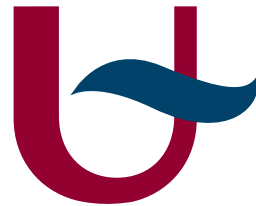


Symbolic-Numeric Methods for Polynomial Algebra

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Outline

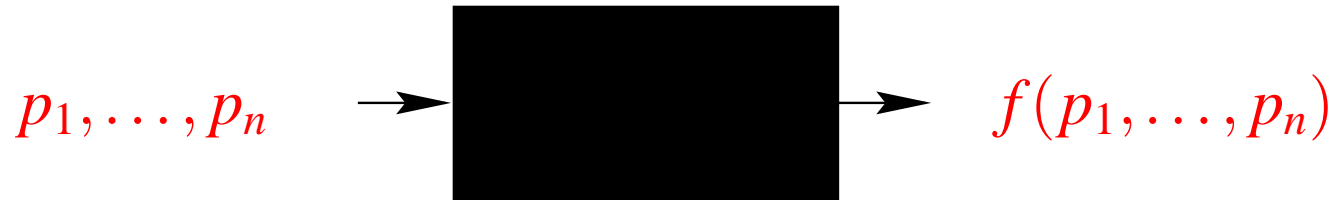
- Sparse Interpolation
- Generators of Polynomial Ideals
- Multivariate Padé Approximants

Outline

- *Sparse Interpolation*
- Generators of Polynomial Ideals
- Multivariate Padé Approximants

Black Box Interpolation

black box function $f(x_1, \dots, x_n)$



Interpolation

explicit representation, for example,

$$\text{polynomial } f(x_1, \dots, x_n) = \sum_{j=1}^t c_j x_1^{d_{j1}} \cdots x_n^{d_{jn}}$$

What if $f(x_1, \dots, x_n)$ is sparse?

Sparse Polynomial Interpolation

When $f = \sum_{j=1}^t c_j x_1^{d_{j1}} \cdots x_n^{d_{jn}}$ is sparse: $f(x, y) = x^5 y^{14} - \frac{1}{5} x y^{11} + \frac{11}{2} y^5 + 1$

There are 4 non-zero terms in $f(x, y)$.

But $\deg_x f(x, y) = 5$, $\deg_y f(x, y) = 14$,
possible terms could be as many as $(5 + 1) \times (14 + 1) = 90$.

Sparse Polynomial Interpolation

When $f = \sum_{j=1}^t c_j x_1^{d_{j1}} \cdots x_n^{d_{jn}}$ is sparse: $f(x, y) = x^5 y^{14} - \frac{1}{5} x y^{11} + \frac{11}{2} y^5 + 1$

In exact arithmetic:

[Zippel 1979] require an upper bound on degree d

[Ben-Or, Tiwari 1988] require an upper bound on t

$2t$ evaluations of f , deterministic

[Kaltofen, Lee 2003] probabilistic early termination

In floating point arithmetic:

[Giesbrecht, Labahn, Lee 2006]

require upper bounds on both d and t

require complex arithmetic (interpolate on the unit circle)

[Cuyt, Lee 2008] *iterative method*

require neither a bound nor complex arithmetic

deterministic!

Numerical qd Algorithm

[Rutishauser 1954]

Determine the poles of $F(x)$ from its Taylor expansion.

$$F(x) = \frac{h(x)}{f(x)}, \quad h(x) \text{ analytic}$$

and $f(x) = (1 - b_1x) \cdots (1 - b_tx) \in \mathbb{C}[x]$ with $|b_1| > \cdots > |b_t|$.

$$\begin{aligned} F(x) &= \frac{h(x)}{f(x)} = \frac{r_1}{1 - b_1x} + \frac{r_2}{1 - b_2x} + \cdots + \frac{r_t}{1 - b_tx} + g(x) \\ &= r_1(1 + b_1x + b_1^2x^2 + \cdots) + \cdots + r_t(1 + b_tx + b_t^2x^2 + \cdots) + g(x) \\ &= (r_1 + r_2 + \cdots + r_t + \rho_0) + (r_1b_1 + r_2b_2 + \cdots + r_tb_t + \rho_1)x + \cdots \\ &= \sum_{i=0}^{\infty} a_i x^i \end{aligned}$$

The i -th Taylor coefficient $a_i = r_1b_1^i + \cdots + r_tb_t^i + \rho_i$

Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

$$f(x, y) = 10x^3y^2 + x - 5y + 22$$

- Pick p_1, p_2 relatively prime: say $p_1 = 2, p_2 = 3$.
Evaluate $a_0 = f(2^0, 3^0), a_1 = f(2, 3), \dots, a_i = f(2^i, 3^i), \dots$

$$\text{In floating point: } a_i = 10 \cdot (2^3 \cdot 3^2)^i + 2^i + 5 \cdot 3^i + 1^i + \rho_i$$

- Approximate non-zero terms at (p_1, p_2) by the qd algorithm.

The numerical qd algorithm

[Rutishauser 1954]

$$\text{Initialize } e_0^{(n)} = 0, n = 1, 2, \dots; \quad q_1^{(n)} = \frac{a_{n+1}}{a_n}, n = 0, 1, 2, \dots$$

$$\text{Continue } e_m^{(n)} = q_m^{(n+1)} - q_m^{(n)} + e_{m-1}^{(n+1)}, m = 1, 2, \dots, n = 0, 1, 2, \dots;$$

$$q_{m+1}^{(n)} = \frac{e_m^{(n+1)}}{e_m^{(n)}} q_m^{(n+1)}, n = 0, 1, 2, \dots$$

- Recover coefficients by solving a Vandermonde system.

Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

the qd algorithm

Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

the qd algorithm

0

0

0

0

Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

the qd algorithm

$q_1^{(0)}$
0
 $q_1^{(1)}$
0
 $q_1^{(2)}$
0
 $q_1^{(3)}$
0

Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

the qd algorithm

0 $q_1^{(0)}$
0 $q_1^{(1)}$
0 $q_1^{(2)}$
0 $q_1^{(3)}$
↓
72

Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

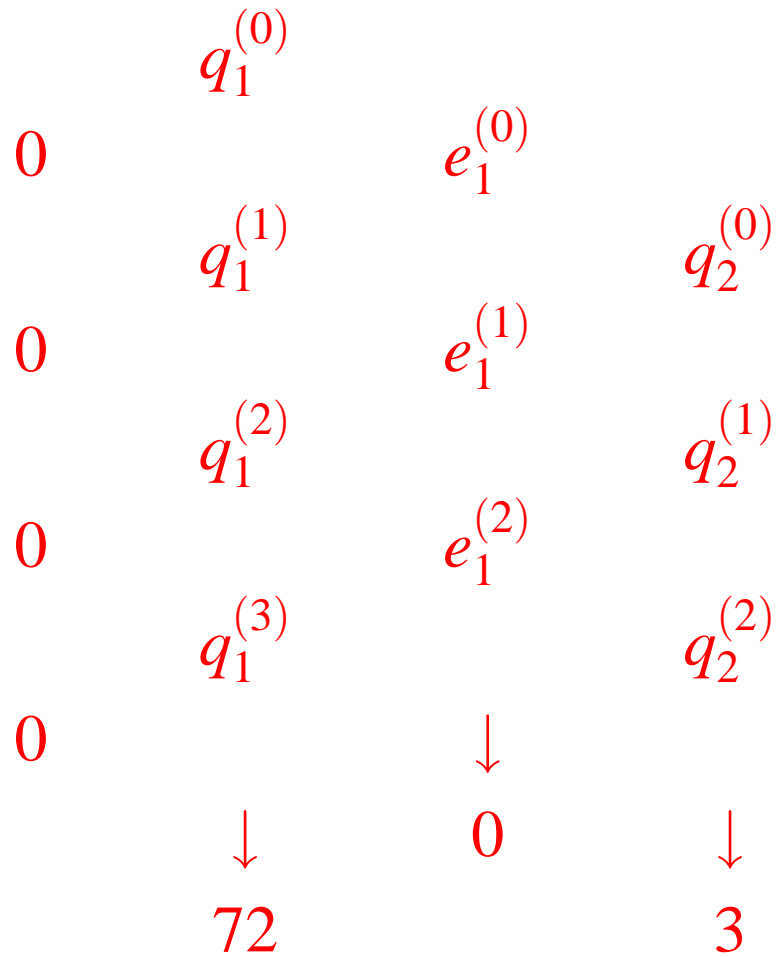
the qd algorithm

$$\begin{array}{ccc} & q_1^{(0)} & \\ 0 & & e_1^{(0)} \\ & q_1^{(1)} & \\ 0 & & e_1^{(1)} \\ & q_1^{(2)} & \\ 0 & & e_1^{(2)} \\ & q_1^{(3)} & \\ 0 & & \downarrow \\ & \downarrow & 0 \\ & 72 & \end{array}$$

Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

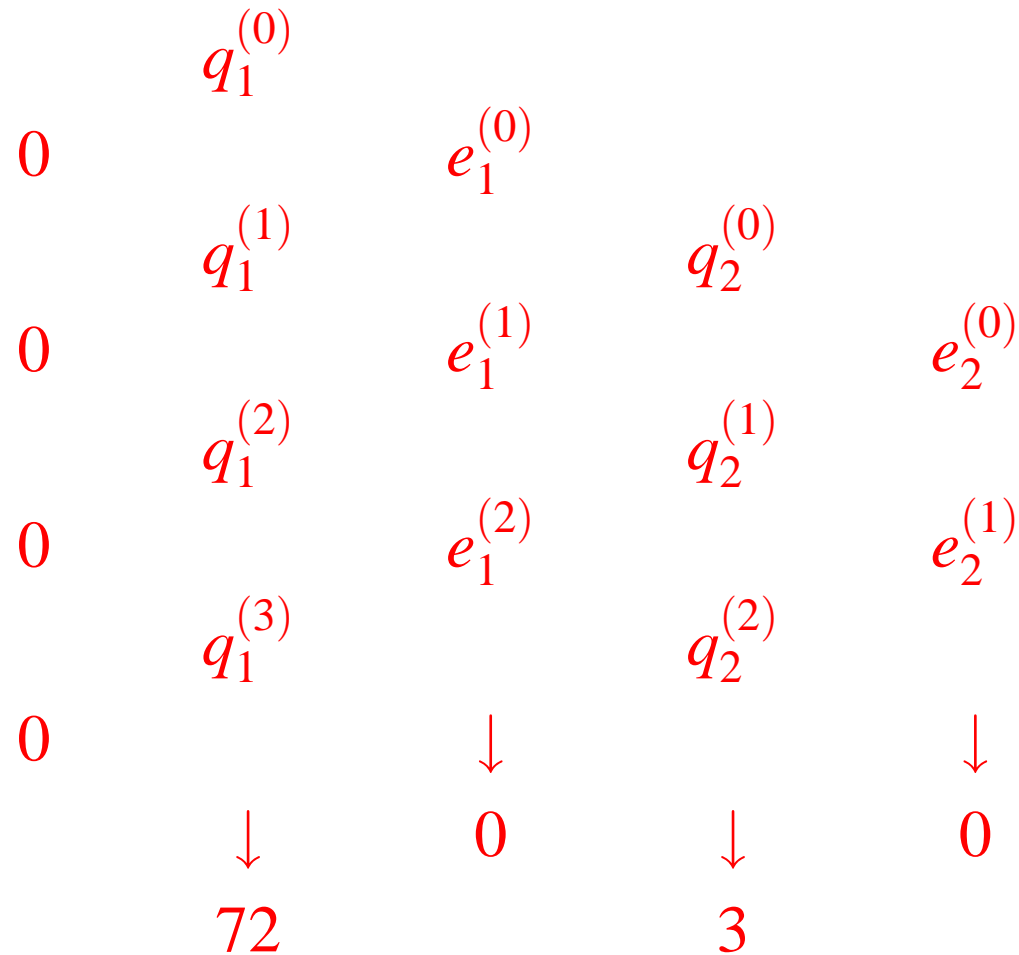
the qd algorithm



Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

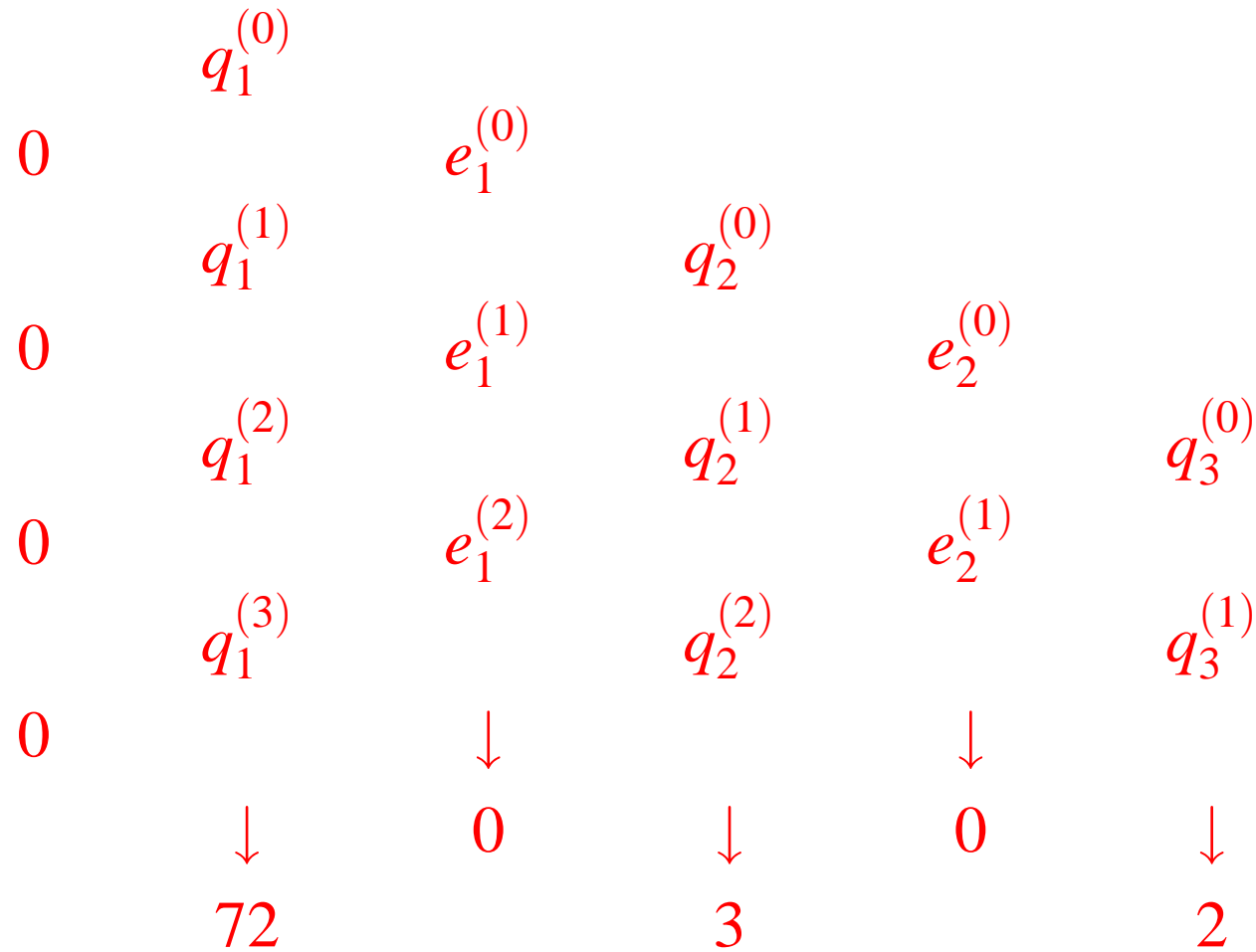
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Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

