STUDENTS' PARTICIPATION IN A DIFFERENTIAL EQUATIONS CLASS:
PARAMETRIC REASONING TO UNDERSTAND SYSTEMS

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This dissertation is dedicated to my husband, Dee Keene and my advisor, Chris Rasmussen. Each of these men provided a tremendous amount of support and advice to allow me to finish the work. Thank you, Dee and Chris.
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ABSTRACT

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Dynamical systems as a content area in mathematics is growing in importance. New technology enables mathematicians and scientists to model the real world with systems of rate of change equations that are interrelated and based on time. One goal of this study is to report the results of an investigation on how students develop and use parametric reasoning as one basis for understanding dynamical systems of differential equations in an inquiry-oriented differential equations class. The need to understand how students reason parametrically with time and grow to understand this new aspect of mathematics provides significance for this study. This research study provides evidence that students already have understandings of time and rate from earlier experience and from their instruction recovering solutions to single ordinary differential equations and they use this to build their conceptions and understandings of solutions to systems of differential equations. The study also provides case studies of two students’ mathematical activity as they learn systems of differential equations. Finally, the study uses a new construct of “advancing mathematical activity” and the mathematical practices of symbolizing, algorithmatizing, justifying, and experimenting to document how students enculturate into the larger mathematical community.
CHAPTER 1 INTRODUCTION

1.1 Setting the Stage

Dynamical systems are an increasingly important field of study in science, mathematics, and engineering (Stogatz, 1994). Real world dynamical systems are situations where varying quantities have mutual effects on each other as time passes (Williamson, 1997). For example, the phenomenon when two animal populations exist in the same environment and the numbers of animals of each species affect the numbers of the other is a dynamical system. The study of dynamical systems includes analysis, interpretation and prediction of the complicated interplay and changes that occur among many different variables as time passes. Furthermore, current technology has provided the computational power to develop the mathematics to describe and predict the behavior of these complex systems that are intractable without it (Kallaher, 1999). Central to dynamical systems is the concept of time, and how quantities behave and interact as time passes. Learners’ conceptions of time are a component of mathematical reasoning within the broader mathematical domain that Kaput (1999) calls the “mathematics of change and variation” [MCV]. Kaput emphasizes that students at all levels need to be exposed to the ideas of dynamic behavior in mathematics and be provided opportunities to develop understandings of the mathematical ideas of change at many different educational levels. Furthermore, mathematics education researchers are advocating that the MCV needs to
hold a more prominent place in the K-12 curriculum (Confrey, 1994; Kaput, 1999; Romberg & Kaput, 1999; Stroup, 2000). This call for more study of dynamical systems extends to undergraduate mathematics as well. The Mathematical Association of America Committee for Undergraduate Programs in Mathematics curriculum guide states that, Mathematical sciences departments have a responsibility to see that all students, definitely including mathematics majors, have opportunities to learn about contemporary topics in pure and applied mathematics...[which includes] dynamical systems, including chaos and fractals.(2003, pg. 21)

Complementing this call for study of dynamical systems at the university level, Romberg and Kaput (1999), Forrester (1993; 1994), and Mandinach and Cline (1994), contend that dynamical systems ways of thinking and analyzing phenomena, especially key qualitative and geometric approaches, are increasingly relevant to secondary school mathematics learning because of the technological advances that have developed in business, industry, and research today.

An essential and under researched form of mathematical reasoning that is foundational to understanding complex dynamical systems, and that extends the MCV, is parametric reasoning. For the purpose of this research, I define parametric reasoning as developing and using conceptualizations about time as a dynamic quantity that implicitly or explicitly coordinates with other quantities, to understand and solve problems. One of the primary purposes of this study is to more precisely characterize parametric reasoning as it contributes to and is enriched by students’ understanding of systems of linear differential equations and their solutions. This characterization of parametric reasoning is both theoretically and pragmatically significant. It is theoretically significant in that I
develop a characterization of parametric reasoning, which, although an important part of mathematical thinking, has until now has been neglected in the literature. Pragmatically, new insights about students’ parametric reasoning while learning systems of differential equations will inform differential equations instructional design as well as contribute to K-12 mathematics educational goals. NCTM states in the 9-12 Algebra Standards, “The study of change in grades 9-12 is intended to give students a deeper understanding of the ways in which changes in quantities can be represented mathematically and of the concept of rate of change” (p. 305). The introduction of increasingly more sophisticated technology makes it possible to study these ideas of change over time at the K-12 school level and students can reason about mathematical situations involving time.

In the following sections, I first discuss examples of dynamical systems and systems of differential equations to illustrate their importance in science and engineering today. In the context of these examples, I provide justification for characterizing parametric reasoning within the domain of differential equations. Second, I relate the context and purposes of this study to the curricular and pedagogical changes in current reform effort in differential equations. Third, I situate the research in the larger field of mathematics education research. Finally, I describe in more detail the research goals for this study.

1.2 Dynamical systems and the dynamical systems approach to differential equations

Dynamical systems are of growing importance in science and engineering. For example, Arney (1998) argues that “we need to understand the complexity of both the human mind and the physical machines holding the data….One of the basic means of
modeling the transformation process of data analysis is through dynamical systems” (p. 2). Moreover, he suggests that dynamical modeling is the new science that we need in order to advance mathematics into the new generation of discovery. Universities have begun offering undergraduate courses in dynamical systems to meet these recently identified needs of university students as they learn mathematics, engineering and science (see examples: http://nlds.sdsu.edu [San Diego State University]; http://www.dean.usma.edu/departments/math/courses/ma153 [United States Military Academy]; http://www.acs.utah.edu/Gencatalog/1023/crsdesc/math.html. [University of Utah]). These documents illustrate how and why the study of differential equations from a dynamical point of view has become important and valuable for many university level students.

The mathematics of dynamical systems can generally be classified into one of three categories: 1) systems of differential equations that model continuous phenomena using two or more rate of change equations; 2) difference equations that model discrete phenomena; and 3) other systems that cannot be modeled with either of the other two types of systems of equations. Although all three of these categories are important, the focus of this study is on continuous dynamical systems of two or more differential equations. Mathematical models that utilize systems of differential equations are used extensively in the description and prediction of real-world phenomenon and are increasingly prominent in undergraduate differential equations courses (e.g., Blanchard, Devaney, and Hall, 2002; Borrelli and Coleman, 1998; Diacu, 2000; Kostelich and Armbruster, 1997). Two examples of such systems are described next.
An example from biology: The following system of two differential equations models the population of a parasite population and its host population:

\[
\frac{dH}{dt} = (.02 - .1P)H \\
\frac{dP}{dt} = (.01H - .03)P
\]

where \( H = H(t) \) is the function modeling the host population, \( \frac{dH}{dt} \) is its continuous rate of change (derivative) equation, \( P = P(t) \) is the function modeling the parasite population, and \( \frac{dP}{dt} \) is its continuous rate of change (derivative) equation (Williamson, 1997). The independent variable in the system is time and the parasite and host populations are the dependent quantities studied over time. In this system, each population will affect the other population in different ways depending on the changes in the number of host and parasite populations. For example, when the parasite population is large, the rate of change of the host population is negative, and so the host population is decreasing. Similarly, when the host population is large, the rate of change of the parasite population is positive, so the parasite population is increasing. Both populations are continually changing over time, affecting the rate of change of the other population and hence the size of the population.

The quantity time is a critical variable when describing and predicting with this set of differential equations; even though \( t \) does not appear explicitly as a variable in the equations, it is essential to the system. Conceptually, understanding how time is inherent and absolutely vital to the system, even though implicit, is necessary to the analysis of the system. Furthermore, understanding and predicting solutions to this system, and others
like it, is an important contribution of mathematics to the biological sciences. Although this and similar systems do not have closed analytical solutions, current technology allows for qualitative and numerical analyses. Such analyses require conceptualizing time-based models of the phenomenon and reasoning about how the parameter time affects and controls the mathematical model.

*An example from astronomy*: The system of Newtonian differential equations that model the orbits of a single planet of mass $m_1$ relative to a fixed sun of mass $m_2$ is another good example of a dynamical system. This system of differential equations takes the following form:

$$
\begin{align*}
x'' &= -kx \quad \frac{1}{(x^2 + y^2)^{3/2}} \\
y'' &= -ky \quad \frac{1}{(x^2 + y^2)^{3/2}}
\end{align*}
$$

where $x''$ and $y''$ represent the second derivatives of $x$ and $y$, respectively, with respect to time and $x$ and $y$ are variables that stand for the location of the planet. $k$ is a value determined by the equation $k = G (m_1 + m_2)$ where $G$ is the gravitational constant and $m_1$ and $m_2$ are the masses of the two bodies. These rate of change equations model the change in the location of the astronomical bodies with respect to time; again, time is implicit in the differential equations (Williamson, 1997). Despite the fact that analytical solutions may be impossible to determine, students can analyze the behavior of this phenomenon with this dynamical systems model (or one similar) by parameterizing the variables $x$ and $y$ in terms of time and finding solutions numerically to the differential equations.
The previous examples offer two mathematical models of dynamical systems that describe and predict real situations. In each of these examples, parameterizing the functions in terms of time is crucial to understanding the phenomenon under study.

The increase in mathematicians’ interest in dynamical systems, which relates directly to the MCV discussed earlier, is one factor contributing to pedagogical and curricular changes in differential equations. I next look at some of the other reasons for these changes and situate my study in relation to these reform efforts and more general mathematics education reform that has occurred in the last twenty years.

1.3 Changes in differential equation courses pedagogy and curricula

During the last ten to fifteen years, a dynamical systems approach to teaching and learning differential equations has become more common in North American universities. This approach involves students’ focusing on the physical processes of phenomena, the complex interaction between variables in a system as they vary over time, and analysis of the systems’ numerical and qualitative (graphical) solution representations. In many differential equation classrooms teachers are reducing class time spent recovering the closed form solutions for special types of differential equations and systems of differential equations (those equations that can be solved analytically) and emphasizing numerical and graphical ways to analyze the solutions to differential equations (Blanchard, Devaney & Hall, 1995). According to Kallaher (1999), “today, differential equations can be taught from the qualitative point of view with emphasis on the underlying mathematics and physical processes that give rise to the equations” (p. vii). This does not mean that analytical solutions are not discussed, but there is a de-emphasis
on those differential equations that have closed solutions, as they comprise a small percentage of the differential equations that are used to model real world phenomena.

A dynamical systems approach is being integrated into the reform efforts for many differential equation classes. These changes are the result of three primary influences: mathematicians’ interest and contemporary work in dynamical systems, K-12 mathematics and calculus reform, and new technology that makes it possible to investigate solutions to systems that were not solvable with analytic methods. In the following paragraphs, I discuss each of these influences and conclude the section with an example of a system of differential equations where I illustrate how parametric reasoning is fundamental to students’ mathematical activity when studying these systems.

Studying differential equations with a dynamical systems approach offers students a look into newly developing mathematics. In the previous section, I presented two examples of dynamical systems providing entry into areas that are of interest to the scientific community. The scientific phenomena are not new, but the tools to investigate them by mathematically modeling them as dynamical systems in order to understand them qualitatively and numerically in considerable detail are relatively new. As Blanchard (1992) notes, “Learning that these systems are not completely understood, or that the mathematical ideas they [students] are using were developed in their [students’] lifetimes, changes their [students’] opinion about the nature of mathematics” (p. 387). As students are actively involved in this less algebraic mathematical domain, they often come to understand that mathematics (especially dynamical systems) is not an ancient collection of methods and tricks, but an exciting and evolving discipline. Their interest
and involvement in the mathematical community grows as they experience the power of these new perspectives in solving and understanding dynamical systems.

The second influence that has supported the new differential equation course curricula and pedagogy is calculus reform. This reform, which began in the 1980’s, involves changes to the standard curriculum and instructional strategies. Changes in calculus content include a decreased emphasis on symbolic differentiation and integration and the inclusion of the “rule of three.” In the rule of three, graphical, numerical, and algebraic techniques for differentiation and integration are all components of instruction and students are exposed to techniques that were previously unexplored at this level. For example, students may learn to approximate derivatives numerically by calculating difference quotients, instead of immediately using symbolic techniques. As far as instructional strategies are concerned, technology such as computer algebra systems and graphing calculators permit analysis of symbolically complex problems previously not accessible to first year calculus students.

Further, influences from K-12 mathematics education reform have impacted university classrooms through increased emphasis on cooperative learning, more active involvement by students, and written communication as an important aspect of learning. For example, some undergraduate mathematics classes have computer labs where students work collaboratively to complete an assignment, and write up a lab report of several pages; these lab reports require students to explain their interpretation of the ideas involved and their problem solving approaches. These undergraduate instructional reform efforts have provided further inspiration for rethinking the teaching and learning of differential equations.
The third impetus for the differential equations reform is the introduction of new technology with its power to investigate graphically and numerically phenomena that were previously not accessible. The plethora of new technology allows scientists to study solutions to differential equations in numerical and graphical forms and understand much more about complex real life phenomena. As Blanchard (1992) notes, “Technology serves as a vehicle for changing the nature of the course from one where students passively receive information to one where students actively participate in their education” (p. 387). In summary, advances in technology that allow for the contemporary investigation of dynamical systems, K-12 mathematics education reform and calculus reform have contributed to differential equation reform and the move to the dynamical systems approach to teaching differential equations.

To conclude this section, I describe an example of a system of differential equations where I illustrate how parametric reasoning is fundamental to students’ mathematical activity when studying dynamical systems. The particular system of differential equations that I use models a spring-mass system. A spring-mass system is a phenomenon typically studied in physics that entails a certain size mass attached to a spring (see Figure 1.1) that is then put in motion by either pushing or pulling the mass.

![Figure 1.1. Sketch of Spring Mass Phenomenon](image)
In this system, $x(t)$ is a function that models a mass’s distance from the center (arbitrarily set as $x=0$) and $y(t)$ is a function that models the velocity of the mass, and thus is the derivative of $x(t)$; they are related by the system of differential equations:

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -2x - 3y
\]

Studying this system of differential equations and the solutions to it provides information to describe and predict the position and velocity of the mass over time. This system is autonomous, meaning that the derivatives $dx/dt$ and $dy/dt$ are dependent exclusively on the time dependent variables. This allows the use of vector fields as a useful tool to study the system as each point on the field represents one $(x,y)$ pair that yields only one tangent vector.

To elaborate, the behavior of solutions to this system can be qualitatively analyzed; this means the behavior of solutions can be examined in terms of increasing, decreasing, long term behavior, points of equilibrium, etc., by studying their geometric representations as directed curves in the phase plane. A phase plane is an $x$-$y$ plane (therefore no time is explicit) where each point represents one $(x,y)$ value. Figure 1.2 shows a phase plane vector field for the given system of differential equations. Each point in this plane represents a pair of values $(x(t), y(t))$ to which a tangent vector is “attached.”
Figure 1.2. Vector Field for Spring Mass System

Each vector is a graphical representation of the instantaneous rate of change (derivative) of $x$ and $y$ with respect to time for any solution curve that contains the point and is determined by the values of the two differential equations using the $x$ and $y$ coordinates of the point. For example, the vector at the point (2,3) [see box on vector in Figure 1.2] is $3\vec{i} - 13\vec{j}$ found by substituting 2 for $x$ and 3 for $y$ in the differential equation. This vector shows the direction of the solution that goes through the point (2,3) in this autonomous system of differential equations. By starting at a given state (point in the phase plane), one can follow the direction of the vector in the plane to succeeding points and continue to create a curve that represents the solution in two dimensions without using numerical or analytical means.

Graphically, solution curves can be sketched in the phase plane by choosing an initial point and following the tangent vectors to create a graph in the phase plane (see Figure 1.3). This sketch is parameterized with time, meaning that the changes in the $x$ and
$y$ values can be used to determine the solutions as pairs of interrelated functions of time $x(t)$ and $y(t)$, dependent on time, or as a curve in three dimensions, $x,y,$ and $t$. Thus, $x(t)$ and $y(t)$ are interrelated functions that have time as the independent variable and so “move” as time changes. Understanding and thinking about the solutions as parametric curves in the phase plane is central to parametric reasoning.

![Figure 1.3. Phase Plane with Possible Sketched Solutions](image)

Students can use the phase plane, sketches of representations of solutions in the phase plane, and/or graphical and symbolic representations of the explicit functions of $x(t)$ and $y(t)$ to reason about the solutions and the phenomenon that is being modeled. In particular, students create the explicit functions $x(t)$ and $y(t)$ both graphically and analytically with time as the independent variable, even though time is not present explicitly in the phase plane or the original differential equations.

To illustrate this, consider the left curve in Figure 1.3. Students can create a graph of $x$ versus $t$, by choosing a point in the third quadrant on the curve and using the $x$
value of that point as the initial value of the $x(t)$ curve on an $x$-$t$ plane. Then, by following the curve “in time,” one can qualitatively track the change in the $x$ value as time moves. The curve moves from the third to the second quadrant, so $x$ becomes positive and rises for a while and then decreases in value (see Figure 1.4).

![Graph of $x$ vs $t$](image)

*Figure 1.4. Graph of $x$ vs $t$ Created From Phase Plane in Figure 1.3*

This spring mass system example illustrates a physical situation and its system of differential equations model where students’ parametric reasoning is an essential part of their understanding.

1.4 **Situating the study in mathematics education research**

This study fits within and will contribute to three areas of mathematics education research: teaching and learning of differential equations, undergraduate mathematics teaching and learning, and the mathematics of change and variation. I next discuss each of those areas.

First, this study will add to the research in the field of students’ learning of differential equations. Studies have addressed student understanding of solutions to
ordinary differential equations, especially student understanding of equilibrium solution functions and the structure of the solution spaces (e.g., Artigue, 1992; Rasmussen, 2001; Stephan & Rasmussen, 2004; Zandieh & McDonald, 1999). Studies have, to a lesser degree, addressed student thinking about systems of differential equations in the context of student reasoning about solutions to differential equations in general (Rasmussen, 1999; Trigeuros, 2000, 2001; Artigue, 1992). Also, the differential equations classroom itself and learning in social context has been a focus of research (e.g., Ju & Kwon, 2004; Stephan & Rasmussen, 2003; Yackel, Rasmussen & King, 2000). This investigation adds to our understanding of students’ reasoning about systems of differential equations and their solutions in the context of the classroom.

Second, this study draws from and contributes to the more general domain of undergraduate mathematics education research. Current research has described student conceptual development in calculus and investigated students’ difficulties and misconceptions in the area of derivative and integral (Kaput & Dubinsky, 1994). For example, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) provide a characterization of covariational reasoning at the calculus level for individuals by proposing levels of mental actions that students enact as they solve tasks about two quantities that vary together. By using this framework for covariational reasoning, analysis of students’ expressions of these actions allow teachers to better analyze student understanding of change in calculus and contribute to instructional ideas in the area of derivative. As another example, Thompson (1994) details students’ mathematical activity with the fundamental theory of calculus and its relation to student images of rate of change.
This research expands these studies by looking at parametric (time-based) reasoning and focusing on the understandings that students use in advanced mathematics classes. Students reasoning at the undergraduate level has been studied to a much lesser degree, so the results of this dissertation will add to an underrepresented body of research.

Finally, this research will contribute new theory about student understanding of the MCV, both at the undergraduate and K-12 levels. Characterizing parametric reasoning will contribute to research on K-12 student understanding of variation and change (e.g., Kaput & West, 1994, Thompson, 1994, Confrey, 1994) by providing new ways to understand students’ intuitions and conceptualizations of quantities that change with respect to time. For example, Stroup (2002) posits that qualitative understandings of rate of change are important to be valued on their own and can be integrated into more formal characterizations of differentiation as well. These qualitative (and often informal) understandings are not viewed as obstacles to students’ learning of traditional calculus but taken as understandings that are valuable in their own right and relate in positive ways to calculus.

To summarize, this study builds on the mathematics education research on student learning in differential equations, undergraduate mathematics, and the MCV. Likewise, understanding how students use parametric reasoning informs our understanding of students’ conceptualizations of mathematical change, which includes ways of reasoning about analytical and graphical solutions to systems of differential equations. I conjecture that the study of informal, intuitive and qualitative ways of parametric reasoning which students use to build understandings of solutions to systems of differential equations, are
valuable to study in their own right and not just as a preliminary and less complex mathematical domain.

In the next section, I formally introduce the goals of the study.

1.5 Research goals

There are three research goals for this study:

1) To develop and elaborate a characterization of parametric reasoning and
2) To develop case studies of two students in a differential equations class in order to detail how they grow to understand linear systems of differential equations (SDEs), paying particular attention to how these understandings are supported by and enable their ways of reasoning parametrically,
3) To develop a situated analysis of undergraduate students’ mathematical activity, as they become part of a larger mathematical community.

**Goal 1.** I define parametric reasoning as developing and using conceptualizations about the dynamic quantity time that implicitly or explicitly coordinate with other quantities to understand and solve problems. Described in another manner, parametric reasoning occurs when students use their understanding of time to reason about mathematical situations. For example, when considering the motion of a spring-mass system, mathematicians often represent the variation of position and velocity with a graph showing position on the horizontal axis and velocity on the vertical axis. Time is not explicit in the inscription, but one can reason about the situation by including time as part of the characterization of the situation. See the sketches in Figure 1.3 as an example
when students reason about the situation by “moving” on the phase plane and then making predictions about the spring mass behavior as time passes.

This characterization of parametric reasoning complements the more conventional view of the use of parameters in mathematics. In this view, parametric reasoning is when students reason about functions and equations and their graphs when one (or more) of the variables are parameters (Drijvers, 2003). The term parameter refers to a variable that changes in an equation or system of equations, therefore changing the situation (Kaput, 1999). For example, in the linear equation $y=mx+3$, students might think about what happens to the line as $m$ is altered. In this case, $m$ is a parameter and varying this parameter creates a family of lines.

One analogy that provides a way to think about these two kinds of parametric reasoning involves the idea of a “sleeping” quantity. A parameter in the more conventional view is a sleeping variable (Freudenthal, 1983), in that it is usually fixed at some value and sleeps while the variables in the problem are viewed as changing. Then the parameter “wakes up,” changes value, and then returns to sleep. In parametric (time based) reasoning the quantity time never sleeps. The parameter time is always active in a dynamical sense as the investigation of the mathematics of a situation and its change is conducted.

**Goal 2.** The second goal of developing case studies of two students is accomplished by analyzing each individual student’s work and discourse as they participate in one-on-one interviews, in classroom discourse, and in small group and individual work. First I explore their ways of reasoning in an interview before they begin study of systems of differential equations. Then, I delineate their participation in mathematical activity
during the five weeks of class time on systems. Finally I analyze their responses to a second final interview after formal instruction.

These two case studies serve the purpose of providing an analysis of students’ understanding of systems of two autonomous linear differential equations. Both their individual mathematical activity and the ways they participate in the classroom activity are part of the analysis.

**Goal 3.** As a way to frame the third goal of describing students’ mathematical activity, I use the idea of “advancing mathematical activity.” According to Rasmussen, Zandieh, King, and Teppo (2005), this construct “is potentially useful for researchers who want to document the development of different types of mathematical practices that emerge in classrooms and for teachers and curriculum developers who want to plan for students’ mathematical growth” and “the notion of advancing mathematical activity lies in the collection of practices as they build and progress through activity. Participation in these practices, and changes in these practices, is synonymous with learning (Lave & Wenger, 1991; Cobb & Bowers, 1999).” The four practices I use are symbolizing, algorithmatizing, justifying, and experimenting. I use these observed practices to describe the ways in which students participate in increasingly sophisticated mathematical activity during instruction on systems of differential equations. I hypothesize that the documentation of the emergence of these practices provides a way to describe, analyze, and theorize about the students’ enculturation into the mathematical community.

To summarize, in this introduction I presented the significance of studying parametric reasoning in the context of the differential equations classroom and provided a rationale for why parametric reasoning is useful and important. I briefly described the
domain of dynamical systems and tendered examples of systems of differential equations where students would use parametric reasoning. I further situated the study in mathematics education research, particularly research in the domain of the MCV and undergraduate mathematics. Lastly, I described the research goals of the study. In the next two chapters, I describe the supporting mathematics education literature and then the methodology I use to achieve the goals.
CHAPTER 2 LITERATURE REVIEW

The goal of this chapter is to situate the dissertation in the context of a larger body of research, as well as provide support for and connections to this study. In the chapter, I review the research literature related to 1) mathematics education research in the area of differential equations, 2) mathematics and science education literature on rate and time, including the calculus education research literature and 3) mathematics and science education research literature about research connected to parametric reasoning. The differential equations research is critical to review and analyze as the dissertation study will offer new theory by analyzing students’ mathematical reasoning in the arena of systems of differential equations. The literature on rate, time and parametric reasoning is significant as well, as parametric reasoning is an unreported area in the literature on the mathematics of change and variation so the analysis depends on much of that research as well as contributing to the body of work.

2.1. Mathematics Education Research in the Area of Differential Equations

Differential equation research that supports or connects with this study is overlapping, but for the purposes of discussion can be divided into five primary areas: 1) students’ reasoning about single differential equations and their solutions; 2) student thinking about graphical and numerical solutions; 3) student understanding of systems to
differential equations and their solutions; 4) technology in differential equations; and 5) student learning in the context of an inquiry-based differential equations classroom (Rasmussen & Whitehead, 2003). The first four areas of this part of the literature review are concerned with student learning from an individual perspective, while in the last area of research, student learning in the classroom, learning is approached from a sociocultural perspective.

