $x \neq x + y$.

Alternative justification: Since $m < n$ in the row echelon form of A there must always be free variables that can be set to different values giving different solutions.

(b) Define the following map on $\mathbb{R}^3 \times \mathbb{R}^3$:

$$\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \rangle = x_1 x_2 - 2 y_1 y_2 + 3 z_1 z_2.$$

Please explain if or if not $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^3$.

NO

$$\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle = -2 < 0,$$

but for $x \neq 0$ we must have by (IP_1) $\langle x, x \rangle > 0$.

(c) The homepage of our course shows a 4-dimensional depiction of the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$:

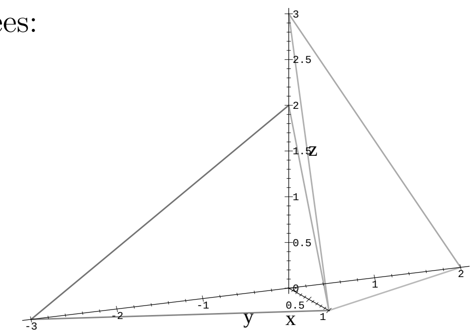
$$F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{bmatrix} s \\ t \end{bmatrix} \longmapsto \begin{bmatrix} s-t \\ s+t-\frac{1}{2} \end{bmatrix}$$

Is F a linear transform?

NO

We have $F(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but any linear map must map the zero vector to the zero vector.

(d) Consider a 3-dimensional rotation around the x-axis (parallel to the y-z-plane) by 90 degrees:



In the above plot, the 3-dimensional triangle with vertices at $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ is shown both in orginal and in rotated position. Please give the 3×3 matrix which when multiplied by any vector in $\mathbb{R}^3$ effects this rotation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 2 (8 points): The running time of the heap sort algorithm in terms of counting the number of comparisons can be expressed as $c_0 + c_1 n + c_2 n \log(n)$, where n is the length of the array and c_i are constants to be experimentally estimated. We have run our algorithm on five different lengths and obtained the following counts.

```
Heap sorted random array of length 20 with 118 comparisons
Heap sorted random array of length 40 with 323 comparisons
Heap sorted random array of length 100 with 1042 comparisons
Heap sorted random array of length 250 with 3320 comparisons
Heap sorted random array of length 500 with 7578 comparisons
```

Please give the least squares model that can be used to determine the constants. More specifically, give a matrix A and a vector b such that $\hat{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ best solves $A\hat{c} \approx b$. You need **not** explicitly compute c_0, c_1, and c_2.

$$A = \begin{bmatrix} 1 & 20 & 20\log(20) \\ 1 & 40 & 40\log(40) \\ 1 & 100 & 100\log(100) \\ 1 & 250 & 250\log(250) \\ 1 & 500 & 500\log(500) \end{bmatrix}, \qquad b = \begin{bmatrix} 118 \\ 323 \\ 1042 \\ 3320 \\ 7578 \end{bmatrix}$$

Problem 3 (10 points, 5 each part):

(a) Consider

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \text{ and } v_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Here v_1 is already orthogonal to v_2, but v_3 is not. Please orthogonalize v_3 by computing $u_3 = v_3 - w_3$, where w_3 is the orthogonal projection of v_3 onto $\mathrm{Span}(v_1, v_2)$.

$$w_3 = \frac{v_1^T v_3}{v_1^T v_1} v_1 + \frac{v_2^T v_3}{v_2^T v_2} v_2 = \frac{1}{4} v_1 + \frac{0}{6} v_2 = \begin{bmatrix}1/4\\1/4\\1/4\\1/4\end{bmatrix}$$

$$u_3 = \begin{bmatrix}-1/4\\-1/4\\-1/4\\3/4\end{bmatrix}$$

(b) Please prove that $\mathrm{Span}(\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}) = (\mathrm{Span}(v_1, v_2, v_3))^\perp$.

$$(\mathrm{Span}(v_1, v_2, v_3))^\perp = \text{nullspace/kernel}(\underbrace{\begin{bmatrix}1 & 1 & 1 & 1\\1 & -2 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}}_{A^T}).$$

Since the rows in A^T are linear independent (because they are orthogonal to one another), the dimension of the nullspace/kernel of A^T must be 4 (number of columns) − 3 (rank of A^T, the number of linearly independent rows/cols) = 1. Now $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$ is an element of this dimenions one nullspace/kernel of A^T, because

$$\begin{bmatrix}1 & 1 & 1 & 1\\1 & -2 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

so it forms a basis for the entire nullspace/kernel.

Problem 4 (8 points): Please give the eigenvalues and bases for the corresponding eigenspaces for the matrix

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\det(\begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}) = \lambda^2 \cdot (\lambda - 3)^2.$$

$$\lambda_1 = 0\text{: nullspace/kernel}(\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}) = \text{Span}(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}).$$

$$\lambda_2 = 3\text{: nullspace/kernel}(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}) = \text{Span}(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}).$$

Problem 5 (6 points): When solving a system of first order differential equations $y_1' = 3y_1 - 2y_2$, $y_2' = -2y_1 + 3y_2$ by Maple we get the following general solution:

```
> dsolve({diff(y[1](t), t) =  3*y[1](t) - 2*y[2](t),
>          diff(y[2](t), t) = -2*y[1](t) + 3*y[2](t)},
>         {y[1](t), y[2](t)});
```

$$\{y_1(t) = \frac{1}{2}\_C1\, e^t + \frac{1}{2}\_C1\, e^{(5\,t)} - \frac{1}{2}\_C2\, e^{(5\,t)} + \frac{1}{2}\_C2\, e^t,$$
$$y_2(t) = -\frac{1}{2}\_C1\, e^{(5\,t)} + \frac{1}{2}\_C1\, e^t + \frac{1}{2}\_C2\, e^t + \frac{1}{2}\_C2\, e^{(5\,t)}\}$$

Here the parameters $\_Ci$ are to be determined by the initial conditions. Please give the eigenvalue problem corresponding to the above differential equations and show how the eigenvalues and entries in the eigenvectors correspond to the numeric constants in the above solution, i.e., explain the constants like $\frac{1}{2}$ and 5 in terms of eigenvalues and entries of eigenvectors of a certain matrix.

$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$, the coefficients in the right sides of the differential equations. However, we do not need to solve the differential equations, because we can read off the answer from the Maple answer.

$\lambda_1 = 1$, derived from e^t. $x_1 = \begin{bmatrix} 1/2 \text{ (coeff. of } \_C1 e^t \text{ in } y_1) \\ 1/2 \text{ (coeff. of } \_C1 e^t \text{ in } y_2) \end{bmatrix}$

$\lambda_2 = 5$, derived from e^{5t}. $x_2 = \begin{bmatrix} 1/2 \text{ (coeff. of } \_C1 e^{5t} \text{ in } y_1) \\ 1/2 \text{ (coeff. of } \_C1 e^{5t} \text{ in } y_2) \end{bmatrix}$

Note: in class we write $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$, while Maple gives as answer $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = (\_C1 + \_C2) e^t \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + (\_C1 - \_C2) e^{5t} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$. If you have determined the coefficients c_1, c_2 from the initial conditions, you can express the Maple coefficients as $\_C1 = (c_1 + c_2)/2$, $\_C2 = (c_1 - c_2)/2$, and vice-versa.