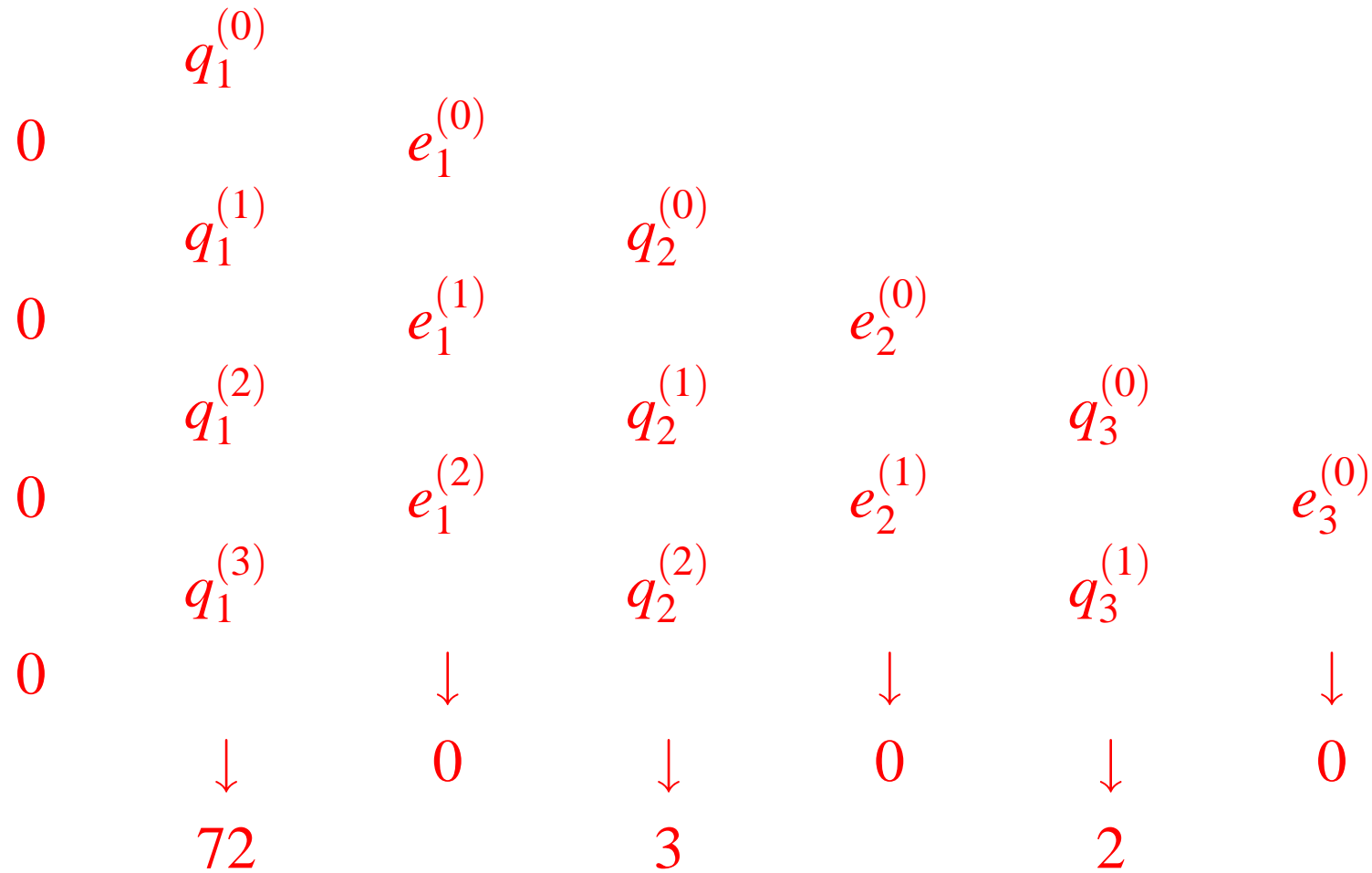
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Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