2.1.1. Student thinking about solutions to single differential equations.

Research on students’ reasoning in differential equations shows that students correctly reason about graphical solutions to differential equations in certain situations, but the ability to simply give correct solutions may not indicate correct conceptualizations at other times. Artigue (1992) provided evidence that after 35 hours of instruction in solving single differential equations, students were able to successfully match solutions to the differential equations from which they were recovered. These students were successful because they were able to use criteria primarily from calculus to decide their responses. These criteria included connections between the sign of $f$ (in which $\frac{dy}{dx} = f(x, y)$) and properties of monotonicity for solution curves, zeros of $f$ and horizontal slope, infinite limit of $f$ and vertical slope, the value of $f$ at a particular point and the slope of a solution curve at that point, and recognizing particular solutions associated with straight lines in the graphic setting and checking them in the algebraic setting.

However, Rasmussen (1997; 2001) found that although students may give an appropriate answer reasoning from ideas from calculus, they might not have correct understandings of solutions to certain types of differential equations. For example,
Rasmussen reported that behind students’ correct answers there might lay an incorrect conception of equilibrium solution. In particular, three of six students in his study, at various points in their solution processes, conceptualized equilibrium solutions as existing whenever the differential equation is zero. While this is true for autonomous differential equations, it is not true in general. For example, when dealing with the equations $\frac{dy}{dt} = y - t$ and $\frac{dy}{dt} = t + 1$, students tended to reason that $y = t$ and $t = -1$ were equilibrium solutions, respectively, and they used this to guide their work on the matching task even if the differential equations are not autonomous. Zandieh and McDonald (1999) also studied students’ underlying understanding of solutions and equilibrium solutions. Much like Rasmussen’s research findings, 7 of the 23 students in their study also included all values for which $\frac{dy}{dt}$ is zero, generalizing incorrectly the idea of equilibrium solutions to autonomous differential equations. Students may have misconceptions about equilibrium solutions, either over-generalizing the concept to situations in which it does not apply, or not seeing equilibrium solutions as a subset of solutions to a differential equation.

Other researchers also have studied how students learn to solve first order differential equations. In his work, Donovan (2002) suggested using the concept of student-developed schemes as a way to study students’ understanding of solutions to first order differential equations; a scheme is a network of connections between concepts as discussed by Skemp (1986). Students that he worked with developed a scheme to understand and find solutions to first order differential equations. He reported one scheme in which high achieving students thought of differential equations as functions and as “objects to be solved” at the same time. This relates to Rasmussen’s (2001)
conclusion that working with differential equations conceptually entails being able to think about “y” as both a variable in the differential equation and as an unknown function. Thinking of “y” as a variable is an important part of being able to create differential equations from physical laws or from data. Thinking of differential equations and their solutions as functions is important for students as they reason about what meaning the curves one sketches from a slope field, for example, intended to convey. Along the same lines, Slavit, LoFarro, and Cooper (2002) also used Skemp’s definition of relational understanding. They report that in a study in which students are learning about solutions to differential equations, students who have used technology to study solutions exhibit a more relational understanding (understanding at a conceptual and not a procedural level) than students who did not.

2.1.2. Students reasoning about graphical and numerical representations

Looking at another research area, Rasmussen examined the connections students make between graphical and algebraic representations. Rasmussen provided students with the autonomous differential equation

\[
\frac{dN}{dt} = -4N \left(1 - \frac{N}{3}\right)\left(1 - \frac{N}{6}\right)
\]

and the corresponding graph of \(\frac{dN}{dt}\) vs. N. He asked students in individual interviews the following questions: (a) What are the equilibrium solutions? (b) Which of the equilibrium solutions are stable and which are unstable? (c) What is the limiting population for \(N(0)=2\), \(N(0)=3\), \(N(0)=4\), and \(N(0)=7\)? All six interview subjects answered parts (a) and (b) correctly but four of the six students were unable to address part (c). The typical student approach to this problem was to figure out the first two parts by creating a sketch like that in Figure 2.1.
Rasmussen discussed the question of why the students did not answer question (c) because the graph they had created is, from a teacher’s perspective, a sketch of various solution functions. The students’ interviews revealed the answer. Students did not view the sketch they had just created as a plot of the functions that solve the differential equation. In the words of one student, his sketch was “just a test for stability.” These students had learned a graphical approach for determining stability in which the graphs they created did not carry the intended conceptual meaning (Rasmussen, 2001). This connects with Donovan’s work in that the students did not think of the graphical sketches as solution functions, as Donovan suggested his better reasoning students did in his schema work.

Finding these qualitative graphical solutions as mentioned in the preceding paragraph has become an important course of action in current differential equations classes, primarily due to the introduction of sophisticated technology that provides sketches of these solutions. The advent of this technology created a change in emphasis in the curriculum for differential equations and provided a venue for mathematicians and
mathematics educators to conduct research about the attitudes students have toward graphical solutions (to differential equations), student reasoning about graphical solutions and numerical approximations they find, and how they think about and use proof when using graphical representations of solutions.

Habre (2000) found that students do not prefer to use graphical methods and techniques and almost always choose analytic solutions when possible. Students from a third semester calculus class at a large northeastern university where the first half of the course covered multivariable calculus and the second half was devoted to differential equations were his subjects. One of the questions in the interview was, “What comes to your mind when you are asked to solve a differential equation?” The initial response from all nine subjects indicated that the students thought of an analytic solution. Their dominant notion of what constitutes a solution remained in the analytic realm even though a significant amount of class time involved learning qualitative methods that relied heavily on technology to look at vector fields and other graphical representations. Habre’s research lent further support to the claim that student conception that solutions must have an analytical form are resistant to change and that moving to the graphical setting to understand differential equations is extremely difficult. Students’ reluctance to value graphical solutions equally with analytic solutions was likely a result of the mathematical culture that they have experienced in many of their previous mathematics classrooms (Artigue, 1992; Habre, 2000). However, this research concerned student attitudes to different representations of solutions and does not address the type of instruction explicitly. Research in inquiry oriented differential equations class indicates that students actually use a variety of representational methods (Rasmussen, Kwon,
Allen, Marrongelle, & Burtch, in review) and it is possible that the students in Habre’s study may have as well, no matter what their attitude.

Although in recent years, more emphasis has been placed on the graphical setting and mathematics educators have suggested that more work in the graphical setting can potentially contribute to greater conceptual understanding, students who are required to do more graphing may not necessarily develop better conceptualizations. Research findings focusing on student understanding in reform courses indicate that graphical and qualitative approaches do not automatically translate into conceptual understanding. Students may use graphical techniques in ways that are not connected to meaning and students may have preconceived notions of what graphs look like which they utilize when looking at graphical solutions to differential equations (Rasmussen, 2001). For example, Rasmussen (2001) arrived at this finding after interviewing students on a task in which they were given a direction field for $dy/dx = y(x-y)$ (See Figure 2.2) and asked to describe how the limiting behavior depends on the initial point in the $xy$-plane.

![Figure 2.2. Slope field for $dy/dx = y(x-y)$](image)
Half of the six students incorrectly reasoned that solutions starting off in the upper left-hand region would tend to zero as \( x \) approaches infinity. Despite evidence to the contrary, evidence they themselves garnered, these students demonstrated strong intuitive notions that asymptotic behavior would prevail. For example, one student concluded that all the slopes in the first quadrant just above the \( x \)-axis would have a positive slope, but rejected this as irrelevant to the long-term behavior of solutions that initially approached the positive \( x \)-axis because the intuition about asymptotes is that graphs that approach a line will get closer and closer to it, but never touch. This tendency to use intuitive notions of asymptotic behavior has been further documented by Rasmussen and Ruan (in press).

This research on graphical understanding suggests that just adding graphical techniques to a differential equations class may not increase conceptual understanding. The learning environment and norms constituted also contribute to student understanding in the class.

Artigue (1992) and Rasmussen (2001) both found that students might not have correct conceptions about using numerical approximations to approximate exact solutions curves. Artigue found evidence suggesting that students’ mental images of Euler’s method was similar to that of a semi-circle inscribed with a series of line segments. In addition to finding further evidence supporting this, Rasmussen (2001) found some students maintained another inappropriate image of numerical approximations, namely, that numerical approximations “track” the exact solution by using the slope of the exact solution at the start of each new time step.
2.1.3 Student understanding of systems of differential equations

Researchers are also beginning to examine students’ understanding of systems of differential equations. In one study, Trigeuros (2000) investigated student learning of systems of differential equations in two differential equations classes at a small private university in Mexico. Her analysis of the interviews revealed that some students had problems interpreting the meaning of equilibrium, interpreting the meaning of a point in phase space, and seeing the dependence of time in the phase space. Students in her study also showed a tendency to focus on just part of the information provided by phase portraits. Only a few students analyzed long-term behavior of solutions in relation to equilibrium solutions.

Trigeuros (2004) also reported on students’ understanding of straight line solutions to a linear system of differential equations. She conducted interviews with 12 students after instruction on solutions to systems and reported that only one had a complete understanding of straight line solutions as analyzed using a framework that places student work into a categorization of inter, intra, and trans modes of understanding. Her primary conclusion was that few students exhibit a strong understanding of solutions to differential equations. Unfortunately, she does not provide more analysis into why this might be, an area that this research report will address.

Whitehead and Rasmussen (2003) proposed that students could reason about and develop conceptualizations for systems of differential equations using mental operations. They documented students’ use of conception of rate as a reasoning tool, students using quantification as a mental operation, and third, students enacting what they called a function-variable scheme in their efforts. This earlier research on student understanding
of systems is included in the dissertation, but with learning refocused as mathematical activity instead of mental understandings.

2.1.4 Technology in differential equations

There is research evidence that graphical representations promote better understanding in a differential equations class. In Habre’s study (2000), students used computer modules designed for specific course goals and intended to introduce students to specific concepts. Although this was not the focus of his research, Habre’s interview data suggests that these modules might have been helpful to students in their development of mathematics in the graphical setting. For example, when given a vector field for the system of differential equations $x'(t) = -x + 4y$, $y'(t) = -3x - y$, all the students he interviewed were able to draw an appropriate trajectory in the $xy$-plane and draw reasonable $x(t)$ and $y(t)$ graphs corresponding to this trajectory. Some students, however, were not able to draw the curve in three dimensions. Habre suggested that the role of such computer modules in student learning warrants further study.

In another study, Andresen (2004) reported on the use of computer algebra systems in differential equations. She studied the development of flexibility in thinking in a class studying differential equations and using the CAS Derive as an integral part of the course. Flexibility is described as a student's ability to change between perspectives and forms of representations as they reason about mathematics; in this case, she studied student’s perspectives of solutions to differential equations. Her research indicated that the use of Derive was a positive influence in students’ ability to develop flexibility,
particularly in their movement between types of representations when solving differential equations.

2.1.5 The Differential Equations Classroom

Finally, research from a sociological perspective has been conducted in inquiry-based differential equations classrooms similar to those used in my study and so provided an important basis for considering the class environment and methodology used in this study. In two different semester long research studies, Rasmussen and colleagues (Rasmussen & King, 2000; Rasmussen & Marrongelle, in press; Rasmussen, Yackel, & King, 2000; Stephan & Rasmussen, 2002) investigated the feasibility of adapting research-based approaches known to be effective in promoting student learning at the K-12 level to inquiry oriented university level teaching. In particular, they considered the theory of Realistic Mathematics Education (see Chapter III for elaboration of this theory). Some of their findings explicated the role of explanation and justification as an important part of classroom discussion in inquiry-based classrooms. The following classroom norms, critical to the success of their project in terms of student learning, were initiated by the instructor and sustained throughout the semester. Students routinely explained their thinking and reasoning (versus just providing answers), listened to and tried to make sense of other students’ thinking, indicated agreement or disagreement with other students’ thinking, and responded to other students’ challenges and questions.

This research team documented how these evolving norms fostered a shift in student beliefs about their role as learners, about their instructor’s role, and about the general nature of mathematical activity. These beliefs shifted from seeing their role as
passive absorbers of information to active participants in knowledge creation. When the classroom is viewed as a dynamic system that includes the way in which students participate in mathematical learning, we can account not only for how student beliefs evolve and develop, we can also promote student beliefs about mathematics more compatible with the discipline itself (Yackel & Rasmussen, 2003).

The research on differential equations learning has significance for this dissertation because it provides some background on student understanding in differential equations. Prior research on individual student learning has primarily focused on first order differential equations conceptions and misconceptions. The dissertation research will examine the lesser documented learning in systems. In addition, the research on learning in social context discussed in this chapter provided support for the research on students’ participation in the class.

2.2 Mathematics Education Literature on Rate and Time

2.2.1 Piaget and others’ research on time and speed

One of the first psychologists to study speed (rate) was Jean Piaget (1971); he conducted several influential studies with young children using experiments that he created. The following is an example of the kind of experiment Piaget performed: Place two trains side by side and start them at the same time at the same speed. Then stop them at different endpoints and ask children of different ages which train a) was going faster or b) went farther or c) went longer in time. Piaget found that many young children decided their answer to which train is going faster by using the information about which train
Piaget defined this as an ordinal notion of speed because it comes from a child’s intuition of objects’ passing each other. Thus, Piaget concluded that judgments of “how fast” (that is, speed) is a more primitive notion than time. He proposed that the perception of speed, different from the notion of speed, comes from the physical act of seeing objects pass each other, or from objects that pass over static locations as our eyes watch the movement. The concept of time comes after the perception of speed and is mentally constructed by children later than conceptions of distance and speed. To sum up, Piaget drew the conclusion that speed was a concept that develops in children before the concept of time, as the children were able to make those judgments about “faster” before “longer” in time (Piaget, 1970).

Piaget was not the only one to do research about when students develop understandings of speed. In the second half of this century, other researchers have repeated Piaget’s experiments (e.g. Siegler & Richards, 1979; Trowbridge & McDermott, 1980). The results of these confirmatory experiments supported many of Piaget’s conclusions, except that the ages that Piaget suggested that children develop understanding of speed, time, and distance were not affirmed; Piaget had suggested their conceptions are present by age 11 and the later research did not confirm this. These replication experiments were developed using more scientific protocols to help counteract the weaknesses that scientists attribute to Piaget’s work. For instance, Siegler & Richards (1979) carefully monitored the experiments to hold constant the questions asked of the subjects and to repeat the same set of questions to each person for consistency in the results, something Piaget did not do. They also used the same children consistently in the experiments as opposed to Piaget, who varied the subjects. Even with the more controlled
scientific protocol, the researchers obtained similar results. Trowbridge and McDermott’s (1980) work also intended to provide evidence about whether students understand time and then distance or distance and then time after speed, but their research indicated that the conceptualizations of time and distance develop together.

The research about rate and time done by Piaget was intended to learn about children’s and adults’ understandings of speed as a part of their human development. Piaget and those who followed him were looking to find at what ages children can make judgments about speed, time, and motion, and not why or how these understandings occurred. In other words, Piaget’s work and those that followed gives us an idea of when, but less insight about why or how these understanding of time, speed, and distance develop.

During the last thirty years, more research has been done to study students’ conceptualizations of rate; interest has centered on how these conceptualizations develop and are used in mathematics and science. For example, thinking of rate as a way to quantify the covariation of two or more quantities in a given situation is one framing of the rate concept. There are other ways to think about rate as well: rate as a quantifiable ratio of two values with different units, rate as an intuitive understanding of a quantity changing over time, rate as derivative. I next consider these different approaches that mathematics educators have used as it connects to this study on parametric reasoning.

2.2.2. Definition of rate and frameworks used to study rate

In this section I look at how prior research has defined the concept of rate and how it compares to the concept of speed from Piaget’s work. According to Thompson
rate can be thought of in two different ways—both as a ratio of two values and as an individual quantity that gives a value for rate of change. This resonates with Schwartz’s (1988) notion that two kinds of quantities exist, extensive and intensive. Extensive quantities are those that are understood by people as single entities existing by measurement or intuitively from experience. For example, extensive quantities might be the distance an object travels or how much time passes or the sweetness of a drink. Intensive quantities are formed by mental operations on extensive quantities and become quantified on their own. The concept of rate is one example of an intensive quantity using this definition—it can be defined as the ratio of two extensive quantities. This definition of rate as an intensive quantity and how it is complementary to rate as an extensive quantity is an appropriate perspective for the frameworks to be discussed next.

Confrey and Smith (1994) examined student conceptualizations of rate as they fit within the perspective of multiplicative reasoning. Multiplicative reasoning was described as student thinking based on experience that occurs when they use the operations of multiplication or division to form new quantities. To illustrate the idea of multiplicative reasoning, Confrey and Smith (1994) offered the action of “splitting” objects into two and understanding this as the dividing of some thing into halves, or “magnification” as some thing getting bigger by a multiplicative factor instead of an addition factor. Traditionally, multiplication has been taught as the composition of repeated addition, but Confrey and Smith suggested that repeated addition is not the only perspective for students to learn multiplication. Unitizing, the process of fusing together processes, graphs, or sets of numbers and understanding them as a unit (Lamon, 1996), can happen with multiplicative reasoning as well as additive reasoning. Students can
understand multiplicative units in terms of splitting units, stretching, and/or shrinking objects or values. Confrey (1994) then extended this idea to that of rate; she defines rate as a “unit per unit comparison” and elaborates this definition as follows:

- *Per* implies a “one unit for one unit” comparison that must involve the recognizing of a rate as a composite unit.

- Covariation is central to understanding of rate when considered using a table approach. Rate involves understanding that as each t-value increases by 1, the y value changes multiplicatively by a set value, namely rate.

- Rate understanding becomes more sophisticated by combining rates, which extends eventually to rational units and an equivalence understanding of \( \frac{p}{q} \) with \( p \) and \( q \) being units constructed themselves. (p. 153)

Confrey and Smith (1994) highlighted two tenets from their findings. First, the \( p \) in the rate discussed as \( \frac{p}{q} \) may be multiplicatively constructed and \( q \) additively constructed in the same situation. For example, when a student considers the rate a car traveling in miles per hour, she may conceptualize the hours going up by one (1 hour, 2 hours, 3 hours, etc.) but think of the distance multiplying as 25, then twice that (50) then 3 times that (75), etc. This idea was different from the more accepted theory that rate is defined as a ratio of two different unit values that are both additively constructed (see Thompson & Thompson’s delineation of speed length, 1994). The authors contended that providing students with instructional activities to help with this multiplicative construction of rate will enhance student understanding of rate. Second, Confrey and Smith’s proposed that rate conceptualization, especially variable rate conceptualization, be grounded in the use of graphs, tables, and students’ experiences which allows the
understanding of rate to be deeper than the more historical view of learning rate as a homogeneous quantity (Kaput & West, 1994).

Carlson, Jacobs, Coe, Larsen, and Hsu (2002) used the concept of covariational reasoning as another way to analyze students’ mathematical activity with rate as they studied university level students’ conceptualizations of rate in calculus. Along with other educators, they proposed that students’ development of covariational reasoning is imperative for understanding in calculus (Thompson, 1994; Saldhana & Thompson, 1993; Zandieh, 2000). Carlson et al. defined covariational reasoning as “the coordination of the two quantities; then tracking either quantities’ values with the realizations that the other quantity also has a value at every moment in time.” Carlson et al. (2002) did not focus on the dynamical notion of time but it is implied in their work, and provides background to the concept of parametric reasoning in this study. The rate relationships appear to use time implicitly as the levels of covariation are presented.

Covariational reasoning, as described in their study can be evaluated using a series of five levels of mental actions. Briefly, students develop images about rate in this approximate hierarchy: Mental action level 1-coordination, level 2-direction, level 3-quantitative coordination, level 4-average rate, and level 5-instantaneous rate. These rate relationships are developmental and involve two varying quantities in which one quantity’s change is determined by the other quantity’s change. This description is meant to have a dynamical aspect to it. Coordination implies recognition by the student that the two quantities are connected. Direction attributes to the student an understanding that as one quantity increases, the other one increases or decreases. Quantitative coordination is the conceptualization that as one quantity changes, there needs to be recognition of “how
much” the other quantity changes. As one quantity changes, the changes in the other are related and varying. The last two mental actions, **average rate**, and **instantaneous rate** involve understanding conceptually how to think of rate of change in two different and important ways. Average rate means thinking of the covariation as a ratio of changes. This connects to the idea of secant lines in calculus. Instantaneous rate is thinking of covariation as existing at each moment continuously and relates to the concept of derivative in calculus.

In one task used in Carlson et al.’s research on covariational reasoning, the students were asked to look at a picture of a flask and sketch a graph of volume versus height, assuming the water is pouring into the bottle at a constant rate. Heid (personal communication, 2005) reports that while using this task in her research, students regularly integrated time into their discourse. The students parameterized volume and height as they created the volume-height graph making each quantity a function of time and imagining what happens as time passes. This reasoning with time is what students in my research used as they developed ways to reasoning about solutions to systems of differential equations.

As another framework for student understanding of rate, Thompson and Thompson (1994) presented the following levels of conceptual development that they posit are required to construct the understanding of speed as a rate:

1. Speed is a quantification of motion.

2. Complete motion involves two completed quantities – distance traveled and amount of time required to travel that distance (this must be available to students both in retrospect and in anticipation).
3. Speed as a quantification of completed motion is made by multiplicatively comparing distance traveled and amount of time required to go that distance.

4. There is a direct proportional relationship between distance traveled and amount of time required to travel that distance (p. 283).

This framework proved to be a powerful construct to analyze individual understanding of rate. By explicitly focusing tasks on the different levels, students may develop their understanding of rate as an intensive quantity that is formed by comparing distance and time. This comparison must be constructed both with whole numbers and parts of whole numbers as well. For example, a student might have a sense of a distance and the time it takes to go that distance. That ratio then is considered a rate. Then according to the framework, the student would understand as well that going half the distance takes half the time, or going 4/5 the distance takes 4/5 the time and the rate remains the same in all those travels.

Evolving from the framework was Thompson and Thompson’s concept of *speed length*. A student often thinks about rate as the amount of distance traveled in one unit of time. For instance, when a teacher says 50 kilometers per hour and asks a student to explain what this means, students often think about it in terms of 50-kilometer units. A 50-kilometer unit is an example of “speed length.” This reasoning can provide a basis for further development of rate in situations with non-unit time increments. Even when more sophisticated notions of rate emerge, however, students may fold back (Pirie & Kieren, 1994) to the concept of speed length. To further elaborate, Thompson and Thompson (1994) analyze data from a teaching experiment in which an experienced mathematics teacher worked with a sixth-grade student and they concluded that
when children have internalized the measurement of a total distance in units of speed-length they can anticipate that traveling a distance at a constant speed will produce an amount of time. This implies that children first conceive speed as a distance and time as a ratio (total length/speed length)” (p. 2).

To summarize, Thompson and Thompson were not suggesting that there was a linear progression or a list of levels such as Carlson et al. suggested in the covariational framework, but rather these delineations are different ways by which students conceptualized rate. They suggested that a student understands speed in a personal way as he or she gives value to motion. The student also must move to an understanding of the relationship of distance, speed, and time that involves proportional reasoning among the three. As a student thinks of an object moving with a constant rate, he or she understands how as time changes, distance changes and in the same proportions.

An analysis of the three frameworks for interpreting student thinking about rate reveals commonalities and differences. One point of comparison particularly relevant to my research is the use of the quantity time. In Confrey and Smith’s framework for understanding rate, time may or may not be involved in the development of rate, as they use the idea of a unit by unit comparison. Thompson and Thompson’s (1994) and Carlson et al.’s (2002) frameworks also do not require time in the statement of their framework; however, their research contains primary emphasis on thinking of rate as speed thus implying the use of time.

All of these frameworks have different perspectives about whether rate is to be studied as an intensive or extensive quality. Thompson and Thompson primarily considered rate as intensive, a relationship between quantities, and their research revealed
how students might construct the homogeneous idea of a rate by taking different ratios of distance and time and eventually conceptualizing the rate as the constant ratio, no matter what time-distance are involved. In a different way, but still as an intensive notion of rate, Confrey and Smith (1995) rejected ratio as an instance of a relationship between quantities, claiming instead that ratios are constructed “by objectifying and naming that which is the same across proportions, i.e., to construct a ratio, one needs to first identify what is the same across more than one instance” (p. 74). Carlson et al. actually did not deal with rate in either way, and so the perception of extensive and intensive quantity is peripheral to their framework, in that both are present but not emphasized.

Another way to compare these frameworks is to look at the level of sophistication of reasoning about rate. Thompson and Thompson’s and Confrey and Smith’s frameworks considered mental actions that organize students’ conceptualization of rate as it develops for children as early as the elementary and middle school levels. Thompson and Thompson’s framework took the view that student’s quantify by first using speed length, which is an additive concept of rate in which speed length eventually evolves into a more ratio based concept of rate. Students come to think about “as time passes, the proportion of time that passes is the same as the proportion of distance, and this proportion is rate.” This understanding develops at different times for different students, however. Carlson’s framework was developed for students studying calculus, so mental actions required for understandings of covariation advance into an understanding of instantaneous rate of change, or derivative conceptualization, a notion which neither of the former two frameworks address. Carlson’s structure is also fairly overarching, as this
idea of covariation at mental action levels one through three can be constructed by children considerably before calculus is learned.

Finally, these frameworks have different purposes, each of which has significance in the research about rate of change. Confrey and Smith’s extended definition of rate elaborates on their construct of multiplicative reasoning. Rate was considered an object that becomes reified (Sfard, 1991) as students grow to understand the ideas of multiplicative reasoning. Thompson and Thompson (1994) also hypothesized that students’ development of rate eventually comes to be about how they reason with a constructed notion of rate and propose the levels of understanding that students experience during this process. Carlson’s framework, on the other hand, was about process, but does not have an object as an end result of the process. Instead, she provided the notion of mathematical mental actions and finding ways to evaluate these actions to determine a students’ mathematical thinking. Rate of change is implicitly present in the sense that the rate of change is always in the process as covariation of functions is articulated and examined in terms of students’ mental actions.

