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Department of Mathematics

NC STATE UNIVERSITY

MA 305 Elem. Linear Algebra, Second Mid-term Examination, October 28, 1997

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Your Name: SOLUTION

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 questions, each question counting for the given number of points, adding to a total of **20 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **two** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later. You will have **75 minutes** to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
5	____
Total	____

If you are taking the exam later, please sign the following statement:

I, ____________________, affirm that I have no knowledge of the contents of this exam.

____
Signature

Problem 1 (5 points, 2 points for (a), 1.5 point for (b) and (c)): Please answer the following questions about vector spaces and matrices. Please, also **justify your answers** briefly.

(a) Consider $\mathbb{R}^2$ with $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ for addition and $\alpha \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \alpha y_1 \end{bmatrix}$ for scalar multiplication. Is $\mathbb{R}^2$ with these operations a vector space?

NO. Multiplication of scalars by vectors does not distribute over scalar addition.
$(\alpha + \beta) \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ (\alpha + \beta) y_1 \end{bmatrix}$,
$\alpha \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \beta \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ (\alpha + \beta) y_1 \end{bmatrix}$,
and $\begin{bmatrix} x_1 \\ (\alpha + \beta) y_1 \end{bmatrix} \neq \begin{bmatrix} 2x_1 \\ (\alpha + \beta) y_1 \end{bmatrix}$.

(b) True or false? Let $A \in \mathbb{R}^{m \times n}, u, v \in \mathbb{R}^n, b \in \mathbb{R}^m$ with the property that $Au = A(u+v) = b$, i.e., u and $u + v$ are both solutions to the linear system $Ax = b$. Then v is in the right null space of A. Please explain your answer.

TRUE
$Au = A(u + v) = b \Rightarrow A(u + v) - Au = 0 \underset{\text{distr.}}{\Rightarrow} A(u + v - u) = 0 \Rightarrow Av = 0 \Rightarrow v \in$ kernel(A).

(c) Let $A \in \mathbb{R}^{3 \times 3}, b \in \mathbb{R}^3$ and consider a linear system $Ax = b$. If the dimension of the right null space of A is 1, what is the geometry of the solution provided the system is solvable?

The solution is a line in 3-dimensional space.

Problem 2 (3 points): Consider the following matrix:

$$A = \begin{bmatrix} a & b & 0 & 1 \\ b & a & 0 & 0 \\ 0 & 0 & a & c \\ 1 & 0 & c & a \end{bmatrix}.$$

As an expression in a, b, and c, please compute by co-factor expansion (also called "minor expansion") the determinant of A. Please show all your work.

Proceeding with co-factor (minor) expansion along the second row, we obtain:

$$\det(\begin{bmatrix} a & b & 0 & 1 \\ b & a & 0 & 0 \\ 0 & 0 & a & c \\ 1 & 0 & c & a \end{bmatrix}) = (-b) \cdot \det(\begin{bmatrix} b & 0 & 1 \\ 0 & a & c \\ 0 & c & a \end{bmatrix}) + a \cdot \det(\begin{bmatrix} a & 0 & 1 \\ 0 & a & c \\ 1 & c & a \end{bmatrix}) \qquad (1).$$

The first minor is computed by co-factor expansion along the first column:

$$\det(\begin{bmatrix} b & 0 & 1 \\ 0 & a & c \\ 0 & c & a \end{bmatrix}) = b \cdot \det(\begin{bmatrix} a & c \\ c & a \end{bmatrix}) = b(a^2 - c^2).$$

The second minor is computed by co-factor expansion along the first row:

$$\det(\begin{bmatrix} a & 0 & 1 \\ 0 & a & c \\ 1 & c & a \end{bmatrix}) = a \cdot \det(\begin{bmatrix} a & c \\ c & a \end{bmatrix}) + 1 \cdot \det(\begin{bmatrix} 0 & a \\ 1 & c \end{bmatrix}) = a(a^2 - c^2) - a.$$

Plugging these expressions into (1) above we get
$\det(A) = -b^2(a^2 - c^2) + a^2(a^2 - c^2) - a^2 = a^4 - a^2b^2 - a^2c^2 + b^2c^2 - a^2.$

Problem 3 (4 points): Show that the four vectors in $\mathbb{R}^4$,

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

are linearly dependent. For that, you have to compute a non-trivial linear combination of the four vectors that equals the zero vector.

Note that the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

has the row echelon form

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Name the four vectors v_1, v_2, v_3, and v_4. We wish to solve $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$. The method proceeds by writing down a matrix with the vectors as columns, and computing a row echelon form. We must then have

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0.$$

$c_4 = c_4$ is the free variable.
$c_3 = -c_4$,
$c_2 = 1/2(-c_3 + c_4) = c_4$,
$c_1 = c_2 - 2c_4 = c_4 - 2c_4 = -c_4$.
With $c_4 = 1$ we have

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 0.$$

Problem 4 (5 points): Consider the vector space P_2 of polynomials $f(x) = a + bx + cx^2$ with real number coefficients a, b, and c and with polynomial addition as vector space addition and multiplication by a real number as scalar multiplication. The set

$$S = \{a + bx + cx^2 \mid a + 2b + 4c = 0\}$$

(those polynomials $f(x) \in P_2$ with $f(2) = 0$) forms a subspace of P_2. Write down a basis for S and show

a) that your basis vectors are linearly independent, and

b) that your basis vectors span S.

$B = \{x - 2, x^2 - 2x\}$.

a) $c_1(x-2) + c_2(x^2 - 2x) = 0 \Rightarrow c_2 x^2 + (c_1 - 2c_2)x - 2c_1 = 0$;
$\Rightarrow c_2 = 0$ (coeff of x^2), $-2c_1 = 0$ (coeff of x^0) $\Rightarrow c_1 = 0$.

b) Let $f(x) = a + bx + cx^2 \in S$. Then $f(2) = a + 2b + 4c = 0$, hence $a = -2b - 4c$.
Express f as a linear combination of the basis vectors:

$$d_1(x-2) + d_2(x^2 - 2x) = -2b - 4c + bx + cx^2.$$

Coefficient of x^0: $-2d_1 = -2b - 4c \Rightarrow d_1 = b + 2c$.
Coefficient of x^1: $d_1 - 2d_2 = b$; checks with both the lines above and below.
Coefficient of x^2: $d_2 = c$.
The linear combination for obtaining an arbitrary $f \in S$ from basis vectors is:

$$(b + 2c)(x-2) + c(x^2 - 2x) = -2b - 4c + bx + cx^2.$$

Problem 5 (3 points, 1.5 points each part): Consider the generic 4×4 matrix A, and the generic 4-dimensional column vectors x and b:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

(a) The entry in row 4 and column 1 of the inverse of A can be expressed as $(-1)^i \det(M)/\det(A)$ (by Cramer's rule), where M is a matrix and i is either 0 or 1. Please write down the matrix M (in terms of the entries of A) and determine the "sign exponent" i.

$i = 4 + 1 \bmod 2 = 1;$

$$M = A \downarrow_{1,4} = \begin{bmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{bmatrix}.$$

(b) Consider the linear system $Ax = b$ with the generic A, x and b from above. The first component of x can be expressed as $x_1 = \det(N)/\det(A)$, where N is a matrix. Please write down the matrix N in terms of the entries of A and b.

$$N = \begin{bmatrix} b_1 & a_{1,2} & a_{1,3} & a_{1,4} \\ b_2 & a_{2,2} & a_{2,3} & a_{2,4} \\ b_3 & a_{3,2} & a_{3,3} & a_{3,4} \\ b_4 & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}.$$