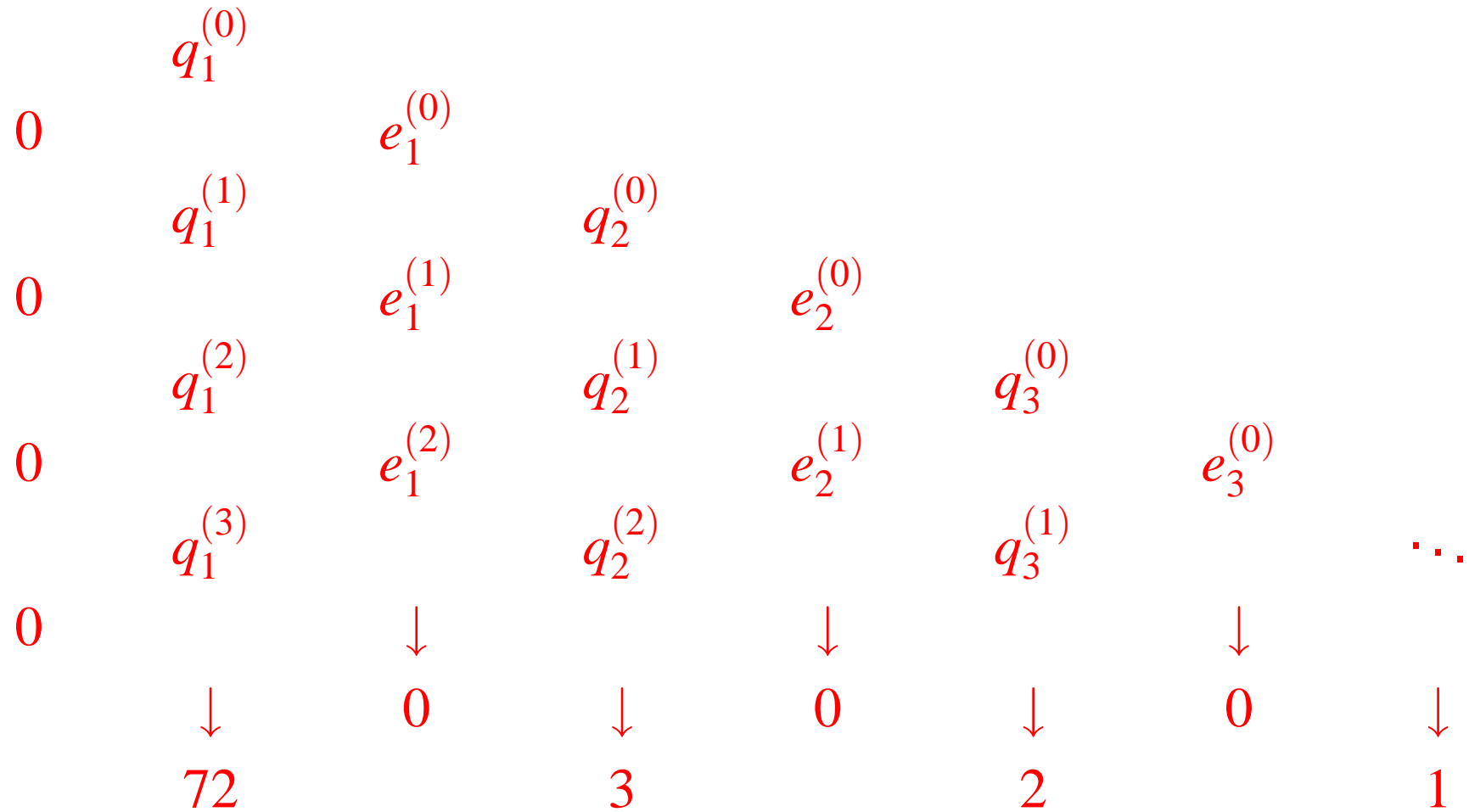
the qd algorithm



Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

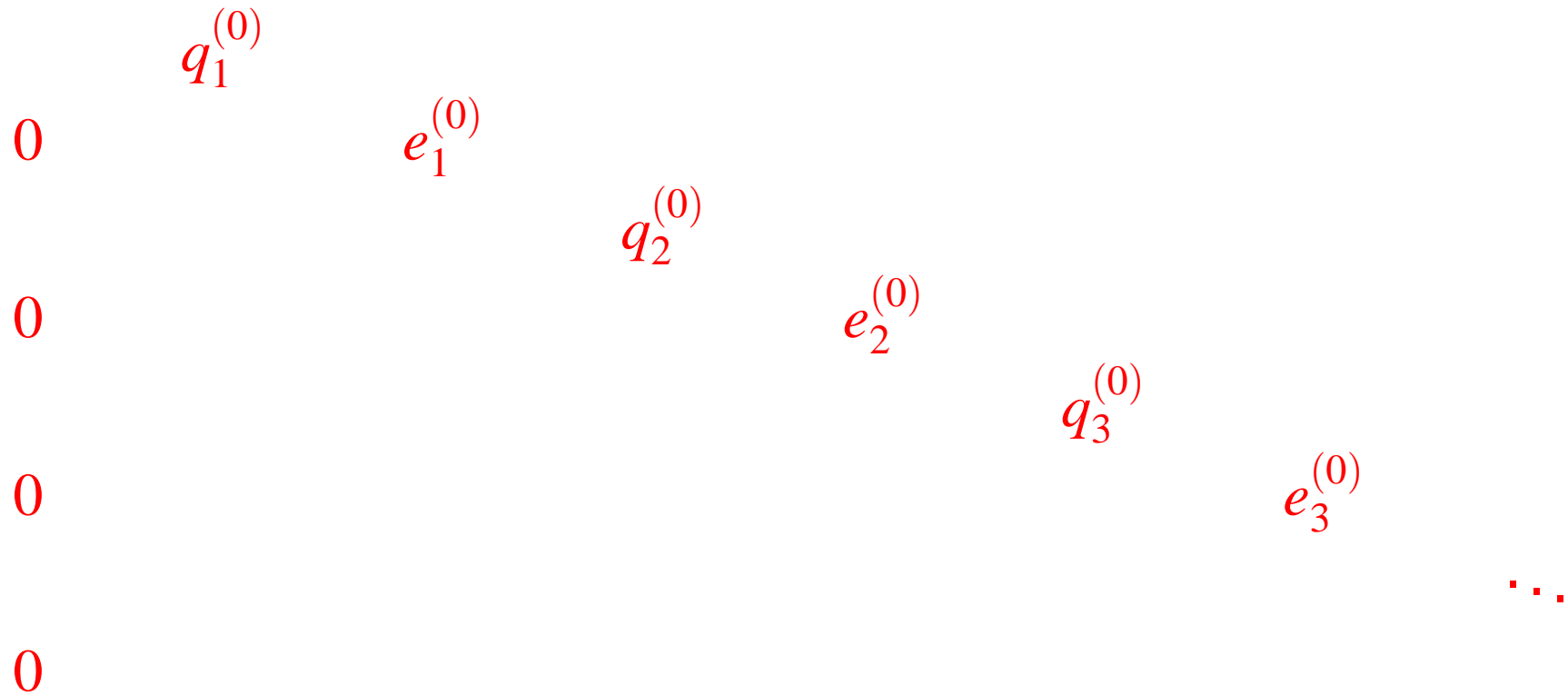
the qd algorithm



Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

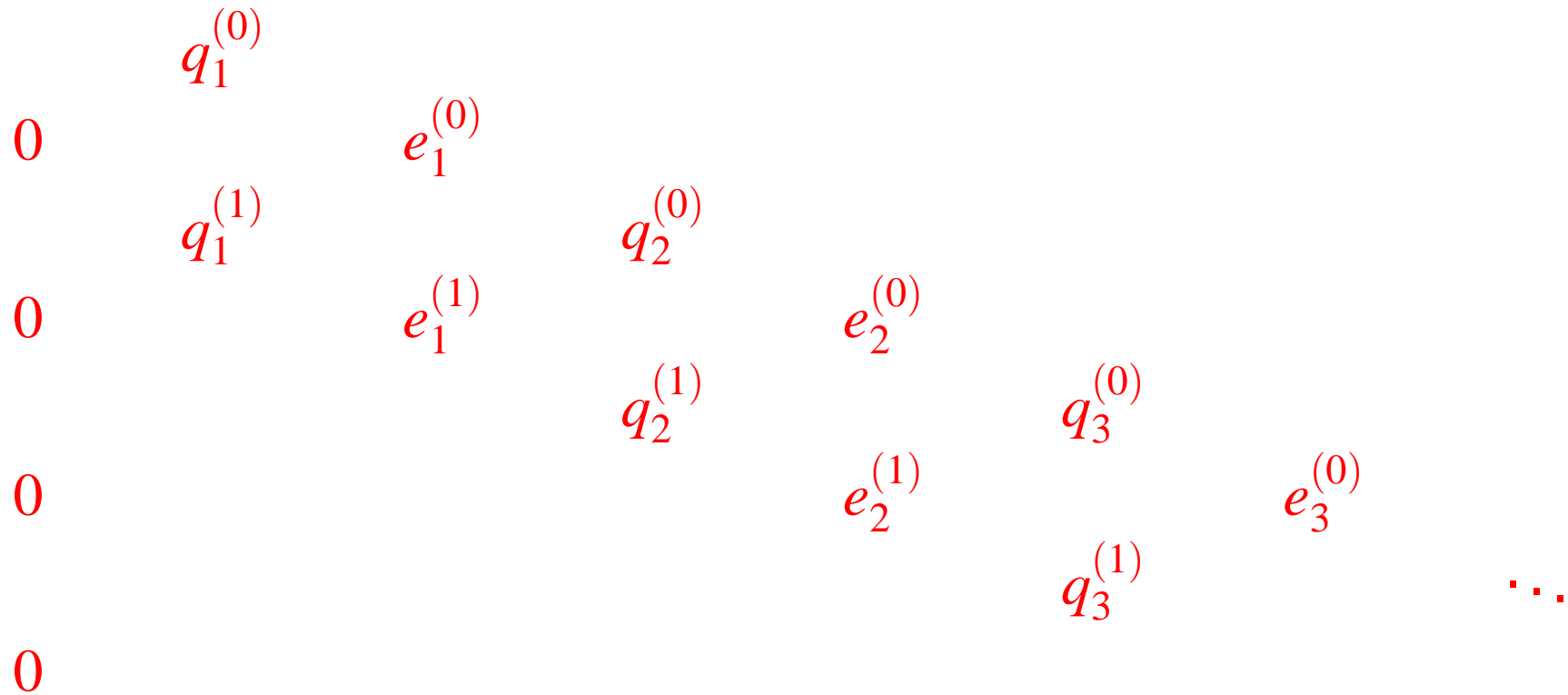