I now turn to other research about students’ understanding of rate that does not use developmental frameworks to interpret and understand their analyses. This additional research falls into three categories: kinematics studies, studies of rate related to graphing and research about student thinking in the mathematics of change.

2.2.3 Kinematics Studies

Several studies in Physics education, (e.g., Brasell, 1987; Beichner, 1990; Hammer, 1995) provided information about students’ conceptualizations of rate in the
area of kinematics, the study of distance, velocity, and acceleration. These studies involved use of the motion detector coupled with either calculators or computers. In this research literature, the students collect data about distance versus time (and then also analyze graphs of velocity and acceleration). The research questions involved whether students can, using a motion detector, develop better understandings of distance and of velocity and the conceptual distinction between the two ideas.

To illuminate this research, I note that these reports documented students’ difficulties with interpreting graphs in physics and mathematics (e.g., Clement, 1985; Halloun & Hestenes, 1985; Minstrell, 1982; Trowbridge & McDermott 1980). Other mathematics education research had shown that misconceptions about graphs of distance and velocity with respect to time are common. Students often conflated steepness and height of a distance versus time graph, which actually corresponds to confusing velocity and distance in real life experiments. Also, students sometimes looked at the graph of an event and interpreted it as the picture of the event, instead of a record of the numerical happening in a graphical form (Fernandez, 1998; Herscovics, 1989; Mevarech & Kramarsky, 1997).

The physics research discussed here provides some opportunities for students that might help them with their distance and velocity conceptualizations identified as difficult. Brasell (1987) reported that the use of motion detectors in real time does help students as they grow in understanding. Her research was based on tasks in which students worked independently to create graphs by walking; she reported on their conceptualizations of distance, velocity, and acceleration from the experience. She found that students who created graphs using their own walking performed better on problems about distance and
velocity than those in a control group who did not. Interestingly enough, she found that those who created graphs but were not allowed to see them being created while walking but saw them after a time delay did NOT do better on test questions about distance and velocity. The implication here is that students who see the graphs as they are created develop a better understanding of velocity.

One might hypothesize that students who experience rate (velocity) along with the representations of such are better able to reason about rate. This is supported by the work of Kaput in mathematics education. He described students’ experience with the mathematics of change and variation and provides evidence of children’s ways of reasoning about distance, velocity and time before it is traditionally taught in calculus (Kaput, 1999). Kaput’s work with middle school students’ using an interactive computer environment to experiment with piecewise linear velocity graphs offered evidence that students’ experience with constant velocity graphs can be helpful for the students as they grow to study rate of change in algebra, and calculus. Students can make connections between velocity in these virtual worlds and the velocity (rate) in other mathematics.

2.2.4 Rate Related to Graphing

Research on graphing and how rate (slope) connects to graphing has extended into the questioning of shapes of graphs. Research indicates that students’ understanding of shapes of graphs is an interrelated concept to students’ understanding of varying slope. Students often conceptualize rates increasing and decreasing by describing how the graph of the function that has the related rate might look. Graphs, for instance, that are concave up indicate to students an increasing slope or rate of change and concave down indicates
to students decreasing slope or rate of change (Stump, 2001). Stump conducted research about individual students’ understanding of slope as measure while learning precalculus. She found that students’ understanding of time grounded their understanding of slope in problems that were time-related, but when using slope and rate of change in functional situations (taken out of a context and using traditional function teaching) their understanding was weaker. Steepness of graphs did not connect well with rate of change. She suggested that her results provide evidence of “a gap in students’ understanding of slope as a measure of rate of change and imply that instruction should be focused on helping students form connections between rates involving time, rates involving other variables, and graphical representations of these relationships.” (p. 87)

Another study indicates that the key to why students can use the motion detector to help develop conceptualizations of distance and velocity is the amount of control that students can exert in determining the activity with the device (Brazelton, 1987). Students who have control are much more likely to develop concepts of rate that are more complete and useable in other situations as they work on ideas and thoughts that they come up with themselves. These experiments support the contention that the experience of motion is an important part of students’ development of rate conceptualization, especially when it involves graphing.

2.2.5 Student thinking about change

Research at TERC has been conducted for almost twenty years in what is referred to as the “mathematics of change;” time based reasoning is a subset of the mathematics of change. Part of the underlying rationale for this long-term project is that calculus, in
which one of the two central concepts is derivative or instantaneous rate of change, should be informally introduced to students in the curriculum much earlier than late high school or college. The researchers at TERC developed instructional tasks and research projects to study situations in which students participate in the mathematics of change. Several of their research reports offered ways for students to experientially understand calculus concepts at elementary and middle school levels. Nemirovsky, Monk, and colleagues’ research suggested that students’ personal experiences with change can grow into understandings of concepts such as velocity, understanding of slope in distance versus time graphs (Nemirovsky, Tierney, & Wright, 1998) and other attributes of change, including rate (Monk & Nemirovsky, 1994). In the same vein as Kaput (described above), these researchers indicated that rate as an extensive quantity experienced in a mathematical environment supports students’ development of a meaningful understanding of rate when used in upper level mathematics classrooms.

TERC is but one research group conducting explorations on ways to provide opportunities for students to learn about rate of change. Researchers in the SIMCALS project have also been conducting studies using software designed to connect students experience and more formal representations of distance, velocity, and acceleration. For example, Bowers, Nickerson, & Kenenhan (2002) examine the use of microworlds to provide instructional activities for students to develop understanding of speed. These researchers focused on the development of rate using SimCalc (an exploratory software tool). Formal instruction on rate has usually involved rate as an intensive quantity defined as the ratio of two different extensive quantities such as miles per gallon or feet per second. Bowers and colleagues suggest that rate as an inherent quantity of some thing
might be focused on as well. They developed activities in which students work on speed in terms of using the students’ own experiences in a dynamic and real world context to develop understandings using technology; that is, the idea of speed has meaning on its own, not as a ratio, but as its own value. Students experience with this microworld led them to develop personally meaningful ways of understanding rate and thus its graphical representation, slope. In the SIMCALC work, velocity versus time graphs are frequently created by students, or used as tools in reasoning. This seems to make a difference in their understanding of velocity as an extensive quality—rate of change as a quantity that can be experienced as a single quantity.

2.2.6 Rate as a constant or rate as a constantly varying quantity

The conventional way to teach rate in school has been as constant and derived from a ratio (Confrey & Smith, 1994), but there is research on student thinking about rate as varying as well. In this section, I discuss both rate as a constant and rate as a varying amount as two harmonizing ways to learn about the concept. I look at constant rate first and then the complementary idea of rate as a constantly changing quantity.

Students are typically taught the \( r = \frac{d}{t} \) formula and learn to find rate using that formula or similar methods. Kaput and West’s (1994) proposition about the reification of this idea of rate is one of the seminal articles on rate. The authors stated that an important part of the understanding of rate is its homogeneity. They define homogeneity to be rate that stays the same no matter what interval of independent variable one is considering. For example, if gas mileage is 25 miles per gallon, then whether one uses one gallon, or ten gallons, whether one drives 40 miles or 100 miles, the rate does not change and this
quantity can be used to reason about quantities in a situation. Kaput and his colleagues provided a way to help students with the idea of homogeneity. They suggested that instructional materials and activities encourage students to think about rate (speed) in terms other than whole numbers. This SIMCALC work allows students to revise their notions of speed to include non-integral numbers. Their research studies suggest that integrating the fractional values for time into the activity helps students with thinking of rate as a homogenous value and not always speed length (Thompson, 1994).

Because of the value of studying rate as it occurs in real situations, researchers offer the possibility that, to conceptualize rate and understand derivative when it arises at a precalculus/calculus level, students might benefit from activities and materials that work with varying rate during instruction before calculus. As mentioned earlier, Confrey and Smith called for more research on student understanding of varying rate. “These analyses suggest that a second understanding of varying rate lies in the coordination of one’s experiential knowledge of slope and changing slope (as steepness) with its visualization on a graph.” (Confrey & Smith, 1994, p. 157).

Confrey and Smith (1994) suggested that when solving routine derivative problems, students are limited in their understanding of rate because the contexts have been limited to situations with constant rate and that instruction needs to involve a broader range of experiences with variable rate. This proposition is restated and justified in another research report, in which Hauger (1997) looked at the rate understanding for precalculus students. In her report, Hauger used rate of change as an attribute of functions and researches how students understood rate of change by interpreting the two types of representations: graphs and tables of values. She found that students did not have
sufficiently strong ways to think about speeding up and slowing down to create graphs or reason about speeding up or speeding down from tables. They also had considerable difficulty in creating graphs to illustrate speeding up and down. Students have traditionally studied rate of change in which constant slope was taught first; thus students had not developed a way to graphically represent these ideas of varying rate. The idea of varying rate is important to study derivative understanding, which I discuss next.

2.2.7 Rate as derivative

Zandieh (2000) elaborated a way to evaluate mathematical activity about derivative by using the traditional definition of derivative, \( \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} \) and its many manifestations. Her research report included a way to evaluate student work by considering which facet of derivative a student uses and at what level of understanding the work shows. She stated “the derivative function acts as a process of passing through (possibly) infinitely many input values and for determining an output value given by the limit of the difference quotient t that point” (p. 107). She proposed different aspects of derivative: graphical, analytical, numerical, conceptual, etc. Then she posited that students show they have reified derivative into an object from a process or may show a pseudo-object understanding instead. Her analysis corresponded informally to understanding rate as a ratio moving from average rate of change to instantaneous rate of change; this also generally matches rate as an intensive and extensive quality. This framework can be helpful for instructional designers and teachers as it posits the process
of how students construct the idea of instantaneous rate of change, with emphasis on context as important for student learning.

Carlson et al.’s (2002) framework included instantaneous rate of change which involved thinking about the formal definition of derivative, but primarily provided ways to analyze student work in terms of thinking about covariation at different conceptual levels. Students’ mathematical activity with routine and non-routine graphically oriented tasks can be analyzed for understanding and evidence of students “running through a list of values for \( x \) and assigning rate values for \( y \)” is evidence of understanding at the what the researches call a mental action at the fifth level, which indicates understanding of instantaneous rate of change (derivative). The researchers note that mental action level 4 involves secant line understanding, which students use to think on intervals, and mental action level 5 is evident when students’ activity is covariation changing continuously.

Schwalbach and Dosemagen’s (2000) work in which they suggested that contextualizing rate is a means to improve understanding of derivative shows evidence of success. In their report in which physics concepts are integrated into calculus, student’s abilities to make valid explanations of derivative improved when use the derivative was embedded in situations.

Another area of research about understanding of derivative is conceptualizing rate as a function or a variable. Rasmussen documented this idea in his work on differential equations (Rasmussen, 2001). He extends the work by Carlson (2002) Zandieh (2001), and Thompson (1994a, 1994b), in emphasizing that rate takes on different forms in different situations and contexts. In differential equations the derivative is often a function, one that is derived from the instantaneous rate of change for a given function.
However, students also need to be able to think about derivative as a variable in its own right. This resonates with research reports that suggest that reasoning about rate as an intensive quantity can be helped by certain mathematical activity. It is slightly different, however, as Rasmussen’s work with rate as a function does not stop at the ratio concept, but considers also rate as the instantaneous rate varying constantly in a functional form.

To understand fully a differential equation and its family of solution functions, students must be able to go back and forth between using the derivative (rate of change equation) as both a function and a variable on its own (Rasmussen, 2001). Rasmussen and Ruan (in press) also offered a framework to think about rate in which they suggested that as students’ reason about solutions to differential equations, sometimes that rate is used as an adjective to describe a function and sometimes the rate of change is an object that exists and can be reasoned about on its own.

To conclude this section, I mention early studies of student understanding of derivative in calculus from a misconceptions standpoint. The early research in calculus indicates that students have difficulty solving problems in which they need a strong understanding of derivative to be successful (Orton, 1983). Benzendoit (1998) documented students’ misconceptions of derivative, suggesting that the understandings they have do not give students ways to reason conceptually about different calculus problems, even if they can use formulas to come up with derivatives.

2.2.7 Contrasting theories of rate understanding

In the next section, I provide an analysis of how contrasting views about rate are seen in the research on students thinking about rate of change. The purpose of this section
is not to suggest that these dichotomies divide the research into correct or not correct, but only that much of the research can be thought of as two sides of the same coin. I look at the complementary sides of three rate issues: rate as an intensive or extensive quantity, rate as a quantity or object or reasoning tool, and rate as constant or varying.

2.2.7.1 Rate as intensive or extensive

The first contrasting theory that structures the research is the question, “Is rate an intensive or extensive quality?” The traditional research on children’s understanding of rate and the derivative literature often treat rate as a derived quantity in which the ratio of two extensive quantities become rate to students. Researchers tend to look at the difficulties and misconceptions students show with this rate construct. Kaput looks at this in his work on rate, as do Lesh (1988) and Lamon (1992, 1994) in analyses of students working word problems about rate. The studies of students learning slope in algebra also considers the perspective of students developing the notion of rise/run, a constructed value of slope on rate. In work on the derivative, the same intensive quantity construction is emphasized. Earlier work in physics education also emphasized rate as an intensive quality, in that it focused on students’ understanding of speed as the rate of change of distance with respect to time, usually with graphing as part of the picture; however, more research in physics on rate that focuses on using technology employs rate as an extensive quality. Finally, the research on how students understand derivative tends to look at derivative as an intensive quantity that is constructed as a limit of the average rate of change, an extension of the intensive quantity perspective. Research such as Bezuidenhout’s work (2001) provided evidence as to whether students’ conceptions
allow them to work traditional and nontraditional problems in calculus, thus reporting on students understanding of the limit definition of derivative.

The idea of conceptualizing rate as the quantification of motion is more apparent in the research on how children come to understand rate as an extensive quality. For instance, Piaget and those that followed him considered speed as a primitive concept that children abstract from their experiences. Such experiential bases for speed continues in more recent work with motion detectors in mathematics and science classes, finding ways to connect students’ experiences with the notions and representations of rate (e.g., Noble, Nemirovsky, Write & Tierney, 2001). Other research looks at adults, specifically preservice teachers’ development of the notion of steepness. Simon and Blume (2001), for example, claimed that even as university students, people do not have a sense of steepness that comes from their experience. Therefore, even though these students may perform well on standard questions, providing them with ways to construct steepness from their own understanding may allow for better understanding. A third research study about derivative concerns a teaching experiment in which high school students are given an environment to experience motion and use it to create representations of motion, particularly instantaneous rate of change (Speiser & Walter, 1994). Students had ways of thinking about motion that eventually grew into ways to think about and reason with derivative.

As a connection between developing understanding of rate as intensive and extensive quantity are the frameworks for student understanding of rate. These frameworks attempted to classify student reasoning about rate in terms of a set of not necessarily hierarchical stages. Both views of rate are presented along with connections
to complete a whole understanding of rate. For example, in Thompson’s inventory of what it means to understand rate, the first concept is that students know speed as a quantification of motion, but the last way of reasoning is the construction of rate as distance/time. Zandieh’s ideas about derivative utilize both contextual and real world settings as well as analytical limit ideas.

2.2.7.2 Rate as quality, object or tool

The second contrasting theory that partitions researchers’ theories of rate understanding for students involves the question of how they think about rate: as a quality, as an object on its own, or as tool to use. For instance, rate is often considered as the attribute of a function or an object in motion. With this conception of rate, rate describes something else, such as distance traveled, or the behavior of a function, or how the population of an animal is changing over time. Students can come to understand rate by mathematical activity with rate in context or by working in experientially real situations.

Rate can also be thought of an object on its own. As mentioned before, this object can be developed as a quantity with or without being conceptualized as a ratio (discussed earlier). Zandieh’s research on derivative, as well as Thompson’s and Carlson’s covariation research is examples of students building understanding of rate as object.

Finally, rate can be a reasoning tool. In their work, Rasmussen and Whitehead (2002) introduced the idea that rate understanding in mathematical activity allows students to reason about mathematics in different ways. They proposed that students use rate as a tool to further their understanding in differential equations; students use their
experiences with rate to reason through understanding of differential equations and solutions to differential equations. Using rate to reason resonates with the other research about rate, but adds to it as well. For instance, students think about solutions to differential equations by using the derivative as the ratio $\Delta y/\Delta x$. The students think of the solution as a set of points that are generated by moving along different paths depending on the rate as a ratio at one point for a length of time to get to the next point (Euler’s method to numerically generate a solution). This can be a powerful way to understand a solution function and is similar to thinking of rate as a ratio in the other frameworks. It also adds the quality of thinking about rate in a way that allows one to make sense of new ideas as well.

2.2.7.3 Rate as varying or increasing

Research on students’ conceptualization of rate as varying is increasing. This research tends to be based on a more experiential perspective. Students develop understanding of rate grounded in their personally meaningful real life and mathematical experiences. This research is connected quite often to the extensive view of rate; this is as opposed to the constant ratio research, which is connected more to the intensive view. The derivative research fits into both of these perspectives and tends to link the two.

2.2.8 Summing up research on rate

There are studies about students’ understanding of rate as part of the study of graphical and table representations of functions. Researchers have studied how students develop an understanding of slope for linear functions, as well as understanding of what
graphs of functions look like for situations of varying slope. The research on students using motion detectors and computer software to experience rate is ongoing. There is research on derivative and the conceptions and misconceptions that students have in that area.

Stroup (2002) introduced the idea of qualitative calculus to talk about rate in different ways. He stated, “Perhaps most distinct from the standard approached to calculus, is the way in which qualitative calculus centers on the construction of an intensive understanding of rate that is both powerful, yet not organized as a ratio of change” (pg. 120). As he pointed out, mathematicians have traditionally thought of students understanding of rate as “maturing,” that is, changing into an analytic conception of rate of change as derivative that is the “correct understanding.” He suggested that “these discussions of rate can be see as building on a similar set of intra-related ideas and that qualitative calculus is a powerful, even foundational, form of reasoning about how much and how fast.” Stroup stated that then these two kinds of rate can exist side by side and are both important to a strong and deep conceptualization of rate. In working with differential equations and student understanding of derivative as a function and a variable, both of these conceptions need to come together and support each other. One kind of understanding does not have to retreat to the background in terms of the other. In fact, instructional activities can pull on both of these ideas as students have opportunities to develop understandings of rate and differential equations, particularly differential equations.

At this point, there is a significant body of work on students’ conceptualization of rate. Rate as a ratio of two quantities takes up much of the research in the last 20 years.
Research on student thinking about rate as a constant ratio that exists as a construction from two other quantities includes ideas of proportional reasoning, as well as research on students solving conceptual word problems. Not exclusively, but in general, this research often connects to the more traditional instruction about rate. Many of these researchers begin by defining rate and then using this formal conceptualization as part of the research perspective.

The general findings from all the rate research are extensive and focus on how students come to understand rate and its variations, as well as difficulties students have with learning rate. Students have intuitive notions of rate and some of the research reports that students have qualitative intensive understandings of rate that can be built on. This dissertation will report on further elaboration of this idea, as parametric reasoning is characterized as a type of mathematical activity that emerges from students’ intuitions and experiences. I use this rate research to support and enhance the parametric reasoning characterization as I add to this body of literature.

2.3 Parameter and Parametric Reasoning

Students understanding of parameter is an issue that has received relatively little attention in mathematics education research. The literature occasionally mentions parameter as one kind of variable and discusses how students come to use parameter. In this section, I will discuss briefly the research on parameter and students’ understanding of parameter. Then I show how my research study contributes to this literature by offering a different way to consider parameter by thinking of it as a dynamic variable and describing how students reasoning with parameter.
2.3.1 Kinds of Parameter

There are two kinds of parameter, each of which are complementary to the other and connect with a kind of parametric reasoning. The two types of parameter are: parameter as a higher order variable (sleeping) and parameter with time as variable (dynamic).

2.3.1.1 Parameter as a sleeping variable

As discussed earlier, Freudenthal (1983) used the idea of a sleeping variable when he described parameter. He suggested that a parameter is a variable that lies quietly, not changing during the course of one problem or situation; then when the situation changes, it wakes up and changes to a different value, then falls back asleep. This idea of parameter as a sleeping variable was expanded by Drijvers (2001). Drijvers proposed that a parameter can be thought of in four different ways

- Parameter as *placeholder* represents a position, an empty place, in which a numerical value can be filled in or retrieved from. The value in the ‘empty box’ is a fixed value, known or unknown, which does not change. This is the ground level.

- Parameter as *changing quantity* concerns the systematic variation of the parameter value. The parameter acquires the dynamic character of a ‘sliding parameter’ that smoothly runs through a reference set. This variation affects the complete situation, that is, the formula as an object, and the global graph whereas variation of an ordinary variable only acts locally.
• Parameter as *generalizer* generalizes over a class of situations. By doing so, this ‘family parameter’ (van de Giessen, 2002) unifies such a class, and represents it. This generic representation allows for ‘seeing the general in the particular’, for the generic algebraic solution of categories of problems, and for formulations and solutions at a general level. This general solution of all concrete cases at once by means of a parametric general solution requires the reification of the expressions and formulas that are involved in the generic problem-solving process.

• Parameter plays the role of *unknown* when the task is to select particular cases from the general parametric representation on the basis of an extra condition or criterion. This often requires a shift of roles and of hierarchy (Bills, 2001).

Drijvers framed student thinking about parameter in terms of the van Hiele levels (van Hiele, 1986). The van Hieles considered the placeholder level as the ground level of understanding the concept of parameter that is the basis for the second level. The three roles of changing quantity, generalizer and unknown share the property that formulas that contain them are considered as objects. Therefore, they are part of the second level understanding. The most important of the three was supposed to be the generalizer, as generalizing is a key activity in algebra. The targeted higher-level understanding, therefore, involved the jump from placeholder to the other parameter roles.

An example of parameter as a sleeping variable, and fitting into the category of parameter as unknown, changing variable or generalizer would be the m in

\[ y = mx + b \]

or the \( \alpha \) in the differential equation...
\[ \frac{dP}{dt} = .1P(1 - \frac{10}{P}) - \alpha \]

In these two examples, m and \( \alpha \) could serve any of the above three functions depending on the activity and the level of students’ mathematical work.

2.3.1.2 Parameter as a dynamic variable

In contrast, a parameter as a dynamic variable is identified as “always awake” or dynamic. It is a quantity that changes continuously and its change causes other quantities to change as well. A parameter may not be explicit in the situation, but plays an important role behind the mathematical situation of the change as it may influence and even define other quantities. One example of this might be the traditional parametric equations that are studied in calculus; two or more functions are defined in terms of time (\( t \)) and then graphed on \( x-y \) planes (with or without technology) without the appearance of the variable \( t \). Another example of the implicit (time) variable comes from differential equations. Students utilize a sketch on a graphical representation called a phase plane (an \( x-y \) plane) in which time is not one of the axes to create a moving and dynamic function in three space where time becomes the third axis; they do this by integrating the concept of constantly increasing time and creating the illusion of movement on the phase plane that is dependent on time. See Figure 2.3.
Another context [happening in algebra reform] involves the control of simulations, in which students need to set algebraic parameters as part of the process of exploring the phenomena of the simulation. For example, when controlling the motion of synchronized swimmers in a pool, students must determine how to distinguish between a positional and a temporal head start; furthermore, in some circumstances they must deal with as many as 20 coordinated swimmers, each of whom is to be offset in their initial position by a fixed distance from the swimmer to their left, say. In this case, to achieve efficient and systematic control of the swimmers begs parametric thinking, in which each swimmer’s motion is a particular function of time, but in which the functions themselves vary systematically across the swimmers.

However, there is overlap between parameter as changing quantity and the dynamic time as parameter. In my research I focus explicitly on the second parameter type with an emphasis on time as an implicit changing quantity in addition to the support and the practices that students participate in when parameter is dynamic. Therefore, the definition of parametric reasoning I use is in this research is: parametric reasoning is developing and using conceptualizations about time as a dynamic quantity that implicitly or explicitly coordinates with other quantities to understand and solve problems.
CHAPTER 3 METHODOLOGY

The objective of this chapter is to describe the methodology I used to address the research goals of my study. Methodology includes the theoretical framework used in the research study as well as the methods used to conduct the analysis (Cobb & Steffe, 1983). This chapter is organized as follows: first, I provide a brief review of the pilot study I conducted to learn about the classroom setting, identify research goals, and develop interview protocols; second, I describe the setting of the study and the data sources; third, I address the interpretive framework that was to be used for data analysis and why some parts were not successful; fourth, I describe the instructional theory underlying the development of instructional activities including a brief description of the activities used in this project; fifth, I present the data analysis procedures.

3.1. Pilot Study

During January through April of 2002, I conducted a pilot study in a differential equations class at the same university where the dissertation research was conducted. The purposes of the pilot study were as follows:

- Become familiar with the research setting.
- Become familiar with the instructional materials.
- Become familiar with the use of technology in this type of differential equations instruction.
- Develop and refine research goals.
- Develop and pilot protocols for interviews.

The setting for the pilot study was an 80-minute bi-weekly differential equations class at a midsized midwestern university. A mathematics educator using an inquiry-oriented approach for the second time taught the class. In an inquiry-oriented class, students are expected to think about the mathematics, articulate their mathematical thinking and make sense of each other’s mathematical thinking. The typical lecture dominated by teacher talk is rare in this approach to teaching differential equations. Instead, students actively reinvent, with the support of the teacher, important ideas and methods by participating in classroom discourse. The validity of mathematical claims primarily comes through student reasoning, rather than from an external authority such as the teacher or a textbook.

As well as being a typical inquiry oriented classroom, there was considerable emphasis on technology. Students participated in this differential equations class by using technology individually and as part of small groups or learning pairs. The TI-92+ calculator and the computer program Nucalc (Pacific Tech, 2000) were used regularly in the class after the first third of the semester. The purpose of technology was to promote student reasoning and conceptual development, not to simply perform difficult calculations or complicated symbolic or graphical manipulations. For example, instructional activities included students creating tangent vector fields using a special calculator program designed to promote student understanding of solutions to differential
equations and experiencing tangent vector fields with the interactive programs such as Nucalc.

