North Carolina State University is a land-grant university and a constituent institution of The University of North Carolina

Department of Mathematics

NC STATE UNIVERSITY

MA 305 Elem. Linear Algebra, Second Mid-term Examination, October 28, 1997

`kaltofen@eos.ncsu.edu` (email)
`http://courses.ncsu.edu/MA305/Fall97/index.html` (URL)

919.515.8785 (phone)
919.515.3798 (fax)

Your Name: _____

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 questions, each question counting for the given number of points, adding to a total of **20 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **two** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later. You will have **75 minutes** to do this test.

Good luck!

Problem 1	_____
2	_____
3	_____
4	_____
5	_____
Total	_____

If you are taking the exam later, please sign the following statement:

I, _____*, affirm that I have no knowledge of the contents of this exam.*

_____

Signature

Problem 1 (5 points, 2 points for (a), 1.5 point for (b) and (c)): Please answer the following questions about vector spaces and matrices. Please, also **justify your answers** briefly.

(a) Consider $\mathbb{R}^2$ with $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ for addition and $\alpha \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \alpha y_1 \end{bmatrix}$ for scalar multiplication. Is $\mathbb{R}^2$ with these operations a vector space?

(b) True or false? Let $A \in \mathbb{R}^{m \times n}, u, v \in \mathbb{R}^n, b \in \mathbb{R}^m$ with the property that $Au = A(u+v) = b$, i.e., u and $u + v$ are both solutions to the linear system $Ax = b$. Then v is in the right null space of A. Please explain your answer.

(c) Let $A \in \mathbb{R}^{3 \times 3}, b \in \mathbb{R}^3$ and consider a linear system $Ax = b$. If the dimension of the right null space of A is 1, what is the geometry of the solution provided the system is solvable?

Problem 2 (3 points): Consider the following matrix:

$$A = \begin{bmatrix} a & b & 0 & 1 \\ b & a & 0 & 0 \\ 0 & 0 & a & c \\ 1 & 0 & c & a \end{bmatrix}.$$

As an expression in a, b, and c, please compute by co-factor expansion (also called "minor expansion") the determinant of A. Please show all your work.

Problem 3 (4 points): Show that the four vectors in $\mathbb{R}^4$,

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

are linearly dependent. For that, you have to compute a non-trivial linear combination of the four vectors that equals the zero vector.

Note that the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

has the row echelon form

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 4 (5 points): Consider the vector space P_2 of polynomials $f(x) = a+bx+cx^2$ with real number coefficients a, b, and c and with polynomial addition as vector space addition and multiplication by a real number as scalar multiplication. The set

$$S = \{a + bx + cx^2 \mid a + 2b + 4c = 0\}$$

(those polynomials $f(x) \in P_2$ with $f(2) = 0$) forms a subspace of P_2. Write down a basis for S and show

a) that your basis vectors are linearly independent, and

b) that your basis vectors span S.

Problem 5 (3 points, 1.5 points each part): Consider the generic 4×4 matrix A, and the generic 4-dimensional column vectors x and b:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

(a) The entry in row 4 and column 1 of the inverse of A can be expressed as $(-1)^i \det(M)/\det(A)$ (by Cramer's rule), where M is a matrix and i is either 0 or 1. Please write down the matrix M (in terms of the entries of A) and determine the "sign exponent" i.

(b) Consider the linear system $Ax = b$ with the generic A, x and b from above. The first component of x can be expressed as $x_1 = \det(N)/\det(A)$, where N is a matrix. Please write down the matrix N in terms of the entries of A and b.