progressive form of qd algorithm



Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

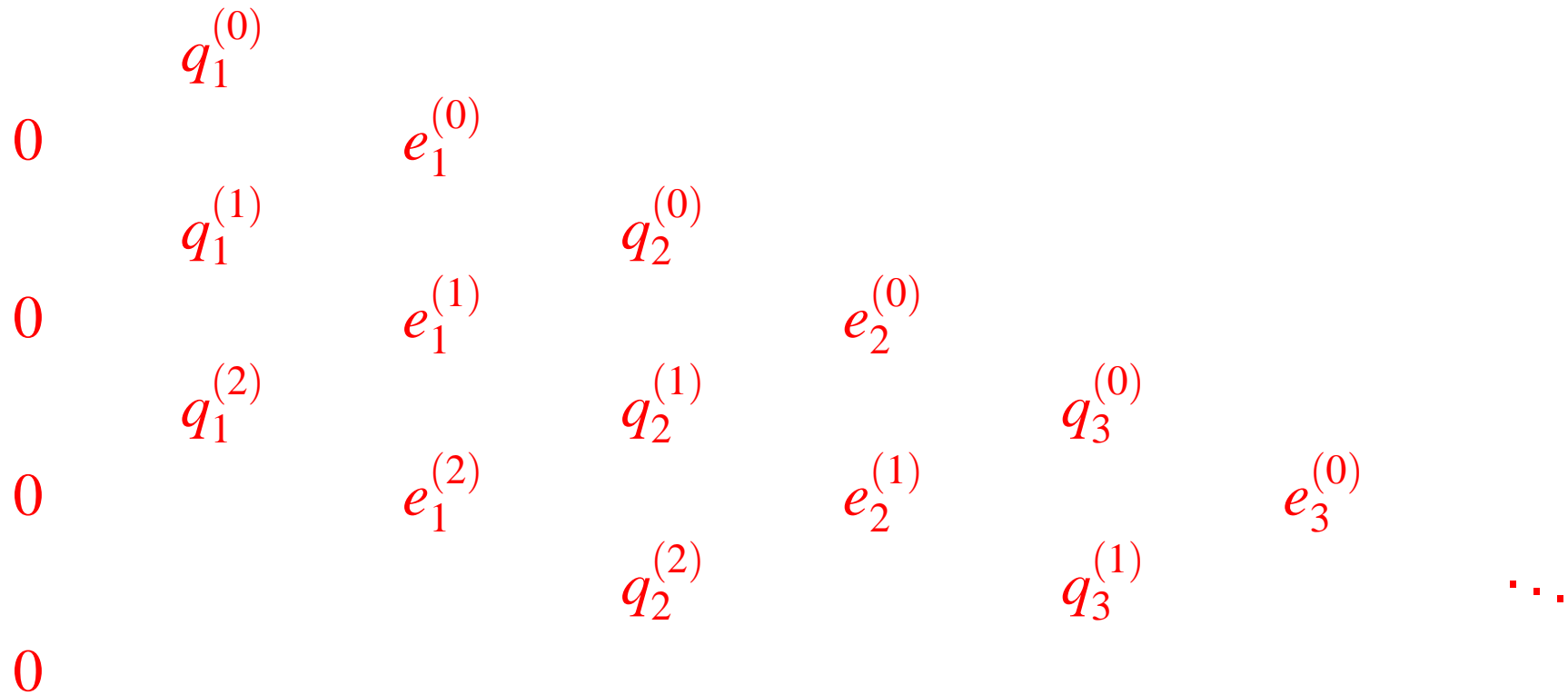
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Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