Students in the class were primarily mathematics and engineering majors taking the course as a requirement for enrollment in future courses. There were 23 students in the class, 19 males and 4 females. We used one video camera located in the rear of the classroom to record each class session for a total of 24 sessions. The camera was focused on the instructor when he was speaking and on students in the classroom during whole class discussion. Every attempt was made to capture the speaker in these discussions. During small group work, the location of the camera and microphone varied among the small groups.

To help develop the protocol for the dissertation study interviews, I conducted two semi-structured task-based one-on-one interviews with eight students during the course of the pilot study semester. The eight students were selected from those who volunteered by choosing those that could come at the available times and all interviews were videotaped. In the first interview, students worked on tasks that involved the concepts of function and slope and on a task that foreshadowed some of the problems in the course. These tasks were used to investigate conceptual resources available to students when beginning the differential equations course. The second interviews were conducted toward the end of the course as a follow-up to the first interview to further investigate students’ understandings and to pilot additional tasks for subsequent formal research. The interviews were transcribed and analysis of individual student mathematical activity revealed the ways the pupils reasoned about derivative and function (see Whitehead & Dost, 2002).
The pilot study provided me with experience in the learning environment and tools used in an inquiry-oriented differential equations class. I gained greater familiarity with the instructional materials, which were based on the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1995) (discussed later in this chapter). RME served as the guiding framework under which was developed a differential equations local instructional theory. In this approach, students participated in mathematical activity that began with experientially real starting points that serve as a basis for building conceptual understandings of differential equations and solutions to these equations and systems of equations. Co-emergent to this, students developed analytic and qualitative strategies for finding and understanding solutions and spaces of solutions.

The pilot classroom teaching experiment resulted in modifications and refinements of the instructional materials and in the development of instructor notes for the instructional materials for differential equations. It provided me with experience in working in a differential equations classroom that would be similar to the one I would use in my study and support for the classroom teaching experiment design. A classroom teaching experiment (Cobb, 2000) is a research design where individual and social learning are studied by cycles of research and revision; theory informs practice which informs theory. The pilot led to the creation of protocols for the interviews for the fall 2002 classroom teaching experiment where the research study in this proposal was carried out. Finally, during the pilot study, I developed and refined my research goals after observing the class and interviewing the eight students. The students’ mathematical activity as they enacted the instructional tasks focusing on systems of differential
equations provided me with the goals of developing a characterization of parametric reasoning, describing how students come to understand solutions to systems of differential equations in an inquiry class similar to the pilot, and how to describe the ways that the students’ participated in the mathematical community of the classroom.

3.2. Theoretical Framework: The Emergent Perspective

I used the emergent perspective as an initial lens for making sense of the complexity of teaching and learning in the classroom. As stated by Cobb and Yackel (1996), the emergent perspective provides a perspective of “exploring ways to account for students’ mathematical development as it occurs in the social context of the classroom” (p. 176). The emergent perspective merges the theoretical perspectives of constructivism, a psychological theory of learning, and interactionism, a theory about meaning as it is created in social context (Cobb & Yackel, 1996; Wood, Cobb, & Yackel, 1995). Cobb and Yackel (1996) state, “The social perspective is an interactionist view of communal or collective classroom processes. The psychological perspective is a psychological constructivist view of individual students’ activity as they participate in and contribute to the development of these communal process” (p. 176). Further, Blumer (1969) states,

Symbolic interactionism sees meanings as social products, as creations that are formed in and through the defining activities of people as they interact… Accordingly, interpretation should not be regarded as a mere automatic application of established meanings but as a formative process in which meanings
are used and revised as instruments for the guidance and formation of action (p. 5).

The emergent perspective seeks meaning as it is established in the interactions of students and their teacher. Therefore, studying this interaction is a way to understand the collective classroom mathematical activity.

To complement the interactionism theory of meaning making, constructivism is a theory of learning for analyzing an individual’s thinking and learning. The constructivist learning theory is based in the philosophy that people develop knowledge by constructing it in their own minds. Piaget’s ideas of assimilation and accommodation have been expanded and developed in this theory (Piaget, 1971). Reconceptualization of ideas occurs for a person when there is a conflict between a person’s mental structures and an experience in the world. The person then modifies personal knowledge systems so that it fits with sensory experiences. Von Glaserfeld elaborated Piaget’s ideas of constructing knowledge in the educational realm. Von Glaserfeld’s theory proposes that knowledge cannot be mirrored from the world by a person’s brain, but that a person can only build understandings that appear to fit the world as it is experienced. This idea has been integral to the radical constructivist learning theory (von Glaserfeld, 1987). In the emergent perspective, researchers meld the psychological perspective with interactionist view in order to analyze student individual learning by their participation in the classroom.

The interpretive framework shown in Table 3.1 was developed by Cobb and Yackel (1996) as an analytic framework. Each of the six cells of the chart provides a possible analytic lenses. These six constructs, social norms, personal beliefs about roles,
sociomathematical norms, mathematical beliefs, classroom mathematics practice, and mathematical conceptions are each discussed in the next section.

Table 3.1

The Interpretative Framework

<table>
<thead>
<tr>
<th>Social Perspective</th>
<th>Psychological Perspective</th>
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<tbody>
<tr>
<td>Classroom social norms</td>
<td>Beliefs about own role, others’ roles, and the general nature of mathematical activity in school</td>
</tr>
<tr>
<td>Sociomathematical norms</td>
<td>Mathematical beliefs and values</td>
</tr>
<tr>
<td>Classroom mathematical practices</td>
<td>Mathematical conceptions</td>
</tr>
</tbody>
</table>

Classroom social norms, sociomathematical norms, and classroom mathematics practices are reflexively related to an individual’s beliefs about his/her role in mathematical activity, his/her beliefs about mathematics, and his/her mathematical conceptions (Cobb & Yackel, 1996). Being reflexively related means one does not exist without the other (Cobb & Bauersfeld, 1995, p. 296). Next, I describe the three social constructs and of the interpretive framework and briefly relate them to the coordinated psychological constructs.

Classroom social norms. “Social norms are characterized by regularities in communal or collective classroom activity and are considered to be jointly established by the teacher and students as members of the classroom community” (Cobb & Yackel,
Social norms are not behaviors and patterns of discourse that a teacher can institute, but are negotiated by the classroom members, including students and teacher, together during the class. Social norms are not unique to mathematics classrooms; all classes have norms that are negotiated through interaction. For example, the expectation that a student raises his hand before speaking in the class is a social norm. Examples of social norms pertaining to argumentation that emerge in an inquiry classroom are expectations that students will justify their answers, make sense of others’ answers, and indicate agreement or disagreement with other students’ explanations. The social norms that are negotiated and renegotiated in the classroom can be seen to delineate the classroom participation structure (Erickson, 1986). Social norms are reflexively related to students’ personal beliefs about their role in classroom activity and how they see mathematical activity in general. A basic premise of the emergent perspective is that classroom norms and students’ beliefs co-emerge during classroom mathematical activity and cannot be separated (Cobb & Yackel, 1996).

*Sociomathematical norms.* Norms that specifically relate to the mathematical activity in the classroom are called sociomathematical norms. Like social norms, sociomathematical norms are interactively constituted by the teacher and students on a day to day basis. Examples of sociomathematical norms include: what constitutes an acceptable mathematical solution and what constitutes an appropriate mathematical solution (Yackel & Cobb, 1996). In the inquiry-oriented differential equation classrooms, sociomathematical norms are constituted primarily through discourse. An example of a sociomathematical norm is what constitutes an appropriate justification of a mathematical statement. More specifically, students reason using rate of change equations to justify and
explain their statements about solutions to differential equations (Yackel, Rasmussen, &
King, 2000). Sociomathematical norms are reflexively related with students’ beliefs
about mathematical activity in the same way as social norms and beliefs about students’
roles in classroom activity.

*Classroom mathematics practices.* Classroom mathematics practices are a
community of learners’ ways of thinking (Stephan & Rasmussen, 2003). This construct is
motivated by researchers’ interest in the mathematical development of a classroom
community as a collective (Yackel & Cobb, 1996) and involves taken-as-shared
mathematizing in the social setting of the classroom. Classroom mathematics practices
evolve in this setting and are those mathematical ways of thinking that are taken-as-
shared; they are the sociological counterparts to the psychological construct of individual
mathematical conceptions and ways of reasoning. For example, in a case study of one
classroom studying first order differential equations, one of the classroom mathematical
practices that Stephan and Rasmussen (2003) identified was that students reason about
solutions to autonomous differential equations as being horizontal shifts of each other
when viewed in a Cartesian plane.

This theoretical framework is an appropriate tool to study individual and
collective parametric reasoning as well as students’ development of understanding in the
classroom. The assumption that neither perspective on learning is primary and that each
perspective contributes to research and practice of learning mathematics in the classroom
provides a way for the analysis to examine both individual cognitive reorganizations and
collective understanding. Parametric reasoning can then be framed from both
perspectives.
3.3 Formal Research Study

3.3.1 Research Site

The research was conducted in an inquiry-oriented differential equations classroom in a midsized midwestern university. The university is an extension campus of a large public state university and located in an urban setting where all students commute to class. The course in differential equations is a required course for engineering and mathematics majors and generally enrolls a few science majors as well. All of the students in the class had completed or were completing three semesters of calculus and some had completed a course in linear algebra. The class met for 80 minutes twice a week in the evening (6:30 p.m. – 7:50 p.m.) and was taught by a research mathematician at the university. The mathematician’s research area was partial differential equations, and he had taught differential equations many times. However, this was the first time he used an inquiry-oriented approach, in contrast to a lecture-oriented approach, to the course. The differential equations class was one of a series of differential equations classes in an ongoing undergraduate mathematics education research project at the university level to build theory about individual and collective learning in inquiry oriented differential equations. (Rasmussen, Kwon, Marrongelle, Allen, & Burtch, in review).
3.3.2 Class Format

This differential equations class used an inquiry-oriented approach to instruction, which means that students were expected to participate regularly in the mathematical activity of the classroom by making sense of mathematics, communicating their own ideas to others and making sense of others’ ideas (Cobb & Yackel, 1996; Richards, 1991). The students were actively involved in thinking about mathematics, and the teacher’s role was that of guiding the mathematical activity of the students as they worked on the instructional tasks by listening to students and using their reasoning to support building of new conceptions. Explanation and justification of mathematical thinking was expected by students and negotiated as normative in the classroom by both the teacher and students. This class had many of the same characteristics as reform oriented undergraduate mathematics classes being taught elsewhere, including the use of small group learning, an emphasis on graphical, numerical, and analytical techniques, and utilization of technology. However, in this course, the development of students’ conceptualizations of solutions to differential equations was paramount, while the use of technology and small group work were used as means to achieve these goals, in contrast to other reform classes where the changes in instructional techniques was the primary reform.

A typical lesson in this differential equations class was as follows: 1). Students came into the class and turned in their daily journal entries in which they responded to questions the teacher had asked to probe their understandings and concerns. 2).The teacher initiated a new task. This involved a short discussion or activity to set the stage for the tasks. 3). Then the class alternated between small group work and whole class
discussions in which the students thought and talked about the ideas connected to the
tasks. The teacher and students listened to and tried to make sense of other students’
ideas; the teacher’s role was to facilitate the mathematical discourse in ways that
advanced the mathematical agenda. The students discussed their ideas and listened to
others during both the small group and whole class work. The statements made were
justified and students made decisions about the validity of the ideas by listening and
evaluating the reasoning offered. Class often did not end with the completion of a task. It
sometimes continued into the next class period, or students finished the task on their own.

Students completed written work in a variety of ways. At the beginning of each
class session, students turned in a journal entry that was a reflection on their
mathematical ideas, their thoughts about the class, and their concerns. The teacher read
these journals between the classes, briefly responded in written form to their thoughts,
and returned them in the next class period. Each week, homework was assigned and
collected. This homework might consist of completion of the tasks begun in class, or
additional problems that were conceptually connected to the class tasks. Homework
problems were graded and returned the next week. Two exams were given during the
course, a midterm and a final exam. These exams were a combination of problems that
required the students to reason about mathematics as well as solve problems that assessed
their skill with procedures. Finally, students developed two portfolios that were turned in
at the same time as the midterm and final exams. These portfolios contained self-selected
work, which might be homework, or a classroom task, that the students chose to best
express the new conceptualizations that they had developed in that portion of the
semester. With each selection, the student wrote a reflection on its value in his learning
connections to other concepts. The portfolios were finished with dictionaries that students created to define important concepts in the semester.

3.3.3. Use of Technology

Technology played an integral role in the differential equations class. Students used programs developed for the TI-92+ and also had access to computers for using the program NuCalc (Pacific Tech, 2000). This technology provided students with tools to reason about the concepts in differential equations, including but not limited to solution, equilibrium solution, and solution space. Students used the technology as a means of developing new understandings. Their experience with these programs was involved with investigating ideas, as opposed to solving complicated problems by asking the computer to perform computational calculations and receiving a solution.

3.3.4. Role of the Teacher

In addition to technology and the classroom design of small group-whole class discussion cycles, the teacher’s role as instructional leader was vital but significantly different than in a traditional presentation style mathematics class. In an inquiry-oriented classroom, one role of the teacher is to facilitate negotiation of the classroom social and sociomathematical norms that provides an environment for students’ mathematical practices to advance. In this class, the teacher did not present material in a lecture format. Rather, he initiated the instructional tasks and facilitated the mathematical activity in the classroom. Another role, then, was to promote explanation and justification in the classroom, leading to sense making for the students. He used different pedagogical
strategies: constructing representations on the blackboard that illustrate or structure the ideas of the students, repeating statements of students in different ways to clarify thinking, asking questions of students, or asking them to interpret and evaluate other students’ statements.

As an example of the teacher’s role in the classroom, I discuss one pedagogical tool he used. Pedagogical tools are symbols, words, or methods that a teacher may use to enhance the classroom sense making (Rasmussen, Marrongelle, & Keynes, 2003). One pedagogical tool is a generative alternative; a generative alternative is an idea, solution, or inscription presented for the purpose of encouraging students to make sense of the mathematics, both collectively and individually. For example, in one lesson, the professor was leading a discussion about the development of a mathematical model for a real life phenomenon and when the students offered their ideas, he offered one as well. The model he suggested was purposefully not correct, but it advanced the students’ thinking about mathematics through argumentation.

3.3.5. Participants

Eighteen students were initially enrolled in the class where the research was conducted, and of these 18, 11 completed the course. The 11 students that completed the semester were all male mathematics or engineering majors with a wide variety of ages. There were 2 females and 5 males who did not complete the course. Several of the students were considered non-traditional in that they were older and had returned to school to pursue degrees, not having entered university directly from high school. More than half of the students were working full time and attending class in the evening. The
students expressed considerable anxiety at the beginning of the inquiry-oriented instruction because the teacher was not “telling them the answers.” They often wrote in their reflective journals (some even towards the end of the semester) that they were not “learning anything.” One student mentioned that this was a way for the teacher to get out of doing work. Another felt that he would not be able to know what he needed to know in his future classes.

By the end of the semester, however, the majority of the students were comfortable with the classroom environment. They came to rely on themselves to decide if a conjecture was correct and listen to the discussions to make sense of the mathematics. The conjecturing and justifying that was part of the class was present to some extent, but there was evidence that during the last six weeks of the class there was less time spent in the conjecture-justifying activity.

I selected two students for in-depth case study analysis at the end of the semester. Both Adam and Brandon had been in small groups that were videotaped during the entire semester and had been interviewed before and after instruction on systems of differential equations. Adam was chosen as one of the two students for the following reasons. He was a mathematics major, what might be called a “traditional student” and was one of the academic leaders of the classroom. He was very comfortable with his ideas and not afraid to come to the front of the class to present his arguments. He also was the person who shared his thinking the most in his small group, although the other group members did challenge and push his thinking during those small group times. He often asked questions of other students on the other side of the classroom. Adam would often begin to say something and end with “I don’t know.” Finally, he was not afraid to make a claim about
his thinking even if he was not sure he was correct and publicly argue for his claim while being comfortable with changing his statement when provided with an argument that convinced him.

Brandon was chosen to be the second case study for different reasons. First of all, Brandon was a non-traditional student that had returned to university to obtain a bachelors degree in engineering, even though his job involved significant engineering tasks already. He was taking the course at night as he worked full time and this was one of two night classes in which he was enrolled. His ways of participation in the class varied significantly from Adam. His mathematical activity was much more private in general, as was observed by the amount of homework he turned in compared to his class participation. He spoke less than half as many times as Adam did and when he did, it was often to offer concerns or lack of confidence in his own mathematical thinking. He was more involved in the mathematical activity in his small group, although one of the members of his small group, Jay, was more vocal.

3.4 Research Design

Classroom research is a vital and rapidly growing arena for mathematics and science education research. As stated by Brown (1992),

The classroom must function smoothly as a learning environment before we can study anything other than the myriad possible ways that things can go wrong. Classroom life is synergistic: Aspects that are often treated independently, such as teacher training, curriculum selection, testing, and so forth actually form a systemic whole (p. 141).
The research design for this project was based on classroom teaching experiment methodology (Cobb, 2000). The classroom teaching experiment (CTE) was developed as a way to study student learning in the classroom and is a modification of the teaching experiment developed by Steffe (1983) in response to increased interest in studying learning as a social process to better investigate student learning. Learning does not take place by students in a vacuum, and Cobb and colleagues developed this methodology to investigate learning from both the individual and collective perspective.

In a classroom teaching experiment, researchers first envision how the teaching-learning process might be realized in the classroom (Gravemeijer, 1995). This envisioning is called the hypothetical learning trajectory.

This process [the teaching learning process] includes the formulation of a hypothetical learning trajectory that is made up of three components: learning goals for students, planned learning or instructional activities, and a conjectured learning process in which the teacher anticipates how student’s thinking and understanding might evolve when the learning activities are enacted in the classroom (Simon, 1995, in Cobb, 2000, p. 316).

Following the development of the hypothetical learning trajectory, instructional tasks developed are used with the students in the classroom and researchers analyze the students’ activity and work. The analysis is two fold. The instructional tasks and teacher strategies are evaluated regularly during the CTE to make minor revisions and share researchers ideas and observations of the students. Also, the researchers analyze the CTE
at the end of the classroom enactment. This allows the researchers to revise the hypothetical learning trajectory as well as interpret the results of the CTE. Frequently, after revision, one or more classroom teaching experiments may ensue in a cyclical pattern.

Figure 3.1 shows a schematization of the developmental research cycle. This design cycle methodology is based on the importance of the theory and practice each being influenced by the other. The theoretical base guides the analysis, which then informs the theoretical framework (see Saldana, 2004; Rasmussen & Marrongelle, in press; Cobb, Stephan, McClain, & Gravemeijer, 2001 for examples).

The research team consisted of three individuals, two researchers (myself and another researcher) and the professor. The research team met regularly throughout the
semester. First, the team met weekly for project meetings. In these meetings, the team discussed proposed instructional strategies and anticipated student reasoning. We also reviewed video recordings of the previous week’s classes. We discussed student discourse in the previous week and instructional strategies that might be used to build on student conceptualizations that had developed. We modified, added, or deleted instructional activities if it was appropriate as we considered the students’ thinking in the class.

Also, at the end of each class, the team met for a debriefing session in an audio-recorded meeting. In these meetings, we discussed the teacher’s pedagogical activity in light of student thinking and the effectiveness of the instructional materials. We considered strategies that the professor used which worked well to advance students’ mathematical explanation and justification. These debriefings and the weekly meetings exemplify the developmental cycle in Figure 3.1 as the classroom activity informs the research into the learning process and research informs the classroom activity.

I next look at the materials used in the classroom teaching experiment as they play a very important role in the students’ learning. The instructional design theory that guided the development of instructional materials was that of Realistic Mathematics Education. The theory and the materials used are described in the next section.
3.5. Instructional Design and Materials

3.5.1. Realistic Mathematics Education

Freudenthal (1971) provided the original basis for Realistic Mathematics Education (RME) by proposing the idea that mathematics is a human activity (1971). For Freudenthal, the main activity of mathematics is mathematizing. Freudenthal characterizes mathematizing as “the entire organizing activity of the mathematician, whether it affects mathematical content and expression, or more naïve, intuitive, say lived experience, expressed in everyday language” (Freudenthal, 1991, p. 31). Freudenthal gives two justifications as to why mathematizing is the key process in mathematics teaching: 1) mathematizing is the major activity of mathematicians and as such, enculturates students into mathematics by using a mathematical approach to everyday life and, 2) formal mathematics emerges as the end product of mathematizing; students should begin at a meaningful starting point and mathematize to more formal mathematics (Gravemeijer, 1995). “The objective of the RME approach is that students experience formal mathematics no differently from informal mathematics” (Gravemeijer, 2000). With the perspective of mathematizing as human activity as a foundation, RME offers a way for developers of instructional materials to design instruction so that students can begin with personally meaningful mathematical interpretations and develop more sophisticated mathematical understandings.

There are three primary tenets of the RME instructional design theory: guided reinvention, didactic phenomenology, and emergent models (Gravemeijer, 1994).
Guided reinvention: Instructional activities provide opportunities for mathematics to be developed by the students. The word guided is important, as the principle does not imply that students will “discover” mathematics on their own. The teacher must be constantly engaged in order to appreciate student reasoning and provide guidance in suggesting directions for students to participate in reinventing mathematics. The activities afford students opportunities to participate in mathematical activity in appropriate ways so they are guided toward instructional goals as they mathematize while working on the instructional tasks. Freudenthal (1991) provides the warrant for reinvention when he states, “If the learner is guided to reinvent all this, then valuable knowledge and abilities will more easily be learned, retained, and transferred, than if imposed” (p. 49). Gravemeijer (1995) suggests that guided reinvention can also be described as progressive mathematization (p. 90). Progressive mathematization from the researcher’s perspective describes students’ different levels of mathematical activity. Students start developing mathematical ideas in a specific experientially real situation, and then build from that informal reasoning to eventually come to understand mathematics in a more formal structural way. Gravemeijer (2000) states

Within this perspective, we can distinguish between formal and informal by denoting formal mathematics reasoning as a form of reasoning that builds on arguments that are located in the newly formed mathematical reality. Seen this way, the distinction between informal and formal is a relative distinction: a distinction that is relative to a certain topic… (p. 160).

The historical development of mathematics and the informal reasoning of students are two possible guiding ideas for introducing starting points and developing activities
that provide students with reinvention activity. The starting point for students entering into mathematical activity should be experientially real to the students. This starting point may be a real life experience of students, a technological experience or a mathematical experience that is not necessary real life, but is personally meaningful. In differential equations, for example, students have a personally meaningful way to reason with rate of change. The instructional materials then provide tasks that allow students to enter into mathematical activity and refine, revise, and build new understandings of rate of change as related to differential equations.

Didactic phenomenology: Freudenthal’s philosophy of mathematics as a human activity grounds the tenet of didactic phenomenology. He advocated that traditional mathematics instruction is actually anti-didactical when teachers use the final results of mathematization, the formal concepts and structures of mathematics, to teach children when they are first learning. Students need to mathematize by entering the mathematical development process at a place that is realistic to them and then participate in increasingly formal levels of mathematical activity. Freudenthal states that didactical phenomenology is “a way to show the teacher the places where the learner might step into the learning process of mankind” (Freudenthal, 1982, p. ix). To conclude then, didactical phenomenology is a construct that refers to instructional designers locating starting points for students to enter into mathematical activity and that will lead to progressive mathematization. Cobb (2000) discusses this when he states:

In addition to taking account of students’ current mathematical ways of knowing, the starting points should be justifiable in terms of the potential endpoints of the learning sequence. This implies that students’ initially informal mathematical
activity should constitute a basis from which they can abstract and construct increasingly sophisticated mathematical conceptions as they participate in classroom mathematical practices. At the same time, the situations that serve as starting points should continue to function as paradigm cases that involve rich imagery and thus anchor student’s increasingly abstract mathematical activity (p. 318).

Emergent models. Mathematical objects of all kinds may begin as models of reasoning about informal mathematics but with more mathematical sophistication emerge as models for more formal mathematical reasoning as they [students] build a mathematical reality for themselves (Gravemeijer, 2000). As an example, Streefland (1991) uses this shift in students’ activity with models in his instruction on fractions. Students initially solve problems about sharing pizzas by drawing and partitioning circles. In this case, students are using the circles as models of their thinking about sharing pizza and fractions as fractions relate to fair sharing of the pizza. Later, students return to these partitioned circles and used them to reason more formally about relations among fractions, deepening their understanding by using the circles as models for mathematical reasoning.

These models of-models for transitions are viable in both the psychological perspective and the social perspective. Students individually use these models to grow in their mathematical reasoning abilities as their mathematical understandings develop. Simultaneously,

In social terms, this third tenet implies a shift in classroom mathematics practices such that ways of symbolizing developed to initially express informal mathematical activity take on a life of their own and are used subsequently to
support more formal mathematical activity in a range of situations (Cobb, 2000 p. 319).

Overlying the tenets of RME, a relationship exists between the larger instructional design theory and the local instructional design theory that is enacted in different settings and with different content. Freudenthal and colleagues originated RME as an evolving instructional design theory. The principles of RME are enacted in local settings and the local and larger instructional design theories inform each other. For example, the differential equations local instructional theory is the hypothesized learning trajectory developed by researchers for the differential equations class that is the setting for this study.

