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Department of Mathematics

NC STATE UNIVERSITY

MA 305 Elem. Linear Algebra, Final Examination, December 9, 1997

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Your Name: ________________________

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 questions, each question counting for the given number of points, adding to a total of **42 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **three** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later. You will have **two hours** to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
5	____
Total	____

If you are taking the exam later, please sign the following statement:

I, ____________________, *affirm that I have no knowledge of the contents of this exam.*

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Signature

Problem 1 (10 points, 3 points for (b) and (d), 2 points for (a) and (c)): Please answer the following questions about rank, inner products, and linear maps. Please, also **justify your answers** briefly.

(a) Is the following situation possible? $A \in \mathbb{R}^{m \times n}$, $m < n$, $b \in \mathbb{R}^m$, and there is a single, unique $x \in \mathbb{R}^n$ such that $Ax = b$.

(b) Define the following map on $\mathbb{R}^3 \times \mathbb{R}^3$:

$$\left\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right\rangle = x_1 x_2 - 2 y_1 y_2 + 3 z_1 z_2.$$

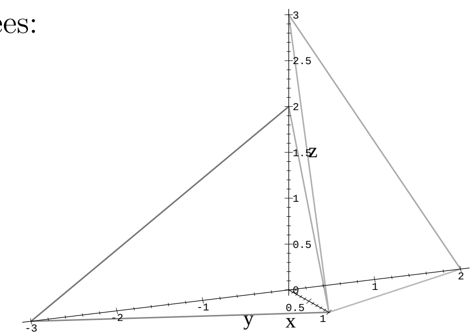
Please explain if or if not $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^3$.

(c) The homepage of our course shows a 4-dimensional depiction of the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{bmatrix} s \\ t \end{bmatrix} \longmapsto \begin{bmatrix} s-t \\ s+t-\frac{1}{2} \end{bmatrix}$$

Is F a linear transform?

(d) Consider a 3-dimensional rotation around the x-axis (parallel to the y-z-plane) by 90 degrees:



In the above plot, the 3-dimensional triangle with vertices at $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ is shown both in orginal and in rotated position. Please give the 3×3 matrix which when multiplied by any vector in $\mathbb{R}^3$ effects this rotation.

Problem 2 (8 points): The running time of the heap sort algorithm in terms of counting the number of comparisons can be expressed as $c_0 + c_1 n + c_2 n \log(n)$, where n is the length of the array and c_i are constants to be experimentally estimated. We have run our algorithm on five different lengths and obtained the following counts.

```
Heap sorted random array of length 20 with 118 comparisons
Heap sorted random array of length 40 with 323 comparisons
Heap sorted random array of length 100 with 1042 comparisons
Heap sorted random array of length 250 with 3320 comparisons
Heap sorted random array of length 500 with 7578 comparisons
```

Please give the least squares model that can be used to determine the constants. More specifically, give a matrix A and a vector b such that $\hat{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ best solves $A\hat{c} \approx b$. You need **not** explicitly compute c_0, c_1, and c_2.

Problem 3 (10 points, 5 each part):

(a) Consider

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Here v_1 is already orthogonal to v_2, but v_3 is not. Please orthogonalize v_3 by computing $u_3 = v_3 - w_3$, where w_3 is the orthogonal projection of v_3 onto $\text{Span}(v_1, v_2)$.

(b) Please prove that $\text{Span}(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}) = (\text{Span}(v_1, v_2, v_3))^{\perp}$.

Problem 4 (8 points): Please give the eigenvalues and bases for the corresponding eigenspaces for the matrix

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 5 (6 points): When solving a system of first order differential equations $y_1' = 3y_1 - 2y_2$, $y_2' = -2y_1 + 3y_2$ by Maple we get the following general solution:

```
> dsolve({diff(y[1](t), t) =  3*y[1](t) - 2*y[2](t),
>         diff(y[2](t), t) = -2*y[1](t) + 3*y[2](t)},
>        {y[1](t), y[2](t)});
```

$$\{y_1(t) = \frac{1}{2}\_C1\, e^t + \frac{1}{2}\_C1\, e^{(5\,t)} - \frac{1}{2}\_C2\, e^{(5\,t)} + \frac{1}{2}\_C2\, e^t,$$
$$y_2(t) = -\frac{1}{2}\_C1\, e^{(5\,t)} + \frac{1}{2}\_C1\, e^t + \frac{1}{2}\_C2\, e^t + \frac{1}{2}\_C2\, e^{(5\,t)}\}$$

Here the parameters $\_Ci$ are to be determined by the initial conditions. Please give the eigenvalue problem corresponding to the above differential equations and show how the eigenvalues and entries in the eigenvectors correspond to the numeric constants in the above solution, i.e., explain the constants like $\frac{1}{2}$ and 5 in terms of eigenvalues and entries of eigenvectors of a certain matrix.