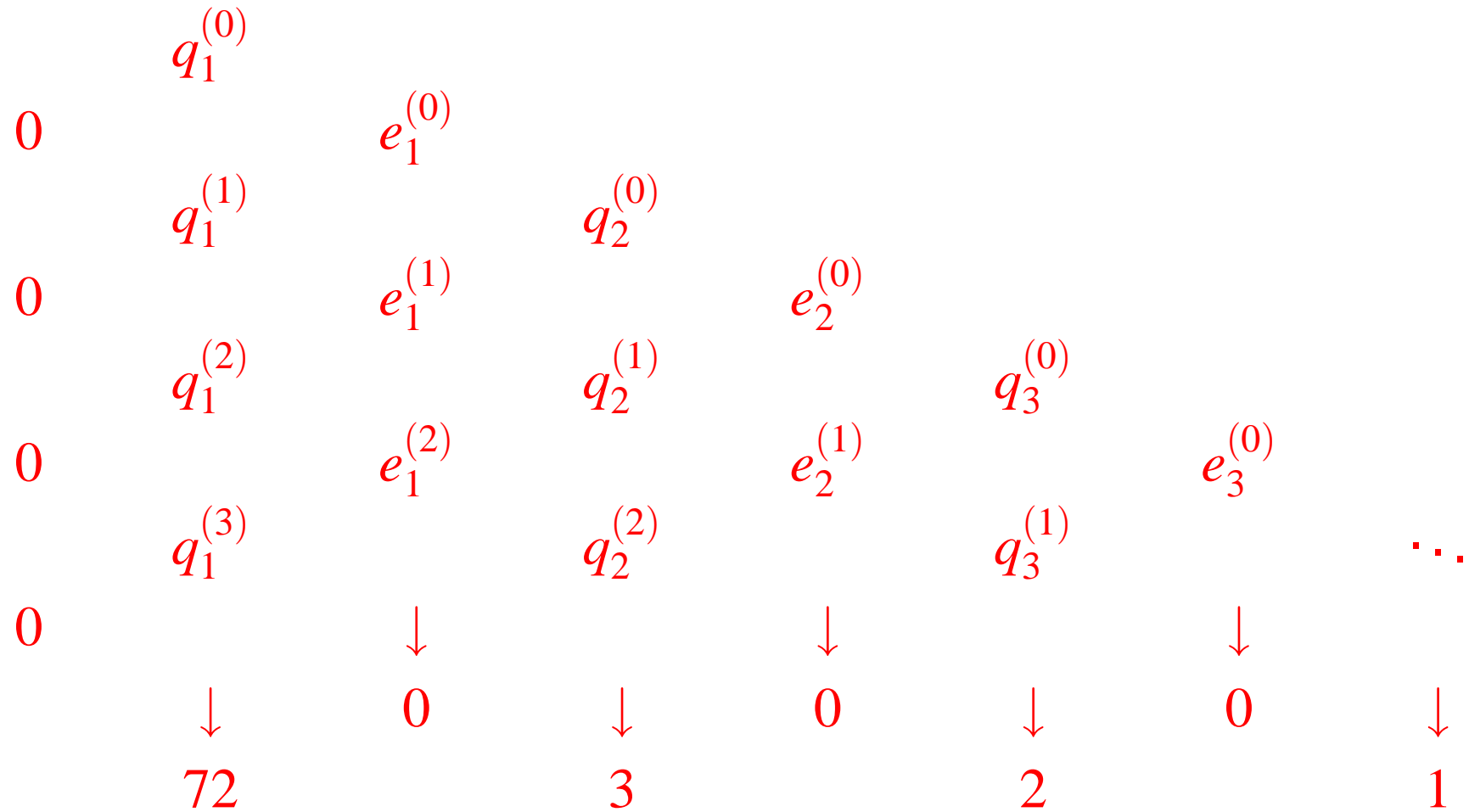
progressive form of qd algorithm



Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

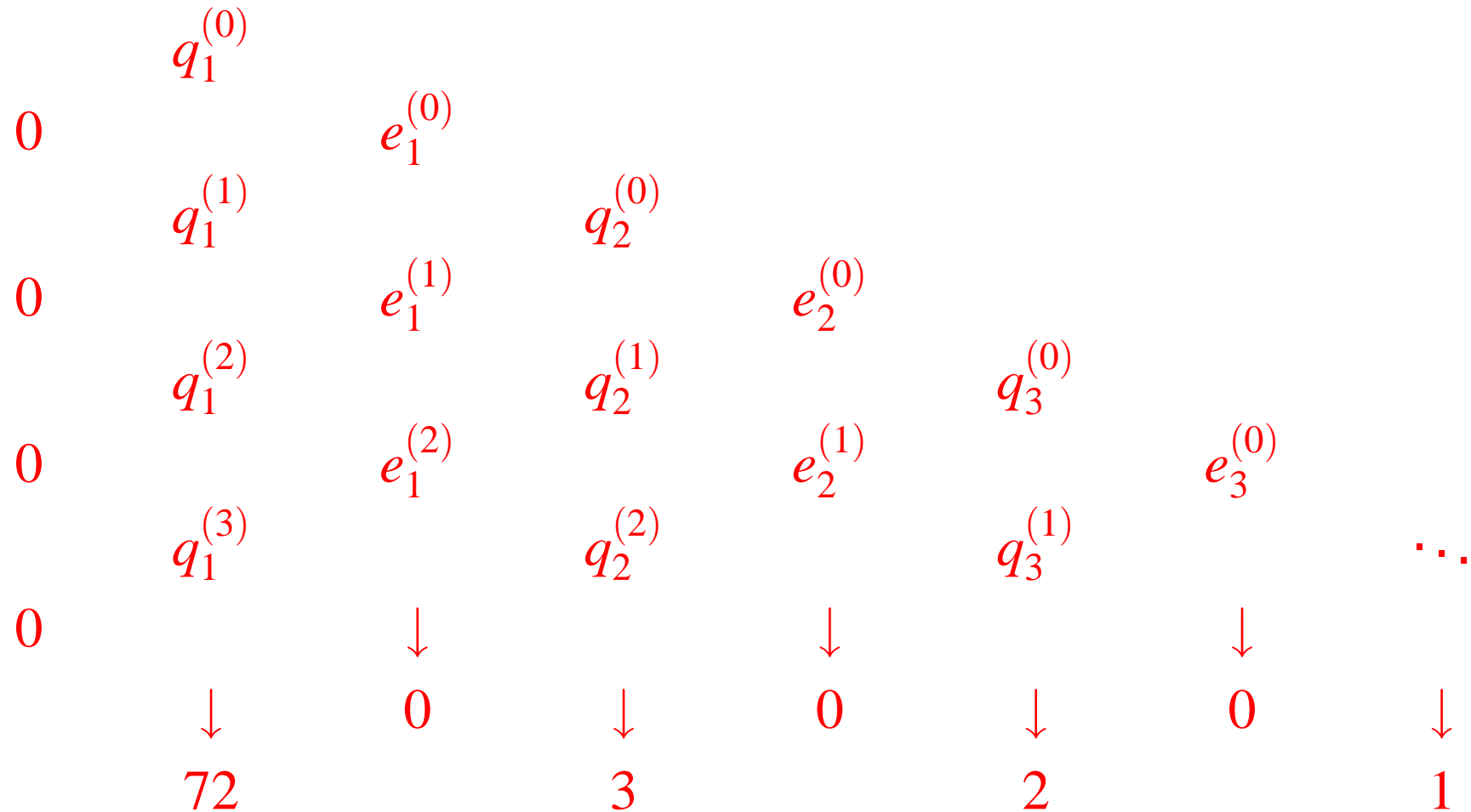
progressive form of qd algorithm



Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

progressive form of qd algorithm

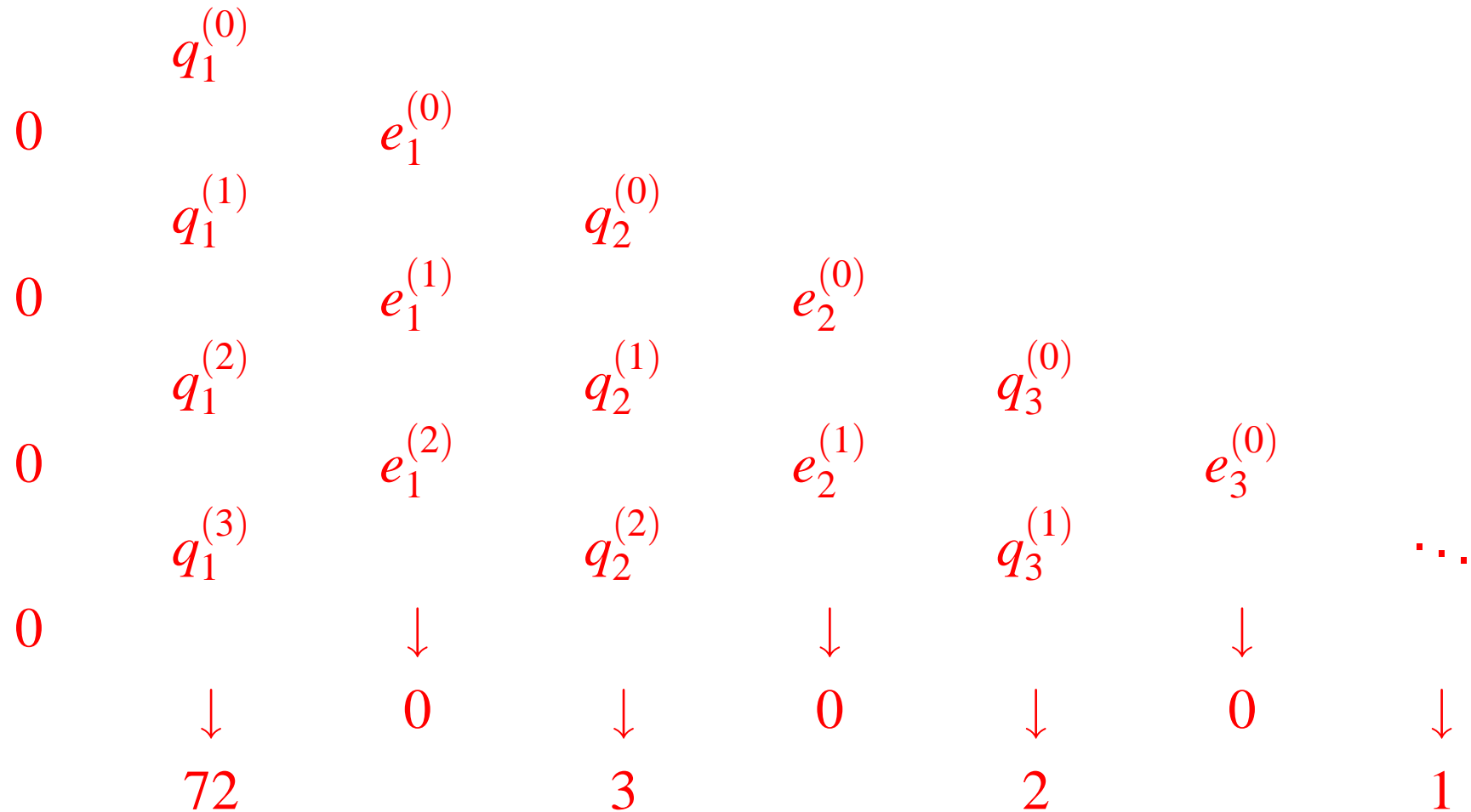


\implies Numerical sparse rational interpolation?

Numerical Sparse Polynomial Interpolation

[Cuyt, Lee 2008]

progressive form of qd algorithm



\Rightarrow Numerical sparse shifts?

Outline

- Sparse Interpolation
- *Generators of Polynomial Ideals*
- Multivariate Padé Approximants

Multivariate Poles

$$F(x_1, \dots, x_n) = \frac{h(x_1, \dots, x_n)}{f(x_1, \dots, x_n)}, \quad h(x_1, \dots, x_n) \text{ analytic}$$

$$\text{and } f(x_1, \dots, x_n) = f_1 \cdots f_t \in \mathbb{C}[x_1, \dots, x_n]$$

The poles of $F(x_1, \dots, x_n)$ are where $f(x_1, \dots, x_n) = f_1 \cdots f_t = 0$.
Consider the multivariate Taylor expansion

$$F(x_1, \dots, x_n) = \sum_{(i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}}.$$

Can $f(x_1, \dots, x_n)$, or f_1, \dots, f_n , be determined from $a_{\vec{i}}$?

Until recently, not really....

Multivariate Poles

$$F(x_1, \dots, x_n) = \frac{h(x_1, \dots, x_n)}{f(x_1, \dots, x_n)}, \quad h(x_1, \dots, x_n) \text{ analytic}$$

$$\text{and } f(x_1, \dots, x_n) = f_1 \cdots f_t \in \mathbb{C}[x_1, \dots, x_n]$$

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Can $f(x_1, \dots, x_n)$, or f_1, \dots, f_n , be determined from $a_{\vec{i}}$?

What is missing? Generators: $\mathbb{C}[x]$ is a principal ideal domain;
 $\mathbb{C}[x_1, \dots, x_n]$ is a finitely generated ideal.

The poles of $F(x_1, \dots, x_n)$ form an affine variety defined by a set of generators that are the irreducible factors f_1, \dots, f_t .

Single Multivariate Pole

Let f be an irreducible polynomial, with p having zero constant:

$$f(x_1, \dots, x_n) = 1 - p(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$$

Consider $\frac{1}{f} = \frac{1}{1-p} = 1 + p + p^2 + \dots = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}}$

Expand $p = b_1 x^{\vec{d}_1} + \dots + b_m x^{\vec{d}_m}$ in the linear recurrence $p^{i+1} = p^i \cdot p$