Simon’s definition of a learning trajectory offers a way to hypothesize the development of a class’s understanding of specific mathematical content. He proposes that learning trajectories ideally include “the learning goal, the learning activities, and the thinking, and learning in which students might engage” (Simon, 1995, p. 133). The specific systems of linear differential equations instructional tasks as they were used in this research are attached as Appendix A. A summary of the learning trajectory and a brief presentation of the instructional sequence are described next.

3.5.2. Hypothesized learning trajectory and instructional materials

In this section, I describe a hypothesized learning trajectory constructed and revised after each cycle of use by the instructional designers (Rasmussen, 2003). The instructional tasks used in the course during instruction on solutions to systems of
differential equations are integrated into the trajectory and provides readers with a background for the analysis in the study as the students’ parametric reasoning as well as other forms of mathematical activity emerge partly from their participation in the enactment of the tasks. Students’ conceptualizations of systems of differential equations and their solutions emerge as they participate in the class. Students have already built understandings of first order differential equations and their solutions and solution spaces when they begin the mathematical activity that involves systems.

The primary trajectory for the mathematics of solutions to systems of differential equations students is as follows. Students participate in mathematical activity that supports them in the following progression: 1) understanding what a linear system of differential equations and its solution is, including conceptualization of phase planes, curves in space as solutions, and pairs of functions as a solution; 2) developing a conceptualization of straight line solutions to autonomous linear systems of differential equations; 3) developing a conceptualization of the structure of solutions spaces (the entire set of solutions to an SDE) including straight line solution functions, equilibrium solutions functions, where they occur, and the behavior of the solutions near them or other places in the phase plane; 4) developing a way to solve and interpret the general solutions to system of differential equations; and, 5) extending understanding of solutions of linear systems to non-linear systems.

In general the concept of rate of change is taken as experientially real to students and thus provides a reasoning tool that permeates the learning trajectory for systems of differential equations. The other two starting points for the learning trajectory are population models and spring mass physics context, and I discuss both of these.
However, I present the trajectory in an assumed sequential order for ease of understanding, even though research suggests that learning in differential equations does not proceed sequentially (Stephan & Rasmussen, 2003).

The students’ learning trajectory initially begins with students considering a population model (see Appendix A) to begin thinking about how to interpret a system of differential equations. The students analyze and interpret the rabbit-fox system of differential equations. The expectation is that students think about how the two differential equations interact and reason about the solutions qualitatively and then modify Euler’s method to create numerical solutions. This provides students with one way to find solutions and motivates an interest in developing more ways to visualize, interpret, and represent solutions to systems of differential equations.

The next activity entails students mathematizing a set of points in three-space by considering a red dot on a slowly revolving propeller on an airplane and the trace it makes in space as the airplane taxis down a runway (see Appendix A). The students use pipe cleaners to construct the (imagined) trace of the dot in three dimensions, known as a curve in space, and examine its two dimensional projections, interpreted as the $x$-$t$, $y$-$t$, and $x$-$y$ projections. These projections provide three representations of the curve that students grow to understand and use for mathematical reasoning. The mathematization of curves in three space and their two dimensional projections are then elaborated in tasks that use rabbit fox (predator prey) systems of differential equations. The solutions to these systems that model interacting populations are represented as curves in three space and related to this, functions $x \, (t)$ and $y \, (t)$ that make the systems true. As students continue to engage in mathematical activity about systems, they revisit the airplane
instructional task as grounding imagery for how a three dimensional curve coordinates with various two dimensional projections. The intention is that what would be the equivalent of the phase plane view for the airplane task emerges for students as the most useful and informative way to depict the traces of the red dot. Thus, rather than imposing the phase plane on students from the expert perspective, the phase plane is grounded in what students see as useful.

Students use the population systems of differential equations to reason about the concept of a solution that is projected onto the $x$-$y$ plane. Students then develop the idea of a tangent vector field in the $x$-$y$ plane as a way to symbolize the infinite number of pairs of $x$-$y$ values and the direction that the solutions take at these values. Part of the development of the phase plane (tangent vector field) is students’ reinvention of nullclines and revisions of their conceptualization of equilibrium solution functions in this new setting. Nullclines are curves in the phase plane that represent patterns of points where one or the other differential equations in the system is constantly 0. For autonomous differential equations, points where both equations are constantly equal to zero are points where the system obtains equilibrium. Analytically, an equilibrium solution for a system of two differential equations is a pair of constant functions that satisfy the system. Nullclines and equilibrium points are important for students to create as they provide structure to the graphical solution spaces and are useful for qualitative analysis of spaces of solutions to systems.

Students work on tasks in order to investigate nullclines and equilibrium solution functions and eventually develop conceptualizations of these objects and ways to find
them analytically and graphically. This development is integral to the qualitative analysis of systems of differential equations and the students’ development of a phase plane.

The phase plane originally serves as a model of student reasoning about the differential values at specific $x$-$y$ quantities. During the course of instruction, it becomes a model for more formal reasoning about phase planes and solutions represented in those planes. The instructional materials provide for this transformation in accordance with RME design theory. In the case studies in Chapter 5, descriptions of how students’ growth in understanding leads them to reasoning about straight line solutions using the phase plane will be presented.

Using the spring mass context and the system of differential equations developed in the class to model the context, the students use the phase plane to conceptualize straight line solutions and then invent a way to think about any solution to a system of linear differential equations. I discuss this trajectory in the next paragraph. The teacher provides guidance as the students reinvent a system of differential equations to model the system where $x$ and $y$ are the distance from center and velocity of the spring, respectively.

The first task in this reinvention involves students creating distance-velocity graphs of a spring mass solution. The expectation is that these graphs will be either circles or spirals depending on whether friction is considered. After the graphs are drawn, the system that models the spring-mass system is created with the professor’s help and phase plane vector fields are explored. In the exploration, students are encouraged to find a phase plane where the solutions do not appear to spiral or form a circle. Students discover this phase plane and then are motivated to understand how this is possible and learn more about the situation. They then discover a line where it appears that if the
spring-mass initial condition is on the line, the spring just moves to the middle and does not oscillate. When asked to find that line, the students reinvent an algebraic algorithm to find where the vectors and the line have the same slope. With that algorithm, they are again surprised by the result that there are two straight lines. The instructional sequence has provided them support to reinvent and grow to understand the idea of straight line solutions.

The straight-line solution is a pair of functions that when represented in a phase plane appear to lie on a straight line. Understanding of straight-line solutions then evolves into understanding of eigenvectors, one of the important conceptualizations in systems of differential equations. As students work with these straight line solutions, they are guided to rediscover eigenvectors as well and use the concept of linear combinations as a way to understand and then analytically solve homogeneous linear systems (pairs) of differential equations.

Finally, the spring-mass system is modified, and students consider the non-linear system of differential equations that models a swaying skyscraper. They investigate this non-linear system qualitatively by extending their work on phase planes from the spring mass system. Eventually, they use this model to generalize to other non-linear systems. The teacher presents the analytic technique of linearization as the final component of the sequence and students build on their conceptualizations of the spring-mass and swaying skyscraper to understand and apply linearization to non-linear systems of differential equations.

The above described hypothesized learning trajectory is the expectation of how students come to understand linear and non-linear systems and their solutions and
solution spaces. The learning trajectory involves the tasks, teacher moves, and other tools that are used to support the students’ progression into understanding of systems of differential equations and their solution functions. Using the principles of didactical phenomenology, guided-reinvention and the model of-model for transition, the instructional designers of the materials developed and revised the instructional sequence to follow the hypothesized learning trajectory for the differential equations class.

3.6. Data Collection

Data collection consisted of video recordings of the differential equations class, both small group and whole class discussions, video recordings and student work from one-on-one student interviews, copies of students’ written work, and researcher field notes.

*Video recordings:* Each of the nine classes was videotaped using two cameras. During whole class discussions one camera focused on the students and captured the discourse and one camera focused on the teacher. During small group discussions, each of the video cameras recorded student participation and individual written work as they actually produced it for two different identified small groups. Microphones were placed in the middle of the group to record all the discourse. For the purpose of constructing case studies, the same small groups were recorded in each class session.

*Interviews:* All students were invited to participate in one hour interviews three times in the semester: at the beginning, in the middle and at the end. Eight students were interviewed at the beginning, six of those students returned for the middle interview, and five of them participated in the end of semester session. Students were paid for
participating in the interviews and all interviews were video recorded by a second researcher for later analysis. These semi-structured task-based interviews used a protocol developed by the research team (Creswell, 2003). Interviews followed a flexible interview script and questions to probe the students responses were used to ensure that essential points were covered. A key element in the interviews was understanding why things were done, not simply recording what was done (Hackos & Redish, 1998).

Some of the problems were modifications from the pilot study interview protocols. The protocol for the interviews was developed by identifying mathematical concepts of interest and then creating tasks that would have the potential to reveal the students’ thinking in those areas. This dissertation specifically reports on analysis of students’ mathematical activity during the middle and end of semester interviews because it is these interviews that dealt with systems of differential equations. These will be referred to as the first and second interviews in this dissertation.

The first interview was conducted immediately after students had finished the instructional sequence for first order differential equations and before systems of differential equations was begun. The purpose of the interview was to glean insight into the conceptual resources the students had available to reason about systems of differential equations and their solutions before formal instruction began. The second interview was conducted at the end of the course and was designed to revisit some of the tasks that students worked in the second interview to gain insight into how the students’ understanding of systems of differential equations had evolved. There were also several more general questions to probe students more general understandings of differential equations.
Artifacts: All written work from all students was collected and copied. This work includes homework assignments, three exams, two primarily consisting of conceptual questions and one consisting of routine problems, all in-class work, students’ daily written reflections and portfolios turned in by the students at midterm and final time.

Field notes: I took detailed field notes during each class period. I used a field note log that was organized into three columns. The first column consisted of the start and end time of the episode for which I was making notes. The second column summarized the mathematical activity that was occurring and whether it was in small group discussion or whole class discussion. The third column consisted of notes on discourse that occurred in whole class discussion. During small group work, the field notes were directly related to the mathematical activity of one small group. The field notes were used only to check if there were difficulties in analyzing the data of the discussion and student work and not as a source for analysis.

3.7 Data Analysis

3.7.1 Case Studies and Parametric Reasoning

I first analyzed two individual students’ mathematical activity and progress in their ways of reasoning parametrically and their growth in understanding of SDEs using a grounded theory approach. The theoretical ideas that developed are grounded in the data analysis process and not developed a priori (Glaser & Strauss, 1967). Analysis of the data collected provided hypotheses for the characterization of parametric reasoning. The analysis also provides theories about the progression of the students’ mathematical
activity as they participated in the instructional sequence, the interplay between parametric reasoning and students’ understanding of solutions of SDEs. As analysis continued, the hypotheses were revised or discarded by triangulating the data of the written work and discourse that was recorded in the videotaping. The results then revealed new insights about parametric reasoning, the students’ enactment of the mathematics involved in solving systems of differential equations and the connection between the two.

Following is a more detailed description of the analysis process.

Class discourse: The transcriptions of the classroom discourse served as the primary data source for this study. I transcribed the video recordings of the whole class discussions and the small group discussions in which Brandon and Adam participated. I noted Brandon and Adam’s statements in whole class discussion and small group discussion in order to describe their participation in the classroom discourse. I also noted ideas for further consideration at later steps in the analysis. Table 3.2 is an example of the table created for each day of class discussion. I also created similar tables that included all of Adam and Brandon’s utterances to provide a refined source of data for the case study analysis.

Table 3.2 Example of Table Created for Analysis

<table>
<thead>
<tr>
<th>Date: 11/13/02</th>
<th>Discourse</th>
<th>Mathematical Content</th>
<th>Research Notes about things of interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time and Person</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9:00 Professor</td>
<td>I see two graphs Let’s talk about that. Okay, why do you think it goes through?</td>
<td>Teacher setting up the situation</td>
<td>He doesn’t provide reasoning on if</td>
</tr>
</tbody>
</table>
Table 3.2 (cont)

<table>
<thead>
<tr>
<th>Brandon</th>
<th>... I was thinking you would have this one because if the original resting position right there is the origin, that means that when you pull the string, it’s going to have to go back before the origin. If there is no spring to it at the origin, it would come back there and stop. It has to go beyond the origin to compress to build up some more energy to bounce back this way,</th>
<th>Student is using his physical experience to reason about the graph asked for—when a spring-mass system works</th>
<th>Note how he is fusing the two ideas together—the movement of the spring in two dimensions the graph “going” on the coordinate plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>It doesn’t really matter because that is just drawing it with the origin being with the spring completely compressed</td>
<td>Continuing to discuss the graph</td>
<td></td>
</tr>
<tr>
<td>Jerry</td>
<td>Exactly, so you just change the first graph all you did change initial position</td>
<td>Continuing to discuss the graph</td>
<td></td>
</tr>
</tbody>
</table>

**Coding Scheme:** After the log was done for each of the class discussions, I studied the discussions for types of thinking and other themes that were present. Then, because I was primarily focusing on Brandon and Adam, I used open coding to develop a coding scheme for all the discourse for the two case studies (Strauss and Corbin, 1990). During open coding, I identified and named the conceptual categories into which the reasoning observed could be placed. The goal was to create descriptive categories which form a framework for analysis. Words, phrases or events that appear to be similar can be grouped into the same category. These categories may be gradually modified or replaced during the subsequent stages of analysis that follow. Table 3.3 shows the codes that emerged from the open coding of the transcripts.
Table 3.3

Coding for Student Reasoning

<table>
<thead>
<tr>
<th>Coding Description</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>using ideas from first order differential equations</td>
<td>(F)</td>
</tr>
<tr>
<td>thinking of the physical context</td>
<td>(C)</td>
</tr>
<tr>
<td>visualization</td>
<td>(V)</td>
</tr>
<tr>
<td>ideas about differential equations</td>
<td>(D)</td>
</tr>
<tr>
<td>graphical interpretation</td>
<td>(GI)</td>
</tr>
<tr>
<td>graphical visualization</td>
<td>(GV)</td>
</tr>
<tr>
<td>prior discussion</td>
<td>(PD)</td>
</tr>
<tr>
<td>phase plane reasoning</td>
<td>(PP)</td>
</tr>
<tr>
<td>algebra concept</td>
<td>(AC)</td>
</tr>
<tr>
<td>fictive motion metaphor</td>
<td>(FMM)</td>
</tr>
<tr>
<td>ideas about exponential functions</td>
<td>(E)</td>
</tr>
<tr>
<td>vector algebra</td>
<td>(VA)</td>
</tr>
<tr>
<td>linear algebra</td>
<td>(LA)</td>
</tr>
<tr>
<td>physics</td>
<td>(Ph)</td>
</tr>
</tbody>
</table>

All of Brandon and Adam’s statements in whole class discussion were coded for these themes, which I then used to identify characterizations that appear across the coding and the previous analysis of themes related to parametric reasoning. The case study chapter (Chapter 5) will elaborate on the themes.
Interviews: I next created a complete transcript of all six of the first interviews and a complete transcript of Brandon and Adam’s second interview. As I transcribed the interviews, I made notes of things that might be of interest for further analysis. The students also produced written work as they solved the tasks and this work was used to assist in understanding the transcription when the discourse alone was not clear. Using the transcriptions I created written summaries of each student’s mathematical activity on each of the tasks. Using the written summaries and the students’ written work, I looked for patterns in student reasoning; these patterns may be observed in one student’s work on different tasks or across the same tasks enacted by several students. I then used the patterns to create themes in a characterization of parametric reasoning.

I used Brandon and Adam’s second interview to primarily identify and find evidence for the progressions of their conceptualizations about solutions to systems of differential equations and ways that parametric reasoning was contributing to and being enhanced by these new conceptualizations. Again in this second interview, the written work was used to assist in understanding the transcription when the discourse alone was not clear

3.7.2. Students’ advancing mathematical activity

In the proposal for this dissertation, I stated that I would identify classroom mathematics practices that emerged from the mathematical activity of the students during instruction on systems of differential equations. I intended to use the conceptualization of classroom mathematics practices as developed by Cobb and Yackel (1996). Yackel (1997) developed a methodology for analyzing discourse to identify and describe
classroom mathematics practices in inquiry-oriented classrooms using Toulmin’s argumentation theory modified for use in collective argumentation and introduced into mathematics education by Krummheuer (1995). In this methodology, discourse is analyzed for its argumentation and the CMPs are identified from the analysis. It is essential that this methodology be used in an inquiry-oriented classroom where students’ raise questions and challenges when they disagree or do not understand. Without these norms, the method of analysis is not viable.

I first attempted a process that was based on this methodology. First, I watched the video recordings and follow the transcripts simultaneously and attempted to identify when a student made a claim in the discourse of the classroom. I also asked another researcher to look at this data. Unfortunately, as I began to identify the data, with the warrant and backing that emerged from the dialogue, I had extreme difficulties. Therefore, after consideration, I abandoned the process of identifying classroom mathematics practices. There is evidence that the social norms that allow argumentation and justification to be the primary activity in the classroom did not emerge as planned. The teacher often made the first or only claims in the classroom. Often, students did not back up their claims with warrants that were identifiable. Occasionally the teacher eventually provided the warrant and backing, or more frequently, I did not find warrants and backings present in the discourse.

This attempt at analysis to identify CMPs led me to consider other ways to analyze and discuss mathematical practices as they occur in the classroom. One way to describe mathematical learning is to frame it as participation in different types of mathematical practices. In their work on advancing mathematical activity, Rasmussen,
Zandieh, King, and Teppo (2005) propose that mathematizing includes the mathematical practices of symbolizing, defining, algorithmatizing, and justifying. As students in classrooms become more sophisticated in their reasoning, their participation in these kinds of activity is evident. For instance, in first order differential equations, students in the differential equations class that is the setting for this study created an algorithm called Euler’s method to create numerical solutions to any first order differential equation. This activity is one that mathematicians often use and by engaging in it in the differential equations class, the collective group of students became more enculturated into the general mathematical community.

To detail these mathematical practices, I used the whole class transcripts for the nine days of instruction on systems of differential equations. I first read through and identified in general the four previously discussed practices of mathematizing. These were not present all the time or necessarily in every class. As I identified an example, I marked it in the transcript and created a separate log of these times— with more formal descriptions. This log resulted in a document that indicated practices of algorithmatizing, defining, justifying, and symbolizing with short descriptions of each as show in Table 3.4

Table 3.4

*Example of Log of Mathematics Practices*

<table>
<thead>
<tr>
<th>Time</th>
<th>Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>11/18 TCA 33:00</td>
<td>Algorithmatizing—reasoning about system solution as function</td>
</tr>
<tr>
<td>11/25 TCB 15:15</td>
<td>Symbolizing</td>
</tr>
<tr>
<td>11/25 TCB 20:15</td>
<td>Experimenting- students look at the sum of</td>
</tr>
</tbody>
</table>
In Chapter 4, I present a characterization of parametric reasoning. In Chapter 5, I present the case studies of Adam and Brandon’s mathematical activity as they learn about solutions to systems of differential equations. In Chapter 6, I discuss the advancing mathematical activity in the classroom, provide a summary of parametric reasoning and the case studies, significance of the results, limitations of the study and plans for future research.
CHAPTER 4 MATHEMATICAL ACTIVITY THAT CHARACTERIZES AND SUPPORTS PARAMETRIC REASONING

4.1 Introduction

Parametric reasoning as defined in Chapter 1 is depicted as developing and using conceptualizations about the dynamic quantity time as it implicitly or explicitly coordinates with other quantities to understand and solve problems. Described in another manner, the use of parametric reasoning occurs when students use their understanding of time to reason about mathematical quantities. One of the goals of this research is to identify the mathematical activity that characterizes and supports parametric reasoning in the context of systems of differential equations. This characterization may be useful more broadly because reasoning about and with time as a dynamically changing parameter supports mathematical activity related to graphing and working with parametric equations in precalculus and calculus as well as mathematical activity in other courses that involve time as the independent variable; this includes problems about motion that occur in middle school and high school.

In this chapter, I characterize parametric reasoning as it supports students’ participation in the classroom mathematics as they learn to solve systems of autonomous differential equations. In Chapter 3, I described the method that I used for developing my ideas. After I characterize parametric reasoning, I discuss the question of growth in students’ parametric reasoning as they learn to find solutions to systems of differential
equations. In other words, is there evidence of more or less parametric reasoning after learning about solutions to systems of differential equations, or is there a change in the depth of reasoning? I also discuss the idea that parametric reasoning “drops off” as the instructional sequence proceeds. The end of the chapter will then relate this characterization of parametric reasoning to the more standard conception of parameter as a sleeping variable and will discuss how the analysis here is related to Drivers’ (2003) work.

The following is a list of five types of mathematical activity that supports and characterizes parametric reasoning which were identified during the analysis of the data. These activities are not in hierarchical order and in fact often appeared together in the whole class discussions or interviews. The conceptualizations that the activities encompass are complex and dynamic. For purposes of clarity for this analysis, however, each will be discussed separately and examples provided.

A. Making time an explicit quantity

B. Using the intuition from real life that “time never stops”

C. Using both quantitative and qualitative reasoning

D. Using both discrete and continuous imaging of time

E. Imagining the motion

In this chapter, I characterize each of these mathematical activities, make connections to other research when appropriate, and present examples to illustrate the mathematical activity. The majority of these examples are from Brandon and Adam’s interviews and their classroom participation but two examples are from other students’ participation and interviews; I do this is to provide evidence that parametric reasoning is
not confined to just the two students’ mathematical participation that I analyzed in the case studies. The examples were chosen for a variety of reasons to support the characterizations and the reasons will be presented in each case with the examples.

4.2 Making Time an Explicit Quantity

The first characterization of parametric reasoning involves making time an explicit quantity. The basis of “making time an explicit quantity” relates to the following quote from Nemirovsky and Tierney (2001):

Making selected features directly visible builds on the difference between something that can be recognized directly, and that which requires an intermediate process of inferences and calculations... Learning mathematics when specifically connected to representations is a matter of learning to “seeing as and recognizing in” (pg. 69).

The point I draw from this quote is that students from elementary age through university level make shifts in their perceptions and learn to see mathematical objects differently, which allows them to build new conceptualizations. As such, students make new features “visible” in their work, and learn to focus and be explicit about their representations of these features. This point is particularly salient in students’ activity in the instructional sequence of this study.

When looking at graphs in two dimensions that do not have time explicitly represented, students often introduced time into the situation as an additional quantity. Sometimes this inclusion consisted of the verbalization of time and reasoning with that
introduced quantity. I coded 10 such instances. Many of these instances occurred during the sequence that involved the motion of a mass on a spring.

At other times the explicit inclusion of time results in the creation of a third axis to represent time. I next provide an example of this introduction of time as a third axis. I chose this example because of its clarity and because it occurs before instruction to systems of differential equations and thus represents a spontaneous and student generated use of time as a third axis. There are several other less explicit examples of student use of time as a third axis, but these examples occur later, when instruction is explicitly focused on the time axis.

The Skateboard Interview Task (see Figure 4.1) was posed to students in the individual interviews conducted after instruction on single differential equations and just prior to instruction on systems of differential equations:

You are up in a hot air balloon looking down at a skateboarder. He has a piece of chalk on the bottom of his board that is drawing a line on the ground as he moves. Below is an example of what you in the hot air balloon see on the ground. Sketch possible graphs of the $x$ vs. $t$ and $y$ vs. $t$ graphs for the skateboarder.

![Figure 4.1. Skateboarder Interview Task, Part 2](image-url)
In Figure 4.2, the graph that Brandon drew is showing.

**Figure 4.2. Brandon’s Sketch for Skateboard Task, Part 2.**

(from Brandon, Interview 1, November, 2002)

Brandon: As long as time advances, it's staying at four. So. That would just be a straight line.

Karen: Uh, huh.

Brandon: Where. For these two graphs.

Karen: Okay. I understand what you're saying. Reasonable. [To Other Researcher.] Questions about that one?

Other Researcher: How would you respond to somebody about? Who says: Well, I don't know about that. I think it's just a point. \(x\) equal to one. I just think it's a point there. \(y\) equal to four.

Brandon: Well, it is a point at \(y\) equals four. But, then, what about at? After two seconds? Where is that point? Because this, this has. This bottom axis is \(t\). Or, this, this axis is \(t\). So. At \(t\) equals one second, where's he at? He's at \(x\) equals one. At \(t\) equals two seconds, he's at \(x\) equals one. At \(t\) equals infinite seconds, he's at \(x\) equals one. So. Yeah. He's, uh, always at that point. And, that. But, that point is. There's, uh, fourth [sic third] dimension.

Karen [Laughs.]

Brandon: Time. And, that's what you're moving through. You're on third-dimension time. And, you're only seeing two dimensions. You're seeing the third dimension here.

Other Researcher: That's helpful.

Brandon: I mean, you could also say there's, uh, a \(z\)-axis. And, actually what's happening is, uh. With time, he's, he's moving up that \(z\)-axis. Because he's coming up toward me. And, that's why I'm only seeing one point.
After Brandon draws the x(t) and y(t) graphs, one of the researchers asks him why it is not just a point. He responds, “You’re on third-dimension time. And you’re only seeing two dimensions.... you could also say there’s, uh, a z axis...” It is interpreted that Brandon “sees the situation as” involving this third dimension and brings it into consciousness and uses it to explain his thinking. Connected to the mathematical activity of making time explicit is students’ use of the intuition that time never stops.

4.3 Using the Intuition that “Time Never Stops”

The research related to students’ understanding of time, does not report the conceptualization of time that can be stated as “time never stops.” This belief is part of students’ understanding of the parameter time as they learn about solutions in three dimensions, as evidenced by Adam and Brandon and other students’ work and use of language. Specifically, the coding indicates 8 number instances of students explicitly or implicitly reasoning using the fact that time never stops. The following is an example from Adam’s interview before the instructional sequence on systems of differential equations and illustrates this intuitive idea. I use this example because Adam clearly articulates his intuition about time. Many of the other examples I coded for this mathematical activity were less explicit. The task is given in Figure 4.1.
Karen: Sketch me an $x$ versus $t$, and $y$ versus $t$ graph of that trace from the skateboarder

[Adam draws the graphs as seen in Figure 4.3]

Adam: [Refers to two dots he has put on the two graphs.] Because he’s not going anywhere. In... Oh! No. No, no, no. Wait a second. Let me think. Because time’s still going. Because time’s, time’s going to keep on going. But he’s not going to go anywhere. So [Draws lines along $t$ axes of both graphs in Figure 4.4]

[Karen asks if the point is located somewhere else, would it look different]

Adam: Same graph. Because uh, if he starts at that point, time’s going to keep on going. And, he’s not going to go anywhere

In this example, Adam uses the fact that time never stops to create a constant function to depict the skateboarder’s journey. He initially sketches a dot on the origin for the $x$-$t$ graph and the $y$-$t$ graph—but after a brief hesitation, he revises his idea and draws
a horizontal line on the $t$-axes to the right and indicates that the graphs continue on forever. Adam’s response resonates with students’ understanding of equilibrium solutions as functions that stay constant over time and that go on forever. It is interesting to note that Adam’s answer is actually incorrect—the fact that he drew the horizontal line on the horizontal axis does not take into account the fact that the dot is placed on the point (1,4) in the original picture, so the graphs should actually be $x(t)=1$ on Graph 1 and $y(t)=4$ on Graph 2 in Figure 4.4. However, Adam’s reasoning about time never stopping is clearly expressed in his response to the task.

Brandon’s reasoning on this problem, in terms of time never stopping, is similar to that of Adam’s, as evidenced in the following excerpt, some of which is repeated from the earlier transcription for ease of reading:

(Brandon, Interview 2, November, 2002)

Brandon:  [Writes.] Here. [Finishes writing.] Okay. Now. Let's say, we're saying this is four. And, this is one.
Karen:  My, yeah. Just use the same numbers.
Brandon:  Okay. So. This is just time off to there. Here's one. This guy [Upper right graph.] is not moving anywhere. No matter how. [Now refers to Graph 1 Figure 4.5] As long as time advances, it's staying at one. So. It's going to be like that. Here's $y$. And, here's $t$. Again, like that. And. What's this? At four. [Refers to Graph 2 Figure 4.5] So. We'll put four there. And, the same thing here. As long as time advances, it's staying at four. So. That would just be a straight line.
Karen:  Uh, huh.
Brandon:  Where. For these two graphs.
Karen:  Okay. I understand what you're saying. Reasonable. [To Other researcher] Questions about that one?
Other rsrchr:  How would you respond to somebody about? Who says: Well, I don't know about that. I think it's just a point. $X$ equal to one. I just think it's a point there. $Y$ equal to four.
Brandon: Well, it is a point at $y$ equals four. But, then, what about at 1? After two seconds? Where is that point? Because this, this has. This bottom axis is $t$. Or, this, this axis is $t$. So. At, at $t$ equals one second, where's he at? He's at $x$ equals one. At $t$ equals two seconds, he's at $x$ equals one. At $t$ equals infinite seconds, he's at $x$ equals one. So. Yeah. He's, uh, always at that point.

Figure 4.5. Brandon’s Response to Skateboard Task, Part 2

Brandon draws a sketch similar to that shown in Figure 4.5 and explains that “as long as time advances” this is what will happen. He goes on to say that, “when $t$ is at infinite seconds...” strongly suggesting that he is thinking of time as going on forever to construct the two horizontal $x(t)$ and $y(t)$ graphs.

The other example that I present comes from a classroom discussions where students are reasoning about solutions that, when viewed in the phase plane, appear on straight lines and what those solutions might look like in three dimensions. This example is important to examine, as straight line solutions play a primary role in the instructional sequence as students reinvent a way to find the general solution for linear systems of differential equations using straight line solutions. It is also one of few examples that occur in whole class discussion, and the one most uncomplicated by other issues. In
particular, the class is discussing what the solution would look like and John mentions that the three dimensional solutions curves that are straight lines in the phase plane are actually shifts along the $t$-axis so that the all the solutions (in 3-space) form a plane. The discussion then turns to the issue of time as it is represented in three dimensions and how it relates to the phase plane. Adam statement provides evidence that in this situation time is going on forever, in this case both in a positive and negative direction.

Adam: No, no, when you are looking at the phase plane you are looking from infinity down, you are seeing all of $t$, plus, minus, everything

When Adam says this, it is inferred that he is speaking about the fact that time exists in an infinite way and that when he reasons about and with phase planes that time is infinite, or continues forever. In this case he not only implies that time goes on forever in a positive direction, he implies that time goes on forever in a negative direction. Shortly thereafter, John echoes and confirms this thinking when he says, “This line [the solution as viewed in the phase plane] is coming out [that is, out of the plane] but it is also going back for negative infinity too.”

Students’ thinking about time going on forever connects with Nunez’s (1998) hypothesis that a primary metaphor that people use for time is that time is “unidimensional space” or a line. For example, when people talk about time, they typically use the ideas of “time moves ahead” and “yesterday is behind us.” This language demonstrates how the metaphor is time as space of one dimension which people see as lying in front and behind them. This metaphor is what was used as Brandon and Adam talked about the construction of the $x(t)$ graph when it is actually not moving in
space. Time is moving in a unidimensional space and they see it as going on forever ahead of them.

Thinking about this metaphor of unidimensional space provides insight into how students reason about time as a parameter. When students talk about curves in the phase plane, they may extrapolate $x$-$t$ and $y$-$t$ graphs from the curves in the phase plane. They actually construct these graphs in a way that indicates that the curves go on forever. One specific example of this is when the students are investigating position versus velocity graphs while studying the spring mass system. For example, when the solution graph in the phase plane has a circular shape, students typically say that the spring mass system is “oscillating.” I interpret this to mean that they think of continuous circular movement in the phase plane, corresponding to never ending oscillations for the graphs of $x$ and $y$ as functions of time.

A final situation where students use the intuition that “time never stops” involves students interpreting solutions to a linear system that appear as a straight line in the phase plane. In the phase plane straight line solution curves look something like a half line, ending (or starting) at the origin. In three dimensions, however, the solution graphs are not linear. The curves are exponential and grow larger infinitely in one direction, and are asymptotic to $t=0$ in the other direction, continuing on forever. As I illustrate in the next section, students were able to reason about such three dimensional shapes on several occasions.

4.4. Using Both Quantitative and Qualitative Reasoning

The mathematics education reform movement, including the NCTM Standards (2000), and the move to use technology as a teaching tool for exploration and
investigation, has provided new ways of thinking about mathematical activity. This includes thinking about mathematics from a qualitative point of view. Qualitative reasoning is defined as looking at trends in mathematical behaviors of functions and other objects, not by looking for specific numerical values, but by thinking about the general long term performance and developing intuitions to help understand the phenomena under investigation (Monk, 1994). Monk illustrates the difference between reasoning qualitatively and quantitatively in the following way:

A student who uses the model [meaning a physical small model of a real ladder] Quantitatively, primarily taking specific numerical readings, probably represents the model to her or himself as a series of points, or pairs of values. On the other hand, a student who uses the model Qualitatively, employing it to make more general statements about corresponding behaviors of variables, probably represents, and thinks about these quantities in this way [qualitatively] (pg. 183).

In this section, I use Monk’s ideas about how students use physical models Qualitatively and Quantitatively to characterize the mathematical activity of reasoning both qualitatively and quantitatively when thinking parametrically. As conjectured by Monk, some students only reason with one or the other but often they may move from quantitative to qualitative as they work on a problem. My analysis supports this conjecture. Moreover, not only do the students move from quantitative to qualitative reasoning, they sometimes move back and forth between the two.

Goldenberg, Lewis, and O’Keefe (1994) also provide evidence in their work that reasoning both from a quantitative (numerical) and qualitative perspective has value for students in mathematics courses at the high school level. In their work, students reason
about functions using specially designed software referred to as Dynagraph, a computer program that visually shows two number lines, one for $x$, and one for $f(x)$; a student can watch the changes in $f(x)$ as they physically manipulate $x$ values. The program allows for the actual numerical values to be turned off or on. Goldenberg et al (1994) found that leaving out the numbers in the software provided students with new ways of thinking about function and gave support to students’ mathematical understandings. They state,

We found, however, that when students played with the dynamic representations of functions, they began to refer to functions behaviorally in ways that were far from what Monk calls pointwise and much more like the across time perspective he observes students generally to lack... In fact, we may be encouraged by the tantalizing hint that getting rid of numbers may make some important properties of functions easier to understand." (pg. 252-253).

Goldenberg et al (1994) suggest that students’ thinking about functions behaviorally (qualitatively) allows them to better perceive and characterize function patterns. The conjecture is that seeing a pattern of change “across time” makes it easier to see the invariance in the function pattern. Monk (1994) suggested a similar idea; he notes that using the static idea of functions as points of (input, output) does not lend itself to the big picture relationship in many functions of time, whereas reasoning about the functions behaviorally does.

In the results of my study, this use of both quantitative and qualitative understandings existed in parametric reasoning as well. Students who reasoned from both perspectives had significantly richer ways to think about mathematical situations and solutions to systems of differential equations. As they reasoned about interview tasks
and participated in the mathematical activity of the classroom, they would use both of these points of view, sometimes cycling back and forth between the two kinds of reasoning, and sometimes progressing from quantitative reasoning to more qualitative reasoning. The following are three examples of this reasoning.

In the task-based interviews before formal instruction in systems of differential equations, students were asked to consider the Skateboard Task (See Figure 4.6). It is important to emphasize that this task was given to students after instruction on single, first order differential equations and prior to instruction on systems of differential equations. As such, student responses most likely reflect their, intuitive, informal ways of reasoning as well as ways in which they might adapt their understandings from single, first order differential equations.

You are up in a hot air balloon looking down at a skateboarder. He has a piece of chalk on the bottom of his board that is drawing a line on the ground as he moves. Below is an example of what you in the hot air balloon see on the ground (dark line). Sketch possible graphs of the $x$ vs. $t$ and $y$ vs. $t$ graphs for the skateboarder.

![Figure 4.6. Skateboard Task, Part 1.](image-url)
Adam (and the five others when they did this task) responded to this task by sketching two graphs. As shown in Figure 4.7, Adam sketched an $x$-$t$ graph that started at the origin and continued through the first quadrant at an angle of approximately $30^\circ$. Adam’s $y$-$t$ graph also started at the origin and continued with a slope greater than that of his $x$-$t$ graph. He created the graphs by first assigning values to the points and reasoning about the movement with numbers. For example, when thinking about how the graph changes in the $x$-$y$ plane, he said, “as it moves 1 this way, it will move maybe 3 this way” and “it is moving 4 units this way and 1 unit this way.” He also marked off two units on the $t$ axis for each and the number 2 on the $x$-axis and 5 on the $y$-axis on the respective graphs. This demonstrates that he was reasoning about the situation quantitatively in that he used numerical values to create the graphs. He actually counted off specific values of 1, 3, 4, 1, and so forth. He then talked about the situation in qualitative terms, not using numerical values to imagine how $x$ and $y$ change over time. For example, at one point he said, “As the skateboarder moves, the graph of $x$ moves like this” while sketching in the $x$-$t$ graph. He took specific values of $x$ and $y$ first and then reasoned qualitatively that $y$ was moving faster than $x$ in a qualitative way.

When he moved to discussing the $y$-$t$ graph, he returned to using specific numerical values. Thus, he would cycle in his work between qualitatively thinking about the situation and number based thinking to create and justify the shape of his $x$-$t$ and $y$-$t$ graphs.
Cycling between qualitative and quantitative reasoning was present in classroom discussion by other students as well in the following situation. This excerpt is from the second day of instruction on systems of differential equations. The students were given the following two systems differential equations, each of which were population models, and asked to determine which one models a competing system scenario and which one models a cooperating system scenario:

\[
\begin{align*}
\frac{dx}{dt} &= -5x + 2xy \\
\frac{dy}{dt} &= -4y + 3xy
\end{align*}
\]

\[
\begin{align*}
\frac{dx}{dt} &= 3x - 2xy \\
\frac{dy}{dt} &= y - 4xy
\end{align*}
\]

John explains his thinking on this problem in the following way:

John: Okay, if you look at the first system of equations, it says \(\frac{dx}{dt} = -5x + 2xy\). If there’s no \(y\) population, you have a decreasing system. If \(y\) is 0, \(-5x\) is negative, it’s decreasing, populations is decreasing. If you go to the other one, \(\frac{dy}{dt}\), if \(x\) is 0, then you have the same thing where population is decreasing again. However if \(x\) become large enough, in that second equation, if that population becomes even at
1, you still have a decreasing value, say both \( x \) and \( y \) are equal to 1, you’d have \( -4+3 \) which would still be negative, but if \( x \) was 2, you’d have an actual increasing population, so as more \( x \) interacts with \( y \), you would have population growth instead of decay.

John uses quantitative reasoning when he says “if that population becomes even at 1, you still have a decreasing value, say both \( x \) and \( y \) are equal to 1, you’d have \(-4+3\), which would still be negative...” However, that statement was prefaced with qualitative thinking when he said “However, if \( x \) becomes large enough” and ends with another comment that is a result of qualitative reasoning: “so as more \( x \) interacts with \( y \), you would have population growth instead of decay.”

I note that time is not explicit in this situation, but present in an implicit way. Moreover, John uses both quantitative and qualitative forms of reasoning to infer what happens to the populations as time progresses.

4.5. Using Both Discrete and Continuous Images of Time

In this section, I discuss and illustrate students’ reasoning about time as moving discretely and moving continuously. This characterization involves conceptualizations about time itself and the qualities of time. Thinking about time as if it “jumps” in discrete amounts allows thinking about specific moments in time and their relationship to each other. As students’ reason about time in a discrete manner, they build new understandings of time as a continuous quantity and are able to reason using its continuous behavior as well. For example, consider the task in Figure 4.8 (posed in the interview prior to instruction on systems of differential equations) and Max’s response.
The following is a sketch of $x$ vs. $y$ that is produced over time. Look at each of the following suggested systems of differential equations and decide if any of the choices could describe the graph.

$$\begin{align*}
\frac{dx}{dt} &= x + y \\
\frac{dy}{dt} &= -x + y
\end{align*}$$

$$\begin{align*}
\frac{dx}{dt} &= 2x - 2y \\
\frac{dy}{dt} &= x - y
\end{align*}$$

Figure 4.8. Interview Task About Eigensolutions

(Max, Interview 2, November, 2002)

Max: [Draws slash through choice 1.] This is not true. Because $x$ and $y$ [Refers to graph] value is always positive on this graph... So. This $dy/dt$ give us decreasing rate of change. [Karen: Why?]

Max: Because $x$ and $y$ are positive. So, if we just put those number in. [Refers to choice 1] then $x$ is positive. So. This is not possible... This, any case, on this graph, $dy/dt$ would be negative...Because $x$ value is larger than $y$ value. So this is not true.

Max: [Now points to Choice 2.] This might be true. But, this is saying $dx/dt$ equals $dy/dt$. Their rates of change are the same. [Karen: Uh, huh.]

Max: [Points to graph.] But, it's not. [Karen: Why not?]

Max: [Places pen along slope line.] It's not, um. The slope way's not one.

Karen: Uh, huh. [Pause.] I'm not sure what the slope one would have to do with this.

Max: After one minute, this same change of $x$ is same as change of $y$. So. Saying, after one minute, [Refers to slope line] from here, $x$ is changing from one to three. [Makes notations on x axis.] And, $y$ is changing one, two, three, [Notates on y axis.] too. But. That's, uh, that's not the point on the graph. [Karen: uh huh]

Max: This is saying something, like, after one minute, $x$ has to change two. $y$ has to change, maybe, one. [Karen: Uh, huh.]

Max: And, you can say that, [Refers to graph.] you can tell that slope is not [Swivels pen slightly.] forty-five degree angle....

Max: So this is not right.[crosses out number 2] And, this. [Points to Choice three.] Might be.[Writes number 4 above 2x.] Six. This is. [Circles choice 3] This could be. Right.

[Pause.] Because it's saying it's $2(x-y)$. So $dx/dt$ is two times larger than $dy/dt$. Right? [Karen: Uh, huh.]
Max: Because $x - y$ is always same value. So $\frac{dx}{dt}$ has two times bigger than this number. $2(x-y)$...Which is this. [Refers to $x-y$] So let's say $\frac{dx}{dt}$ equals two times $\frac{dy}{dt}$. [Writes formula on sheet under choice three.] [Karen: Oh. That's nice. Uh, huh.]

Max: And, from this graph, it's not accurate. But, looks like. [Puts right hand over sheet.] At the same time $x$ is moving twice as much as $y$ does.

Max first thinks about the situation in a discrete manner. In the quotation with a single underline, he discusses what happens after one minute. He reasons about the changing graph by thinking in steps. This allows him to develop a way to think about the situation. As he continues to work on the task, he gradually shifts his time based reasoning to a continuous mode. He thinks of $x$ and $y$ as continuous functions of time and reasons about them as time is moving continuously in a more holistic sense. At this point, in the quotation with the double underline, he does not need to actually think of specific discrete values, but reasons parametrically with a continuous image of time to obtain a correct solution. Of the six students interviewed, a total of four reasoned in ways similar to Max.

In this example, time is both thought of as discrete jump and as a continuous motion. Both of these ideas are important as students continue to participate in the classroom to learn about solutions to systems of differential equations.

4.6. Imagining the Motion

The fourth characterization of parametric reasoning is the mathematical activity of imagining the motion of the function, specifically a solution function. Students conceptualize graphs of solution functions as being created over time, create or interpret representations or use gestures as they imagine this motion and ways to think about and
understand the solution functions. Imaging the motion contributes to their understanding of time as the independent variable in the solution functions and strengthens their parametric reasoning. There are three forms of mathematical activity by which students’ talk and gestures centers on imagining the motion: three dimensional spatial visualization, using the fictive motion metaphor, and fusion of context and representation. In this section I present exemplary examples of each of these three activities.

4.6.1 Three dimensional visualization.

Students often are able to visualize and then describe or physically illustrate functions in three dimensions (or in two dimensions when studying first order differential equations). In each day of the instructional sequence, there is at least one example of this visualization, sometimes from looking at the Java applet and using it to help visualization and sometimes using gestures (See Appendix B for Adam and Brandon coding). I next provide an example from the seventh class day on systems of differential equations instruction. This particular example was chosen because it occurred during whole class discussion and the student involved was Brandon, one of the case studies I examine in Chapter 5. Also, all the students in the class were involved in the discussion and several verbally supported Brandon’s gestures and explanations. In the example, students were discussing what a straight line solution would look like in three dimensions. In two dimensions a solution is called a straight line solution if it heads toward or away from the origin along a straight line. Brandon begins to describe the behavior and then is prompted by the teacher to come up to the board and show what he is thinking.
Brandon: Ok, it seems like if you start up here somewhere on this side of this line, it would swing towards that \([t\text{-axis}]\) but coming out and then away this way, all the way moving along \(t\), so coming out this way and then for that way, lets say it starts down here on this side of the line, its coming up and also moving along \(t\), and then its swing out this way [John: Just show where it starts at that line] Right here. Then it’s moving along \(t\) moving towards the line. I think you also have to see some vectors along here because if it is off of the line, it's going to move...

In this piece of dialogue, Brandon is in the front of the class [See Figure 4.9] and shows his idea of the creation of a particular three-dimensional exponential function. He verbalizes his ideas but also physically acts out the function creating itself over time. It is an interesting question of what “it” is in his speech, but I am positing that it is the function that is making itself. His reasoning is qualitative; he is not using time in a...
quantitative manner, i.e., minutes or seconds, but still creates an in-time movement to illustrate the solution.

Janvier (1998) suggests that students intuitively understand time. The analysis here supports that notion and indicates that students build on that intuition to conceptualize solutions of systems of differential equations. Janvier (1998) calls graphs that use time as dependent variable “chronicles” and posits that students have an inherent facility to create dynamic time-based graphs in their mind. He states, “In fact, such changing elements can be represented in the mind as changing. More precisely, the changes of such function-variables are dynamically representable in the mind as the phenomenon is mentally simulated as a temporal event” (p. 82). As students learn mathematics that involves the parameter time they imagine and often physically display solution function movement over time.

4.6.2 Using the fictive motion metaphor.

Lakoff and Nunez (2002) suggest that people use different metaphors to understand and reason about functions. One common metaphor that they have identified is the fictive motion metaphor. Their description of this metaphor suggests that functions are thought of as something in motion. Students use motion (particularly of a graph) to help them develop and deepen their understanding of function. Analysis of the discourse in this differential equation class provides evidence that this mathematical activity is very common as students (specifically Brandon and Adam, since analysis focused on them) reason about solutions of differential equations. Analysis of classroom discussion
indicates that it was extremely common for students’ to conceptualize solutions as being moving objects that create a trace. Examples such as:

“when R is decreasing, F is decreasing”

“it’s going to go…”

“as time passes, the solution increases exponentially”

“as R goes up, rate of change goes down”

“it never passes the origin, but gets close”

were common ways in which students reasoned about solutions to systems of differential equations in ways that involved motion and movement. The previous episode in which Brandon was imagining the 3D graph includes several examples of such fictive motion. For example, Brandon says “its coming up and also moving along t.”

In the whole class discussions, students used the fictive motion metaphor 36 times over the nine class days. I note that the use of this metaphor dropped off during the last three days, primarily because students were developing an analytical algorithm to find the general solutions to a linear system of homogenous differential equations, so the focus was on the algebra. Table 4.1 gives the number of time the fictive motion metaphor (FMM) was used by students in whole class discussion. The counts do does include the common use of this language in the interviews both before and after the instructional sequence.
Table 4.1

_Fictive Motion Metaphor_

<table>
<thead>
<tr>
<th>Class Day</th>
<th>Number of times FMM is used in whole class discussion</th>
</tr>
</thead>
<tbody>
<tr>
<td>11/6/02</td>
<td>3 (not including airplane visualization task)</td>
</tr>
<tr>
<td>11/11/02</td>
<td>1</td>
</tr>
<tr>
<td>11/13/02</td>
<td>0</td>
</tr>
<tr>
<td>11/18/02</td>
<td>0</td>
</tr>
<tr>
<td>11/20/02</td>
<td>3</td>
</tr>
<tr>
<td>11/25/02</td>
<td>23</td>
</tr>
<tr>
<td>11/27/02</td>
<td>3</td>
</tr>
<tr>
<td>12/02/02</td>
<td>0</td>
</tr>
<tr>
<td>12/04/02</td>
<td>0</td>
</tr>
</tbody>
</table>

One specific example of the extensive use of the fictive motion metaphor occurs in Adam’s interview conducted at the end of the semester. The task is the same one described earlier in this chapter (see Figure 4.8). Adam comes to the conclusion that Choice 3 is the best possibility and then Adam and the two researchers continue discussing the straight line on the graph from the task. One of the researchers suggests that the three dimensional solution is concave down. (Earlier in the interview, Adam had discussed the fact that the three dimensional solution function would be exponential in nature, but the concavity was not mentioned).
Adam is then asked to explain why the three dimensional solution curves would be concave down. His response, given after several seconds of thinking, indicates that he is using the fictive motion metaphor as a way to conceptualize the solutions:

(Adam, Second Interview, December, 2002)

Adam: Oh, it is because, no, look, look, when time increases at the same amount of time [Karen, time constantly increases] yeah, but this is going to get bigger and bigger [sketches vectors on the straight line solution- see Figure 4.10] so it is going to start going bigger in this direction [two hands gestures in a concave down motion in the x direction] than it is in this direction [points up in the t direction] so that is why it is concave down..[Karen: That’s right] Otherwise it wouldn’t be a straight-line solution because if it slowed down, it would have to slow down to a point [holds up one finger to indicate point] and then reverse the other direction so it would run into a different equilibrium point!

![Figure 4.10. Simulation of Adam’s Drawing of Larger and Larger Vectors on the Graph](image)

Adam reasoned about time being continuous in the first sentence and then continued reasoning about the vectors growing as time increases. Time is not explicit in the phase plane, but he integrates it into his explanation. In the last sentence, he particularly uses the fictive motion metaphor for function as he discusses the solution slowing down and coming to a stop (slow down to a point) and then reversing direction. For Adam, there is
some entity at the end of the function that is actually moving around on a path that changes. He almost gives the function human properties as it controls its own movements to create the straight-line solutions for the problem.

The six students who were interviewed at the end of the semester used the fictive motion metaphor extensively, especially when the students were asked to reason about both phase plane and three dimensional graphical representations of the solutions. Thus, even though the use of this metaphor dropped off during students’ algebraic work near the end of the semester, it remained a primary reasoning tool for students as they integrated time into the tasks for the interviews.