$$\begin{aligned} p^N &= p^{N-1} b_1 x^{\vec{d}_1} + \dots + p^{N-1} b_m x^{\vec{d}_m} \\ \vdots &= \quad \quad \quad \vdots \\ p^{M+1} &= p^M b_1 x^{\vec{d}_1} + \dots + p^M b_m x^{\vec{d}_m} \\ p^M &= p^{M-1} b_1 x^{\vec{d}_1} + \dots + p^{M-1} b_m x^{\vec{d}_m} \end{aligned}$$

$$a_{\vec{k}} x^{\vec{k}} = a_{\vec{k}-\vec{d}_1} x^{\vec{k}-\vec{d}_1} \cdot b_1 x^{\vec{d}_1} + \dots + a_{\vec{k}-\vec{d}_m} x^{\vec{k}-\vec{d}_m} \cdot b_m x^{\vec{d}_m}$$

Single Multivariate Pole

Let f be an irreducible polynomial, with p having zero constant:

$$f(x_1, \dots, x_n) = 1 - p(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$$

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$$a_{\vec{k}} = a_{\vec{k}-\vec{d}_1} b_1 + \dots + a_{\vec{k}-\vec{d}_m} b_m$$

Single Multivariate Pole

Let f be an irreducible polynomial, with p having zero constant:

$$f(x_1, \dots, x_n) = 1 - p(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$$

Consider $\frac{1}{f} = \frac{1}{1-p} = 1 + p + p^2 + \dots = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}}$

$$a_{\vec{k}} = a_{\vec{k}-\vec{d}_1} b_1 + \dots + a_{\vec{k}-\vec{d}_m} b_m$$

Repeat for m various $a_{\vec{k}_j}, a_{\vec{k}_{j+1}}, \dots, a_{\vec{k}_{j+m-1}}$

$$\begin{bmatrix} a_{\vec{k}_j-\vec{d}_1} & a_{\vec{k}_j-\vec{d}_2} & \cdots & a_{\vec{k}_j-\vec{d}_m} \\ a_{\vec{k}_{j+1}-\vec{d}_1} & a_{\vec{k}_{j+1}-\vec{d}_2} & \cdots & a_{\vec{k}_{j+1}-\vec{d}_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\vec{k}_{j+m-1}-\vec{d}_1} & a_{\vec{k}_{j+m-1}-\vec{d}_2} & \cdots & a_{\vec{k}_{j+m-1}-\vec{d}_m} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j} \\ a_{\vec{k}_{j+1}} \\ \vdots \\ a_{\vec{k}_{j+m-1}} \end{bmatrix}$$

Multiple Multivariate Poles

$$\begin{aligned}
 \frac{1}{f} &= \frac{1}{(1-p_1)(1-p_2)} = \frac{r_1}{1-p_1} + \frac{r_2}{1-p_2} + \frac{r}{(1-p_1)(1-p_2)} \\
 &= r_1 (1 + p_1 + p_1^2 + \dots) + r_2 (1 + p_2 + p_2^2 + \dots) \\
 &\quad + r (1 + p_1 + p_1^2 + \dots) (1 + p_2 + p_2^2 + \dots) = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}}
 \end{aligned}$$

Who dominates

$$\begin{bmatrix} a_{\vec{k}_j - \vec{d}_1} & \cdots & a_{\vec{k}_j - \vec{d}_m} \\ \vdots & \ddots & \vdots \\ a_{\vec{k}_{j+m-1} - \vec{d}_1} & \cdots & a_{\vec{k}_{j+m-1} - \vec{d}_m} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j} \\ \vdots \\ a_{\vec{k}_{j+m-1}} \end{bmatrix} ?$$

and Where?