4.6.3. Fusion of context and representations

The third aspect of “imagining the motion” involves the idea that sometimes students merge their reasoning about the real life context of the problem with the representations of the context that involve time. In the instructional sequence, there are two primary contexts that students use as a basis for their mathematizing: systems of differential equations that model a spring with a mass moving horizontally back and forth and systems of differential equations that model interacting populations. In both of these instructional sets of tasks, the students often combine their discussion of the representations and the situations as they reason about the time parameter. Nemirovsky, Tierney, and Wright (1998) refer to such merging as “fusion.” I offer two examples of such fusing after a brief discussion of the idea of fusion in mathematical activity and discourse.
Fusion, as described by Nemirovsky et al. (1998), involves the linguistic phenomenon in which a person blends qualities of a representation with an action or events depicted by the representation. The point Nemirovsky and colleagues make is:

Fusion is merging qualities of symbols with qualities of the signified events or situations, that is, taking, gesturing, and envisioning in ways that do not distinguish between symbols and referents....Fusion involves the construction of a discourse in which indexical terms, such as I, it, or here, are constantly revised, dissolving boundaries among all the aspects that are relevant to the grapher’s experience. (pg. 141-142)

Students in the research differential equations class occasionally used language where they fused together the action in the context and the representation of the action in a similar way that Nemirovsky et al report in their paper. In my analysis, I coded 4 times that this fusion of context and representation occurred in whole class discussion and 4 times that it occurred during the interviews. The following two examples were chosen to exemplify the fusion phenomenon as it occurred in the spring mass and population scenarios. Each example illustrates students’ reasoning about the parameter time in a way that fuses the context with the representation.

4.6.3.1 Example from spring mass scenario

In the following episode the students were investigating a phase plane representation of the system of differential equations that models a spring mass situation. In this activity, the students were interpreting the behavior of solutions that are drawn as curves in the phase plane. The question was about the graphs of the solutions in the phase
plane, but as the conversation unfolds, the students discussed both the solutions and the actual context as time passes. That is, they fuse these two aspects of the phenomenon together in their language. In addition, time again appears implicitly in the situation and becomes part of their conversations without explicit notation.

(Classroom observation, November 18, 2002)

John: The second and fourth quadrants attract it to the origin. The other two quadrants let you pass... the third quadrant sends you up like that and the first quadrant sends you down and then back up.

Adam: Josh was saying that the first quadrant would send you around but part of the second quadrant would send you around also through the first quadrant into the fourth quadrant into 0

Professor: And third quadrant sends you to second quadrant [Bob: and then into 0] So a lot of things have happened between. It changes from a center to this. Okay, We come to figure out what happens and also what happens in between. Okay, what is the physical situation?

Other student: There is more friction so friction at n=4 is such a big force that actually it only lets the spring go back to the equilibrium point.

Professor: So it is like the spring because the friction is so large, you stretch it and it.

Brandon: Basically, it stops real fast.

In the beginning of the passage the students are thinking about the graphs in the phase plane. John reasons about the graphs that are created as solutions as a person when he says “the third quadrant sends you up like that.” He also verbalizes the metaphor of a quadrant having a way to control the solution and operating on the solutions in some physical way when he says, “the third quadrant sends you up like that and the first quadrant sends you down and then back up”. This is not really fusing context and its representation but the discourse sets the stage for the fusion. Then the Other student states, “There is more friction so friction at n=4 is such a big force that actually it only lets the spring go back to the equilibrium point.” In this quote the students explain how both the spring returns to the equilibrium point and the curve representing its behavior
returns to the origin. Finally, Brandon, in the last sentence, says “basically it stops real fast”. In this quote the “it” dissolves the boundaries between the physical motion of the spring and the graph of the solution in the phase plane.

4.6.3.2 Example from population scenario

On the first day of the instruction on systems, the students are asked to examine

\[
\frac{dR}{dt} = 3R - 1.4RF
\]

\[
\frac{dF}{dt} = -F + 0.8RF
\]

where \( F \) represents number of foxes (predator) and \( R \) represents number of rabbits (prey). In the following excerpt Ephraim clarifies the statement of another student regarding how the rabbit and fox population would evolve over time:

(Classroom observation, November 20, 2002)

Ephraim: He’s saying at some point, there is an equilibrium solution which means both the rate of changes would be not necessarily equal for population but they would produce just enough, and they both produce the same amount and I guess eat the same amount.

Ephraim is interpreting the equilibrium to the system of differential equations and his language integrates (or fuses) the rate of change equation and the population it is modeling. He says “the rate of changes would be not necessarily equal [meaning the differential equations as well as the rate of changes of the populations]” but then says “they would produce just enough, and they both produce the same amount and I guess eat the same amount,” indicating the predator and prey populations from the scenario. His reasoning in this statement fuses together the representation (differential equation) and the context (predator and prey populations).
4.7. Discussion

In this chapter, I identified five types of mathematical activity that characterize parametric reasoning. This chapter focused on predominantly on Brandon and Adam’s use of these five types with additional evidence from other students during interviews and classes. These five types of mathematical activity form a characterization of students’ time based parametric reasoning. In this last section, I offer some more general remarks about the characterization, compare the analysis to Drijver’s work on parameter, and discuss the influence of the instructional sequence on parametric reasoning.

Each of the five characterizations of parametric reasoning described above appears during the nine days of instruction on solutions to systems of differential equations. In addition, there are some trends that appeared in the analysis. First, parametric reasoning was predominant when graphical representations were the object of discussion. Students’ discussion of graphs and what they represent showed many more instances of parametric reasoning than in the last few days of class which were primarily about algebraic representations of systems and their solutions.

Second, the five types of mathematical activity typically did not occur in isolation. Some of the examples I chose to illustrate one of the characterizations could also be interpreted in terms of one or more of the other types of mathematical activity. The fictive motion metaphor, for example, while described as part of the fifth type of mathematical activity is present in every other example. In other sections of the analysis, particularly in the relationship between the mathematical activities of qualitative and qualitative reasoning and use of discrete or continuous images of time, there is an overlap of the different types of mathematical activity. Depending on the perspective the
researcher takes, the data might support qualitative and quantitative reasoning or reasoning with discrete and continuous images of time. In addition, reasoning qualitatively may involve discrete images of time and sometimes it may involve continuous images of time. Likewise, reasoning quantitatively may involve either discrete or continuous images of time. When discussing a problem in which the students modeled the movement of a skyscraper with a non-linear system of differential equations and then analyzed solutions using a phase plane representations, Bob used the following language.

(Classroom observation, November 25, 2002)

Bob: Using the DE applet, instead of just going to 10 in units, I went out to 100 time units and its[values of the curve in the phase plane] getting small like, you know, to the negative 40,50 something like that, but it still didn’t reach it [the equilibrium solution function seen as a point in the phase plane], it was still circling around. So I don’t think it ever reaches, it ever touches. I’m just observing.

Here, Bob is using discrete images of time as he talks about specific values, i.e. “100 time units,” but he is reasoning qualitatively.

In various ways, parametric reasoning of this type is similar to Freudenthal and Drijvers’ (2003) notion of parameter, but in other ways, it is different. Drijvers’ analysis includes four purposes for parameter: placeholder, generalizer, unknown, and changing quantity. The fourth purpose of parameter as a changing quantity is particularly apparent in Drijvers’ work when students are using software to vary a parameter and see how its changing quantity affects function graphs. My characterization of parametric reasoning also involves parameter as a changing quantity, but is somewhat different. Drijvers bases his analysis of parameter on Freudenthal’s notion of parameter as a “sleeping variable”
which wakes up, changes, and then goes back to sleep again. I focus exclusively on time as the parameter and time in this characterization “never sleeps”; it never stops and is usually imagined as continuous. Another difference between my characterization and Drijvers' is that he assumes that parameter can be many different types of quantities, or have no context at all. His research is particularly focused on parameters when used in equations of lines, parabolas, and other polynomial types. I only investigate the use of time as parameter, because my focus is on time as a dynamic parameter.

The analysis in this chapter highlighted the following five kinds of mathematical activity that characterize students’ reasoning parametrically with time: making time an explicit quantity, using the intuition from real life that “time never stops,” using both quantitative and qualitative reasoning, using both discrete and continuous imaging of time, and imagining the motion. These forms of activity are the means by which students conceptualize solutions to systems of differential equations. Also, conceptualizing solutions may deepen the understanding of time as a parameter, but that is an issue for future work.

The results of this study lead to possibilities that mathematics students can use their inherent understanding of the dynamic parameter time as a springboard for conceptualizing the mathematics of change at the university level. This finding stands in contrast to Janvier’s (1998) contention that time reasoning can be obstacles to learning. As science and mathematics continues to include the study of dynamical systems, which are based on time, understanding how students reason parametrically will be of greater interest. Since the results of this study provided evidence that students were able to reason parametrically before the instruction on systems began, parametric reasoning may
be an area to investigate in the 6-12 curriculum. Since students understand time from an early age, it may be that the five types of mathematical activity that I characterized in this chapter may be a part of students’ thinking in earlier grade levels.
CHAPTER 5 CONTRASTING CASE STUDIES OF ADAM AND BRANDON

5.1 Introduction

In this chapter, I describe Adam and Brandon’s mathematical activity as revealed in their interviews and their participation in class. The purpose of the case studies is to illustrate how these students used different ways of reasoning when learning about solutions to systems of differential equations. The chapter is divided into two primary sections.

First, I provide a description of Adam’s and Brandon’s mathematical participation in the interviews and in the class. In qualitative research, this section is called “thick description.” Thick description is a research device traditionally used by ethnographers where description is deep, detailed, and thorough (Geertz, 1973). In the thick descriptions I illustrate ways the different forms of reasoning either promote or constrain growth in the two students’ conceptualizations about systems of differential equations.

Second, I “step back” from the thick description in the case studies and compare and contrast Adam and Brandon’s ways of reasoning. I also identify and discuss cross cutting themes that emerged during the analysis and writing of each of the case studies. These themes relate to these students’ ways of reasoning and how they either support or constrain new conceptualizations for Brandon and Adam. Although parametric reasoning
is one of the ways of reasoning the data analysis revealed, I chose to not include this form of reasoning in the overarching themes, as I discuss it in detail in Chapter 4.

5.2. Case Study Descriptions

5.2.1. Adam

In this section, I present a description and analysis of Adam’s participation in the mathematics before, during and after instruction on systems of differential equations. Adam’s participation provided evidence of his use of mathematical reasoning that he had developed from first order differential equations and other mathematics experiences. Also, as the class continues through the instructional sequence, Adam’s reasoning about solutions to systems increases in sophistication and he is able to show understanding of several of the concepts involved in understanding solutions to systems by using parametric and other ways of reasoning by the end of the instruction.

Adam had attended all but one of the classes during the first nine weeks and had completed his homework regularly, although his written work was graded average. Adam regularly participated in whole class and small group discussions and contributed significantly to the discourse in the classroom by sharing his ideas regularly. The researchers in the class identified him as a leader in the discussions. He was not afraid to suggest ideas and provide justification for the ideas he proposed. However, he did not always suggest reasoning that was correct. By the time the instruction on systems started, Adam was comfortable with the inquiry-oriented pedagogical approach and so it was common for him to provide ideas however tentative.
5.2.1.1 Adam’s first interview

Adam participated in a semi-structured task-based after nine weeks of instruction on first order differential equations where he used both parametric and other types of reasoning. He was able to do some of the interview tasks easily and justify his thinking. However, he had limited ways to reason with success about solutions to systems of differential equations in general, as he was unable to describe or reason about solutions correctly.

The first interview task involved Adam reasoning about two systems of differential equations that model two different types of predator-prey situations (see Figure 5.1). Adam used two types of reasoning, one that I characterize as “reasoning with understanding of the derivative” and the other that I characterize as “reasoning with the context of the situation.” Adam looked at the system in the first interview, and began substituting numbers for the $x$’s and $y$’s in the $\frac{dx}{dt}$ and $\frac{dy}{dt}$ differential equations. From his discourse, it is evident that he has strengthened his understanding of rate of change equations, i.e. differential equations; when he was presented with a similar problem on Day 1, he participated in the group, but neither his group or the whole class were able to come to any conclusion. To contrast his reasoning, here is an example of Adam’s statements from the first day of the class when small groups were asked to solve this problem.
Consider the following systems of rate of change equations:

**System A**
\[
\begin{align*}
\frac{dx}{dt} &= 3x(1 - \frac{x}{10}) - 20xy \\
\frac{dy}{dt} &= -5y + \frac{xy}{20}
\end{align*}
\]

**System B**
\[
\begin{align*}
\frac{dx}{dt} &= 0.3x - \frac{xy}{100} \\
\frac{dy}{dt} &= 15y(1 - \frac{y}{17}) + 25xy
\end{align*}
\]

In both of these systems, \(x\) and \(y\) refer to the number of two different species at time \(t\). In particular, in one of these systems the prey are large animals and the predators are small animals, such as piranhas and humans. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey.

Figure out which system is which and explain the reasoning behind your decision. Try to develop more than one way to reach a conclusion.

**Figure 5.1. Predator Prey Task in First Interview**

(Class observation August 26, 2002)

Adam: One predator eats a lot of prey. In the other one, a lot of predators eat one prey. We are supposed to match which \(x\) is the predator and \(y\) is the prey. I assume the same \([x\ and y]\) for both.

Jerry: See you’d need a lot of \(x\) it its smaller. If \(x\) is smaller then you’d need more \(x\), more predator.

Adam: To take care of one prey. So this is the small predator.

After ten more minutes:

Jerry: The predators are smaller, need more of them to eat prey

Adam: That’s what we are saying. This [first system] is going to be the small predator. This [second system] is the large predator.

... 

Adam: In the first equation, when you increase \(y\), the rate of change of \(x\) goes down. It comes out positive or negative, you can determine if it’s going to take a large number of predators to eat the prey. If you pick \(x\) or \(y\) in the rate of change of \(x\), the larger the \(y\), the smaller the \(x\). In, when you put in a large \(x\), the rate of change of \(y\) is going down, because there is more prey than the predators are going to go up too.
In this early small group discussion, Adam showed evidence of having two ways to reason with small success about the behavior of the system. First, he used the context of predator-prey and his knowledge of this particular real situation. Second, he used his understanding of derivative. He and the other students were comfortable with interpreting the situation using the calculus tool that if the derivative is positive, or negative, the quantity will increase or decrease respectively. This fits with two ideas from the research literature. One, students use understanding of derivative to reason about rate of change equations (Artigue, 1992) and, two, context is useful as a starting point for advancing mathematical activity (Rasmussen & King, 1997). Adam’s reasoning was similar but much more sophisticated when he worked on the task in the first interview. He immediately substituted in values into the two differential equations and used the values to determine the correct answer. Clearly his understanding of differential equations that depend on time had developed through the work with single differential equations. He reasoned with more authority and less hesitation:

(Adam, First Interview, November, 2002)

Adam: This one’s the rate of change of our $x$ (speaking about system A). And this is the rate of change for $y$. If $x$ increases, if $y$ increases in this one $[3x-2xy]$ there’s going to be less, less of a growth of the rate of change of $x$. [Refers to second DE] This is the rate of change of $y$. If $x$ increases, the rate of change of $y$ is going to be less. This is the rate of change of $x$. So in this one, $x$ has a, $y$ as a negative effect. In this one, $x$ has a negative effect. In this one, $y$ has a positive effect. In this one $x$ has a positive effect. On this part. But as $x$ increases over here this is a rate of change of $x$. If $x$ was by itself, if there weren’t any then it would be going down. If $y$ was by itself, if there weren’t any $x$’s, it would be going down. But, when you throw an $x$ in there, I mean, when you throw a $y$ in there, it has a positive effect on $x$. When you throw an $x$ in here, it has a positive effect on $y$. So, I think, in reverse, what I said in the beginning. I think I’m going to say that this, this one is the bad interaction [meaning competitive system]. And this one’s the good [meaning cooperative system] The good interaction where they
help each other. Because this one. Yeah. Because of what I said right there.

He quickly reasoned to the correct answer and his reasoning was more developed in this monologue.

In the third part of the second task, the skateboard problem, Adam was asked to work backwards. He was given various $x$-$t$ and $y$-$t$ graphs for a skateboarder and asked if they were appropriate graphs for the situation given in the task. The first sets of graphs were two exponential looking curves, which he identified correctly as possible. The second set of graphs showed an $x$-$t$ graph as a concave up curve and a $y$-$t$ graph as a concave down curve. Adam originally thought that it might be possible to have these two graphs for $x$-$t$ and $y$-$t$ graphs, but then he changed his mind.

(Adam, First Interview, November, 2002)

Adam: Holy xxx, I can’t do this. As time elapses, it goes faster in the $y$ direction. And, it slows down in the $x$ direction. On this one, I don’t know how to move my pen, Because on that one, they were both, they both increased, On this one, they both looked the same, On this one. [Karen asks what is bothering Adam]

... Adam: Well, this one’s increasing. And this one’s increasing, too. But the rate of this one is increasing. The rate of increase on this one is decreasing. So I’ve got a problem with that because, on that one, they were both increasing. So, I could go faster and faster in both directions. On this one, it looked about the same. So, I could go the same rate in both directions. On this one, it looked about the same. On that one, they look, they’re opposite. Well, no they’re not opposite...

This problem was designed to probe student’s informal ways of thinking about eigensolutions to linear systems of differential equations and what such solutions would look like when viewed in $x$-$t$ and $y$-$t$ planes. Solutions to systems can be represented by
three dimensional curves. Eigensolutions can be represented by three dimensional exponential curves or pairs of $x(t)$ and $y(t)$ equations that are exponential functions with the same exponent if the eigenvalues are real. In this interview episode he expressed his ideas as he discussed the skateboard problem, foreshadowing the subsequent class discussion. He correctly reasoned using understanding of rate and graphs, that, in this particular situation, one curve cannot be concave up while the other is concave down and have the result be a straight line.

5.2.1.2 Adam’s participation in class

In this section, I describe Adam’s participation in class and his interactions with others in his group, and describe his conceptualizing of solutions to systems of differential equations. He used his understanding of the contexts of the tasks; his understanding from first order differential equations, graphical visualizations, and the parameter of time to reason in his participation. In Table 5.1, I provide a brief summary of his mathematical activity in class. Those addressed in the case study are italicized.
### Summary of Adam’s Mathematical Activity

<table>
<thead>
<tr>
<th>Day</th>
<th>Adam’s primary mathematical activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td><em>Used understanding of derivative to reason about increasing and decreasing Conceptualized a three dimensional curve and its projections into two dimensions</em></td>
</tr>
<tr>
<td>Day 2</td>
<td><em>Used visualization with computer program Used understanding of equilibrium solution and parametric reasoning to conceptualize equilibrium Used vector algebra to begin thinking about derivative in the phase plane</em></td>
</tr>
<tr>
<td>Day 3</td>
<td>Not in class</td>
</tr>
<tr>
<td>Day 4</td>
<td><em>Used opposite of parametric reasoning to take two parameterized functions and co-vary them on one graph Used limited physics understanding to participate in development of spring mass system of differential equations Used visualization and context to begin thinking about straight line solutions as friction increases in the spring mass context</em></td>
</tr>
<tr>
<td>Day 5</td>
<td><em>Used parametric reasoning to think about solutions in the phase plane Developed reasons to believe straight line solutions exist when his ideas did not match up</em></td>
</tr>
<tr>
<td>Day 6</td>
<td><em>Used visualization and vector knowledge to understand straight line solutions in the phase plane Used the “e thing” to find solutions in three space of straight line solutions Used several conceptualizations from first order differential equations to confirm understanding of all straight line solutions for one of the lines forming a plane in space</em></td>
</tr>
<tr>
<td>Day 7</td>
<td><em>Used idea of the plane of straight line solutions to imagine what a solution not lying on the phase plane would be Accepted ideas of sum of solutions being a solution to use Participated in analytic justification</em></td>
</tr>
<tr>
<td>Day 8</td>
<td><em>Used algebraic skills to continue to find sums of solutions as solutions Used ideas of linear combination Developed ideas of linear homogeneous solutions</em></td>
</tr>
<tr>
<td>Day 9</td>
<td>Participated in development of complex eigensolutions situation</td>
</tr>
</tbody>
</table>

**Day 1.** On the first day of the instructional sequence, students participate in an analysis of a predator-prey solution. I do not include a description here of that episode, as Adam’s ways of reasoning were almost identical to the predator-prey task description in his first interview. This next task is then introduced to provide support of the students’
conceptualization of a solution to a system, and the many ways to represent that solution

[See Figure 5.2]

This particular task invited reasoning about curves in three space as representations of solutions. Jerry, one of the other students in Adam’s group, verbalized the idea, which Adam, although he did not say it himself, indicated that he understood and supported.

**Figure 5.2. The Airplane Task**
Jerry: Alpha sees the three dimensional view of it where Gamma sees a two dimensional project of that three dimensional view on that plane, Beta sees it from another plane, Delta sees it from a third and Alpha sees the combinations of those three planes to a three dimensional model.

Adam: Jerry rocks, I just want to tell everybody, he’s right.

Adam’s small group regularly communicated their reasoning and though Adam did not talk frequently, he was involved in the conceptualizations as they developed and agreed publicly. Adam and his group’s reasoning in this situation is based on visualization. They visualize the situation and then change perspectives to look at the imagined curve from different directions.

Adam, John, and Jerry worked hard at solving problem (f) and developed a description of the dot on the propellor moving away at a rate proportional to the speed. Adam did not speak frequently, but participated as they concluded that the person in front of the airplane would see a short horizontal line, and that they others would see an arc moving out from the center. This description foreshadows Adam’s later verbalization of what a solution that is represented as a straight line in the phase plane will actually look like as a curve in three dimensions. Adam used a way of reasoning based on his visualization of the curve.

Day 2. In Day 2, the students worked in the computer lab to reason about the solutions as curves in space. Adam immediately connected the computer generated graphs that the students were creating using the computer program to the previous day’s airplane task.

(Adam uses the curser to follow the solution as he had watched it be created)
Adam: That looks something like a pipe cleaner: No look at this, it gets smaller in the back so this thing does have perspective? (see Figure 5.3)

He was referring specifically to the airplane task where they used pipe cleaners to represent the trace of the red line in the air. Here Adam makes connections; he connected the ideas from the airplane task and the DEExplorer task. At this point, Adam had a way to reason about the solution represented as a curve in three space and its various projections using his understanding of time, and visualization skills. The use of the computer program here was particularly important in Adam’s early conceptualizations. Dynamic computer visualization contributed to understanding of systems of differential equations. This visualization was not grounded in an understanding of systems of differential equations or rates of change, but in thinking about the picture that the computer created. The Java applet actually created the picture as you watched, and it

Figure 5.3. DEExplorer Applet with One Solution
created the curve slowly so that students could watch as time passed, although it does not happen in real time. By watching this, time was made explicit for the students, and Adam in particular. This visualization contributed heavily to his reasoning about solutions to systems as he continues in the course. However, making the connections to the system itself and rate of change came later.

The professor next asked the students to think about equilibrium solutions to the given system. Equilibrium solutions to systems of differential equations have very similar behavior as those for single differential equations. Adam’s reasoned about equilibrium solution functions for systems using his understanding of equilibrium solutions for first order differential equations and his knowledge of exponential functions. In his group, the students set each differential equation equal to 0 to find two values. This was exactly the procedure they developed to find equilibrium solutions for single autonomous differential equations. They originally found the wrong values and when they used them in the program, the curves looked exponential in the \(x-t\) and \(y-t\) plane, so Adam knew that they were wrong. Eventually, they determined correct numbers and placed them in the program. The following conversation began after the question was asked, “Are there any equilibrium solutions?”

(Class Observation, November 11, 2002)

Adam: Put 0 over here, then find out? I don’t know what that means
Joe: What I’m going to try and do is put into another spot \(R= .8\) and \(R=1/.8\)
Adam: So why do we have \(R= .8\) and \(R=.8\) No you divide.
Researcher: What do you expect to see?
Adam: Really not sure yet. That’s not right (when they see an exponential curve on the \(R-t\) plane) we set them equal to 0. On equilibrium solutions, the rate of change is 0. . . .
Adam: There you go (sees a straight line when looking at any view)
Joe: It’s a dot (looking in \(R-F\) plane)
Professor: What is an equilibrium solution?
Adam: It’s when the number of foxes and the number of rabbits stay the same for all time.

Here Adam was voicing his understanding of equilibrium solution and this understanding was supported by his visualization in the DEExplorer. Also, shortly after that, Adam noticed the dot and thought about how the phase plane ($x$-$y$ plane) representation of an equilibrium solution function is a dot. This was the first time that anyone in small or large group expressed that an equilibrium solution appears as a dot on what will eventually be called the phase plane.

In the final episode of the day, the students began work on a task where the students developed a conceptualization of the phase plane. The phase plane is a vector field where each point has a vector attached whose horizontal and vertical components have length $dR/dt$ and $dF/dt$ respectively. Adam struggled with the concept of phase plane. Early in the discussion, when one of his group members suggested plugging in a value for $x$ and $y$ into $dy/dt$, he noted that the vertical component of the vector would be the same in many places, but verbalized that he does not understand this.

(Class Observation, November 11, 2002)

John: If you plug in the $x$ and $y$ value, you will get a $dy/dt$ value at these point.
Adam: no, actually you will get here, here, here, here, and sketches vectors all the way across the line. So what the heck is that all about, you’ll get a different one each time, right, like this one is going to be in this direction.

... Adam: So you combine the two-the vector together would be (inaudible) So you just add the two together, so it would be $dy/dt$? Or $d$ something, its not $dr$ or $df$, We’ll call it $A$,
Jerry: $dt$?
Adam: I don’t know, $dt$?
John: cause if one is negative, you are going to subtract them
Adam: if you add them here, then the numbers before [inaudible] zero
write up the formula $dT/dt = dR/dt + dF/dt$

By the end of this short discussion, Adam reasoned about a vector in the phase plane, but his reasoning was incorrect. Reasoning with understanding of vectors was not a particularly successful form of reasoning for the straight line solution conceptualizations that will be developed next. Adam’s visualization of the three dimensional curve had become less influential in his reasoning, and what he knew about vectors from his work in linear algebra became the focus of his reasoning. The fact that he tries to add together the two values of the vectors indicates that he was confusing the vectors and the rates of change, which were actually only the magnitudes of the vertical and horizontal components that make up a vector at a point in the phase plane. Students continued to work on the phase plane on Day 2 and 3. Adam was absent on Day 3.