Multiple Multivariate Poles

$$\begin{aligned}
 \frac{1}{f} &= \frac{1}{(1-p_1)(1-p_2)} = \frac{r_1}{1-p_1} + \frac{r_2}{1-p_2} + \frac{r}{(1-p_1)(1-p_2)} \\
 &= r_1 (1 + p_1 + p_1^2 + \dots) + r_2 (1 + p_2 + p_2^2 + \dots) \\
 &\quad + r (1 + p_1 + p_1^2 + \dots) (1 + p_2 + p_2^2 + \dots) = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}}
 \end{aligned}$$

Who dominates

$$\begin{bmatrix} a_{\vec{k}_j - \vec{d}_1} & \cdots & a_{\vec{k}_j - \vec{d}_m} \\ \vdots & \ddots & \vdots \\ a_{\vec{k}_{j+m-1} - \vec{d}_1} & \cdots & a_{\vec{k}_{j+m-1} - \vec{d}_m} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j} \\ \vdots \\ a_{\vec{k}_{j+m-1}} \end{bmatrix} ?$$

and Where?

If $f = (1 - 2x_1 - 3x_2)(1 - x_1 - x_2)$, then $(2x_1 + 3x_2)^j$ dominates.

Multiple Multivariate Poles

$$\begin{aligned} \frac{1}{f} &= \frac{1}{(1-p_1)(1-p_2)} = \frac{r_1}{1-p_1} + \frac{r_2}{1-p_2} + \frac{r}{(1-p_1)(1-p_2)} \\ &= r_1 (1 + p_1 + p_1^2 + \dots) + r_2 (1 + p_2 + p_2^2 + \dots) \\ &\quad + r (1 + p_1 + p_1^2 + \dots) (1 + p_2 + p_2^2 + \dots) = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}} \end{aligned}$$

Who dominates $\begin{bmatrix} a_{\vec{k}_j - \vec{d}_1} & \cdots & a_{\vec{k}_j - \vec{d}_m} \\ \vdots & \ddots & \vdots \\ a_{\vec{k}_{j+m-1} - \vec{d}_1} & \cdots & a_{\vec{k}_{j+m-1} - \vec{d}_m} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j} \\ \vdots \\ a_{\vec{k}_{j+m-1}} \end{bmatrix} ?$

and Where?

If $f = (1 - x_1 + x_2)(1 - 7x_1)$, then

$(x_1 - x_2)^j$ dominates terms that favor powers of x_2

$(7x_1)^j$ dominates terms that favor powers of x_1

Multiple Multivariate Poles

$$\begin{aligned}
 \frac{1}{f} &= \frac{1}{(1-p_1)(1-p_2)} = \frac{r_1}{1-p_1} + \frac{r_2}{1-p_2} + \frac{r}{(1-p_1)(1-p_2)} \\
 &= r_1 (1 + p_1 + p_1^2 + \dots) + r_2 (1 + p_2 + p_2^2 + \dots) \\
 &\quad + r (1 + p_1 + p_1^2 + \dots) (1 + p_2 + p_2^2 + \dots) = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}}
 \end{aligned}$$

Who dominates

$$\begin{bmatrix} a_{\vec{k}_j - \vec{d}_1} & \cdots & a_{\vec{k}_j - \vec{d}_m} \\ \vdots & \ddots & \vdots \\ a_{\vec{k}_{j+m-1} - \vec{d}_1} & \cdots & a_{\vec{k}_{j+m-1} - \vec{d}_m} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j} \\ \vdots \\ a_{\vec{k}_{j+m-1}} \end{bmatrix} ?$$

and Where?

It is also possible that no one dominates.

$$(1 - x_1 - x_2)(1 - x_1 + x_2)$$

$$(1 - x_1 + x_2)^2$$

Multiple Multivariate Poles

$$\begin{aligned} \frac{1}{f} &= \frac{1}{(1-p_1)(1-p_2)} = \frac{r_1}{1-p_1} + \frac{r_2}{1-p_2} + \frac{r}{(1-p_1)(1-p_2)} \\ &= r_1 (1 + p_1 + p_1^2 + \dots) + r_2 (1 + p_2 + p_2^2 + \dots) \\ &\quad + r (1 + p_1 + p_1^2 + \dots) (1 + p_2 + p_2^2 + \dots) = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}} \end{aligned}$$

Who dominates

$$\begin{bmatrix} a_{\vec{k}_j - \vec{d}_1} & \cdots & a_{\vec{k}_j - \vec{d}_m} \\ \vdots & \ddots & \vdots \\ a_{\vec{k}_{j+m-1} - \vec{d}_1} & \cdots & a_{\vec{k}_{j+m-1} - \vec{d}_m} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j} \\ \vdots \\ a_{\vec{k}_{j+m-1}} \end{bmatrix} ?$$

and **Where?**

some symbolic-numeric (approximate) polynomial algebra

multivariate polynomial factorization

polynomial gcd

polynomial division

Multiple Multivariate Poles

$$\begin{aligned}
 \frac{h}{f} &= \frac{h}{(1-p_1)(1-p_2)} = \frac{\gamma_1}{1-p_1} + \frac{\gamma_2}{1-p_2} + \frac{\gamma}{(1-p_1)(1-p_2)} + \phi \\
 &= \gamma_1 (1 + p_1 + p_1^2 + \dots) + \gamma_2 (1 + p_2 + p_2^2 + \dots) \\
 &\quad + \gamma (1 + p_1 + p_1^2 + \dots) (1 + p_2 + p_2^2 + \dots) + \phi = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^n} a_{\vec{i}} x^{\vec{i}}
 \end{aligned}$$

Who dominates

$$\begin{bmatrix} a_{\vec{k}_j - \vec{d}_1} & \cdots & a_{\vec{k}_j - \vec{d}_m} \\ \vdots & \ddots & \vdots \\ a_{\vec{k}_{j+m-1} - \vec{d}_1} & \cdots & a_{\vec{k}_{j+m-1} - \vec{d}_m} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j} \\ \vdots \\ a_{\vec{k}_{j+m-1}} \end{bmatrix} ?$$

and **Where?**

optimization?

maybe, by introducing analytical part $h(x_1, \dots, x_n)$

e.g., $e^{x_1 + \dots + x_n}$

Outline

- Sparse Interpolation
- Generators of Polynomial Ideals
- *Multivariate Padé Approximants*

Multivariate Padé Approximants

- Generalized qd algorithm: multivariate poles
- Multivariate convergence theorem: “de Montessus” type
- Block structure of multivariate Padé table
- ...

Thank You!