**Day 4.** The episodes on this day involved students reasoning about a new context, a spring mass system whose behavior can be modeled by a second order differential equation, which can be converted into a system of autonomous linear differential equations. Students were asked first to create the model given the conditions for the situation (See Figure 5.4), and then they created the system and reasoned about the phase plane as the coefficient of friction changed in the system.

![Figure 5.4 Sketch of spring mass phenomenon](image-url)
The first activity required students to create a graph that represents velocity and position as the spring moves. Students did not use their understanding of derivative but creating this graph revealed Adam’s way of integrating time into a graph that does not originally include the independent variable time. The group’s discussion follows:

(Class Observation, November 18, 2002)

Adam: Alright, position is right here, when it is at rest, it is at \( x=0 \) [Looking at the sketch of the spring as shown above.] Then when it gets farther away from \( x \), it gets… [Interruption] it’s at rest when it at this one [indicates the middle mark at \( x=0 \)]. Why don’t we just say it is?

Joe: Its maximum position is going to have its smallest velocity. Right, cause it’s going to start going back the other way.

Adam: Or if you push it all the way in…

Joe: But it is still going to have the farthest distance from its rest smallest velocity.

Jerry: Maximum velocity at 0 position.

Adam: So it is at rest at 0, so you pull it to say…[marks point on graph] Velocity is going to go up, then it is going to go down, then it is going to go up and then it is going to go down. [Adam draws a half circle and then redraws the other direction] What the heck!

Jerry: isn’t it going to go negative though?

Adam: Yeah, you are right? So then it goes around like this

Jerry: It would have the same shape.

Jerry: Would it be a circle? It’s just like a nice looking circle, nothing goofy [Joe starts drawing a circle on his paper]

Adam: wait, no that’s going faster, buddy, velocity is going up. Velocity should be highest when you [inaudible] so it just goes around and around and around

Jerry: Frictionless plane

Adam: No air resistance.

This transcript excerpt shows how the three students constituted the circle as a representation of the spring motion assuming no friction of any kind. The students did not actually discuss time as a variable in the problem, but they integrated it into the task as
they drew. As Adam drew the half circle, he animated the graph as he said “it is going to go up and then it is going to up, then it is going to go down.” The graph in this situation did not represent a function, but showed the relationship between velocity and position as time passes. Their reasoning here was the opposite of parametric reasoning. It was combining two quantities that are based on time into a graph of the relationship with time absent from the graph. The instructional design was helpful in that students working on this question could use this to reason about the reverse direction—given a curve of velocity and position, one can parameterize it as velocity/time and position/time. After the class agreed on the circle or a spiral if friction is accepted to be part of the situation, the students developed the actual system of differential equations with direct instruction from the professor.

Day 5. Adam reasoned about the phase plane and the context of the situation to further understand both. The phase plane for Adam was a map of the situation and this was the first time he reasoned about and with the phase plane tangent vector field.
The students entered the differential equation system and varied $n$ between 0 and 4. Vector fields when $n=0$ and $n=4$ are in Figure 5.5 and 5.6.

As Adam watched the graphs as they changed when $n$ changed values from 0 to 4 he articulated the following idea:

(Class Observation, November 20, 2002)

Adam: when $n$ is zero, it’s a circle going back and forth because… it just goes from $a$ to negative $a$ back and forth because it is not slowing down at all [Adam gestures back and forth] then as $n$ gets larger, it goes to 0 faster, it’s kind of like a spiral

... Adam: the larger $n$ gets, it doesn’t even complete a cycle [See Figure 5.6]

Adam was looking at the coordinate plane as it transformed dynamically into different vector fields that would “guide” the solutions and he was able to imagine and gesture about curves that form as time passes in the plane.
Adam expressed concern as class continued on this day, because his reasoning about the behavior of the spring-mass situation and what he observed from watching the transforming phase plane did not match up. He needed to make sense of the situation, and his desire to do so brought to the class an intellectual curiosity that influenced the next episode.

(Class Observation, November, 20, 2002)

Adam: When $n$ gets to four it seems like a line. If it is in this [quadrant] it’s just going to go straight to 0, if its on this side, its going to go straight to zero no matter what your initial velocity is.

... What do you think that line is, like why [referring to a line of vectors that appear straight, when $n=4$, but did not exist when $n$ was $< 2$]? Definitely weird about that line. Let’s stop this where there’s a line? All right in the second quadrant, the line goes straight down to 0.

and a minute later:

Adam: I don’t know. The vectors over here look like they might miss it and then loop both around, like it just goes past equilibrium and then just go a little bit and come back.

Jerry: That’s when I was seeing the vectors go positive, negative, positive negative, above and it and then Quadrant 3, they are always in same direction

It is important to note that in this short exchange, Adam and Jerry looked at the phase plane in two different ways. Adam was thinking of the vectors as a map of the movement of “it”, in this case, probably the curve that was formed as the spring oscillates. His way of reasoning about a solution from a phase plane was helpful to the class conceptualizations and involved reasoning about a curve that was created over time with the vectors “pointing the way.” He was reasoning about a function that was created by the
vectors by the motion of some object. His reasoning was a result of his time based understanding of the solution curves as they are seen in the phase plane.

In contrast, Jerry articulated his conceptions about rows of vectors. He did not see them as leading or pushing something in a direction; rather they were objects that have different directions with a general pattern. He says “the vectors go positive, negative, positive, negative,” meaning as he scans his eyes across the quadrant he saw that behavior. He was not seeing the vector field as a flow map for the solutions. Jerry had much more difficulty reasoning about solutions to systems of differential equations; this lack of reasoning about the phase plane as a flow map may have been part of his difficulty.

The issue of how to identify, and understand this “straight line” was at the forefront of the discussion of the students during the rest of Day 5 and into Day 6. Adam continued to participate in the discussion, but his group, as well as others, appeared to be focused on the vectors, the vector sums and other forms of vector manipulation. The professor tried several times to focus the students on the idea of slope of vector, but Adam and other classmates did not see this as a viable way to think about the situation. The way of reasoning using vector understanding constrained their thinking about the straight line solutions.

**Day 6.** As discussed earlier, the reinvention of eigenvectors and eigenvalues is the most central learning trajectory in this instructional sequence. Understanding how vectors can “lie on a straight line to the origin” and then finding the lines where this happens is vital to the sequence. In this first episode of Day 6, Adam was struggling with the notion of straight line solutions. The students worked in small groups as they attempted to connect
the procedure for finding the straight line solutions to the conceptualization on which the 
procedure is based. Adam moved toward this understanding, but was still somewhat 
vague in his reasoning. Adam expressed what he was asked to do, and worked on 
understanding the method of finding straight line solutions.

(Class Observation, November 25, 2002)

Adam: Rise over run, we are talking about the phase plane here right is it’s the 
Observer: What is the task again? 
Adam: We are trying to figure out exactly what that boxed equation means and 
how we are applying it because we are obviously applying it and it makes 
sense to know what it means before we apply it 
Observer: Okay, so you have the procedure, now 
Adam: What are we doing? 
John: $\frac{dy}{dt}$ is your vertical component and $\frac{dx}{dt}$ is your horizontal component. 
Adam: And we are trying to find a relation between the two, but why is that 
giving us that line? 

... 
Adam: (answers own question) Anywhere else in the plane, besides on that line, if 
you, like pick another line, the slopes are going to change (gestures to 
other points on the plane). Right. The only place on that plane where we 
are going to have a constant line is where the slopes don’t ever change. 
Does that make sense? Otherwise you’d have a curve that would be 
changing constantly. 

Adam’s reasoning was more sophisticated in this episode. He understood that the 
slopes must stay constant to make a straight line. This understanding can be attributed to 
the way he visualized vectors in the plane and the fact that he connected ideas together 
regularly so he could connect the graphical representation and the algebraic method of 
finding where the slope of the vector and the slope of the line must both equal the same 
value. 

In the next episode, the professor asked the students to reason about the equation 

$$\frac{dy}{dt} = \frac{y}{x} = m$$ 

where $y=mx$ is the equation of the line. This led to a class discussion of
the solution space of straight line solutions, specifically what that space would look like.

Adam and the other students grow to understand the idea that all straight line solutions create a plane perpendicular to the phase plane through the straight line. This conceptualization is significant to the class as they continue to grow towards understanding of solutions to linear systems. Following are selected statements

(11/25 TCA) 14:00

Jay: If you take this, this literally is \( y(t) = -\frac{5}{2} x(t) \) because none of our equations depend on \( t \), you can shift both of these along the \( t \) axis so when you are looking at this in two dimensional space, you are assuming if it is coming out at you, if you don’t have \( t \), you are kind of looking at \( t=0 \), then \( y=0, x=0 \), but that is not necessarily true this could happen at any time.

Adam: No, no, when you are looking at the phase plane, you are looking from infinity down, you are seeing all of \( t \), positive, negative, everything.

Jay: But this (points to phase plane) this is constant for all \( t \)... The differential equations do not depend on \( t \), and we have always determined you can shift something along an axis that it doesn’t depend on...

Jay: … So we literally have the plane that’s coming like this (makes a plane with his hands) where this (points to \( y(t) = \frac{5}{2} x(t) \)) literally defines the relationship of \( x \) and \( y \) values.

During this episode, Adam and Jay worked together to reason about what all the solutions to the differential equations that lie on the straight line would be like. Adam’s statement about looking from infinity was particularly insightful and after he made the statement, Jay was able to express the idea that there was a plane and all the points on the plane were represented by the relationship of \( x \) and \( y \) values such that \( y(t) = \frac{5}{2} x(t) \). said that as he “looked” from very far away, he can see the entire \( t \) axis. Although Adam did not say it, he supported it, and this characterization of all the shifts of a straight line solution making a plane was important in his reasoning and he will build on this reasoning later. Also Jay offered to the class that these variables as seen in the differential
equations are actually functions of time, implying that $x$ and $y$ are both variable and function.

In the next short exchange, the class developed the $x(t)$ and $y(t)$ solutions to this particular system based on their algebraic reasoning, their knowledge from first order differential equations, and their visualization skills. When the professor asked about the solution to one of the differential equations in the system, $\frac{dx}{dt} = 3x$, Adam stated that he knows the solution is $e^{3t}$ and when asked, he said he knew it from solving other differential equations and it is “the $e$ thing.” The students appeared to understand and believe this strategy; during the rest of the class sessions they used “the $e$ thing” when solving differential equations of the form $\frac{dy}{dt} = ky$. I interpret this to not be reasoning, but just using a piece of knowledge in a similar situation.

This next exchange occurred in the instructional sequence when the students knew that a solution lies on a straight line

(Class Observation, November 25, 2002)

Jay: Where did that come from?
Adam: What I did was, when you have a system of equations, we haven’t figured out how to get a solution curve yet, but when we have this [points to $dx/dt = 3x$] we know how to get a solution out of that, and that’s what I did over there.
Professor Did you use separation of variables or what?
Adam: I don’t know, it is that $e$ thing
Jerry: Yes, it’s the separation of variables.
…
Adam: Wow, I just used the $e$ thing. I knew we had developed that when you have a differential equation like that.

Here, Adam explained his knowledge from first order differential equations about solutions to $\frac{dy}{dt} = ky$ as having exponential solution functions.
Soon after, Adam explained his understanding of what the solutions look like:

(Class Observation, November 25, 2002)

Adam: But if your $t$ is right here, it is coming out. If we flip it out like this, you are going to have these things right [draws two horizontal shifts of exponential curves see Figure 5.7]

...what he is saying is they are all going to lie on this plane [gestures to plane out of board] coming out here. It is like when this blackboard and rolling it on its side and they are going to stay flat. They can go up or down in either direction.”

Figure 5.7. Adam’s Sketch of Two Exponential Functions

Adam was expressing his understanding of what a set of all solutions that are represented by a straight line in the phase plane would look like—a set of exponential functions, all of which are horizontal shifts of each other and forming a plane coming out of the board in this case. He reasoned about this plane by merging his understanding of solutions as exponential functions, with his understanding of vertical shifts of solutions in three dimensions as he had conceptualized them from his work with the Java applet that created three dimensional representations of solutions.
During the next few minutes, when Joe, one of his group members, algebraically changed the second differential equation in the system into $e^{-2t}$, Adam was quick to notice and explain why both $x(t)$ and $y(t)$ must be the same exponential function. He was using his ideas about what it means to be a straight line solution developed in the class during the previous two sessions and his understanding of how rate of change would affect the curve.

(Class Observation, November 25, 2002)

[Joe plugs $y=-5/2x$ into the $dy/dt$ equation and then gets $y(t) = e^{-2t}$]

Adam: You are right dude!! Because that way it would go along the same line [gestures] wouldn’t it? ...because they have to have this same speed of change for it to be... the reason why it is the line there is that you are on the same path, they are going at the same rate.
[responding to professor’s question]
Then I jumped in and said, seeing this and the $y$. Where’s your $y$? Okay. seeing this and this are related, They are the same thing actually. That’s why it forms that line there because they are changing at the same, like one couldn’t because one would be going this way [gestures] you’d move and you wouldn’t have a line.

Adam made statements that are critical contributions to the classes’ understanding of what straight line solutions $x(t)$ and $y(t)$ would be in three dimensions. He was thinking about the way that exponential curve would have to move in three-dimensional space and visualizing the projections onto the $x$-$t$ and $y$-$t$ planes. In the first interview, Adam verbalized this during his responses to questions about the skateboarder, and now he used that reasoning to understand a solution to a system of differential equations. His way of thinking about an exponential curves “rate of speed” was used here to think about the way that both would have to have the same “rate of speed” to keep the phase plane representation on the line [see interview 1 analysis]. Algebraically, he used that idea and
connected it to the idea of thinking about the plane as being the relationship between $x$ and $y$, in this case $y=-\frac{5}{2}x$ and reasoned about the ratio of the two solutions needing to be always constant to stay on that plane. The professor then picked up on his algebraic justification of the situation and emphasized it to the class.

**Day 7.** The students began conceptualizing solutions to differential equations that are not represented as straight lines in the phase plane during Day 7. The students had looked at computer generated phase planes and observed what happened. The goal of this next episode was to find a way to use the solutions that the students had already found (straight line solutions) to find a solution that does not lie on the straight line. Students did not have ways to reason about this for many minutes, so finally, the professor offers this hypothesis.

(Class Observation, November 27, 2002)

Professor: Okay, a student in another semester noticed that $-4=-2+(-2)$, $6$ is $2+4$, so this $(-4, 6)$ is the sum of these two. She suggested that the solution starting at that point $[(-4, 6)]$ is the sum of these solutions starting at these points $[(-2, 2)$ and $(-2, 4)]$. Do you think that is reasonably possible? Do you have a way to justify or refute it?”

He was providing a way for the students to think about finding solutions that begin at all initial conditions on the plane (not just those that line on the straight lines in the phase plane). This would lead [according to the instructional sequence] to reasoning about two straight line solutions providing a basis for all solutions of the system of differential equations just as the two vectors that begin at the origin and end at two initial conditions on the straight line solutions are a basis for the vectors in the plane. Adam did not reason in those terms. At this point, his (and the whole class) visualization of the vectors on the phase plane and their understanding of the solutions that are recovered from the
differential equations with given initial conditions were not related. Also, Adam was thinking about the phase plane representation of the solution, but did not connect that to the algebraic idea of being a solution. He understood this has value, but the instructional sequence did not provide impetus to “jump the gap” from two vectors forming a basis for the plane to two solutions forming a basis for all solutions.

Toward the end of this episode, Adam realized the value of adding straight line solutions to get other solutions, even though he did not reason why it is true:

(11/27 TCA) 26:00

John: I’m just making sure that is what you wanted, to algebraically justify it [the sum of two straight line solutions with certain initial conditions is the solution for the initial condition that is the sum of the initial conditions]

Instructor: Algebraically or graphically

Adam: If we could do that, then we can get any point on the graph, I mean, on the phase plane.

Instructor: How?

Adam: Just different initial conditions, yeah, adding different points together, like if you want…

John: Just taking a point, well, no, it’s not even the sum of two points

Adam: This plus this, you are adding the $x$-values, well you could get any other point by picking a point down here and a point over here, you add the $x$-values and $y$-values and get a point somewhere. Else. So if this works, we’ve got solutions for the whole thing…. I’m not sure how we go about seeing if it works.

After Adam and his group understood the power of doing this, they were unable to think of a way to do this algebraically. They plotted the sum of the solutions graphically and noted it “appears” to work and the whole class worked on the idea of checking if the slope is correct for the initial conditions. In the next part of the class, one of the other students, Jay, finally suggested an analytically way to check this.

(11/27/02 TCB) 2:00
Jay: If we have the two functions and their derivatives, we ought to be able to plug them simultaneously into the differential equations and get two valid equations at the same time.

Adam: Good job Jay.

Professor: Steve, what did Jay say?

Steve: The left hand side equals the right hand side.

Professor: Of what? Of this equation. So we have two left hand sides. The left hand sides we see if they are the same.

Jay referred back to a mathematics procedure that had been constituted during the instructional sequences for single differential equations. It was called “if it fits, it fits”, a shorthand that Adam initially verbalized as a way to decide if a function is a solution to a differential equation. During first order differential equations, this idea was negotiated for both graphical “fits” and analytical “fits.” Once reminded, the class continued on and algebraically justified that the sum of the two solutions (each of which is a pair of functions $x(t)$ and $y(t)$) was a solution to the system of differential equations.

**Day 8.** Although not described in this case study, at the end of Day 7, the professor and students worked out a way to find the linear combination of two vectors laying on two different straight lines that results in all vectors on the plane. Adam was attempting to make sense of the idea of a linear combination as he verbalized his ideas about finding the solution to the system if the initial condition is (-1,1) The system he was working with had straight line solutions lying on the lines $y=-x$ and $y=-2x$ and the students knew two exact solutions, one that was recovered using an initial condition (-1,1) lying on the line $y=-x$ and one recovered using an initial condition (-1,2) lying on the line $y=-2x$.

(Class Observation, December 2, 2002)
Adam: Our goal in words, Okay, (-1, 1) is a vector pointing in this direction [gestures]. (-1,2) is a vector pointing in this direction right here [gestures] and we are using different combinations of those to get somewhere else and we just tack on this e thing and that will give us the solution at that point.

Adam stated the procedure here, but did not justify it. His ways of reasoning here were insufficient. Graphical reasoning skills did not support his conceptualization and his vector understanding was helpful but not enough to really conceptualize this concept called superposition.

The last episode I discuss in Adam’s class participation concerns algebraic reasoning in this differential equations class. After the class worked together on the justification and agreed that this particular solution is true, the professor suggested that they consider the questions, “Is it true that the sum of any two solutions is also a solution to a system of differential equations?” and “Under what conditions is it true?” Student discussion solely concerned algebraic manipulation. Students were thinking about the method of “if it fits, it fits” to algebraically justify (or prove wrong) the instructor’s conjecture. This is the first time since Day 3 that students thought about what particular forms systems of differential equations can take. Little notice of the importance of the systems being of the form $ax+by$ and $cx+dy$ had been discussed until this episode. Adam had to struggle with the symbolic manipulation and the use of functions/variables in the manipulation. For instance he states:

(Class Observation, December 2, 2002)

Adam: So we are going to add these together, and we want to get $x_3(t)$, $y_3(t)$ and if all of these, now, this is ridiculous, somebody’s going to have to baby me…. This is $x$ and this is $y$, for this one, its going to be $dx(t)/dt$, you plug in this one….
He was not thinking conceptually about the solutions at this point, just the symbols and the system they belong in. Seconds later he affirmed, “I don’t understand, it looks like it is obvious, we are just writing the same thing over and over.” To help him and others with this struggle, and to provide a way to think about the linearity and homogeneity of the system as important, the instructor asked the class how the problem would change if one of the differential equations had a “+x^2” added. No one answered and they continued to manipulate the variables; eventually Adam began to participate. When they were close to the conclusion of the episode, he articulated “we are saying that it is because when you put that in, the two sides are equal and x_1+x_2=x_3, then when you plug it in, you end up with...” and Jay interrupted “d/dt(x_3) = ax_3+by_3 which is exactly what we are looking for,” and Adam [simultaneously] “which is what we have been saying the whole time.”

The instructor then repeated his questions about changing the system by suggesting that “+1” be added to the dx/dt equation and Adam participated in the discussion in a way that showed he has conceptualized the ideas of linearity.

(Class Observation, December 2, 2002)

Adam:    [Goes to Board] It wouldn’t fit the differential equations because you would get a 2 out front once you added. This is the system of differential equations (points to the linear homogeneous version) and this is for that you’d have a 1 here and a 1 here, which is the same form.
Jay:     You’d get the 2 from the quantity you get 1 from 1 side and 1 from the other, there, and that’s why it doesn’t fit.
Professor: Is that clear?
Adam:    What if we change x to x squared?... x is the function so if you square the functions in both x_1 and x_2?...Well, if that is x_1 ands x_2, and that’s equal to x_3, which you square both sides of the equation...[others participate in algebraic manipulation] which is not = x_2 + x_2^2!
Professor: Not equal, sounds good.
In the above exchange, Adam reasoned about the issue of the linearity of the system. He listened to the conversation about manipulating the functions and drew the conclusion from the class talk. Adam was using his algebraic reasoning skills to manipulate the functions in this situation, and he was able to reason to a correct solution.

5.2. 1.3 Adam’s second interview.

To conclude the case study of Adam, I describe three episodes to illustrate Adam’s growth in understanding of solutions to systems of differential equations. In the first two tasks, Adam discusses his understanding of solutions and their relationship to planes and space, as well as his knowledge of “how to” find equilibrium solutions. But during the mathematical activity surrounding the third task, Adam’s response was evidence that the idea of a three dimensional solution in the phase plane is a conceptualization that is well established for him. The task is as follows:

\[
\begin{align*}
  x(t) &= 3e^{-2t} \\
  y(t) &= 5e^{-3t}
\end{align*}
\]

Suppose that \( x(t) = 3e^{-2t} \) and \( y(t) = 5e^{-3t} \) are solutions to a system of two differential equations of the form

\[
\begin{align*}
  \frac{dx}{dt} &= ax + by \\
  \frac{dy}{dt} &= cx + dy
\end{align*}
\]

Tell me everything you can about these solutions.

……

Now suppose also that the initial condition for this same system of DE’s is \( x(0)=3, \) \( y(0)=1 \). Tell me everything you can about this solution.

Figure 5.8. Interview Task about Straight Line Solutions
Karen: Explain everything you can about the equations.
Adam: They’d be these lines and those solutions would not vary from the planes. I’m looking at this as the phase plane and by that I mean the biggest plane, $t$ is coming out at you. So these solutions would look like negative, well something like this. Well it’s something like this [Draws Figure 5.9] on his paper.

![Figure 5.9. Adam’s Exponential Three Dimensional Graph](image)

Karen: What were you drawing?
Adam: What was I drawing? The solution of the curve.
Karen: What kind, I know what you were talking about, but I want you to explain it.
Adam: The solution of $5t$.
Karen: Why did you have it going towards you though?
Adam: Well really it shouldn’t, because as the farther you get away from $t$, the more negative $x$ is going to get, the more positive $y$ is going to get and so it should go out like this (pulls finger away from the plane of the paper). From this point it should go out like this (demonstrates again). No, hold on. The larger $t$ gets the larger $y$ should get and the more negative $x$ should get. So it will go away from $y=0$ and $x=0$ and then its reverse should go the other way, away from the $x$-axis. And then because this one [points to other pair of solutions] is negative it should be going into the $x$-axis.

Adam’s language was not precise, but he was accurately talking about the three dimensional solutions. He looked at the solutions and used the eigenvalues [coefficients
of $t$ in the exponent of $e$] and was able to sketch on paper and in the air what he visualized as the solutions. He used his understanding of exponential functions and the conceptualization of “toward” and “away from” the axis to express this. I interpret his language and gestures here to indicate that this is one of his strongest conceptualizations from the instructional sequence, one that was built from his reasoning before instruction began [See interview 1 analysis]. Continuing to consider this same task, I asked him about the concavity, one more issue of three dimensions solutions. After some discussion, he states the following.

Adam: Well I know that $t$ is time and it’s going forward so I know it’s going up and so if it was negative time then it would be going somewhere else. The reason why I’m thinking of them going up is because time is going up and it would go this way (point coming out of paper) and the other way it would go the other way.

Karen: So I’m going to call those both concave up and so how do you decide that they are both concave up versus say concave down, but still going up?

Adam: Could you guys expand on that question? (All laugh) Okay, now you said concave down, but still going up? What do you mean by that?

Karen: It’s concave down, but still going up. Cause time is still increasing.

Adam: Oh I see what you’re saying now. Well $e^{5t}$ is concave up and $e^{-2t}$ is concave down, so that’s why it’s like it is. And as $t$ increases, both $x$ and $y$ are going to increase at an increasing rate [indicates first e function]. As $t$ increases, both $x$ and $y$ are going to decrease at a decreasing rate [indicates second e function].

Karen: And that’s related to concavity how?

Adam: Well increasing at an increasing rate, well this one [points to function] is increasing at an increasing rate, so it’s going to be increasing and keep on going up. And this one [points to other function] is decreasing at a decreasing rate, so it’s going the other way. Yeah, I think that.

As is seen in this excerpt, Adam continued to express understanding of the three dimensional solution in the previous discussion. This is more evidence that he understood rate of change and exponential functions in space. To end this interview task, I next asked him to discuss the solution to the differential equation when the initial condition is on one
of the two straight lines, which can be identified by examining the solutions that were
given. He first elaborated his understanding of the solution as it is represented in the
phase plane, and gestured a curve. Then he is encouraged to discuss the process of
finding the solution algebraically:

Karen: Okay. suppose with the initial conditions, under new initial conditions
\(x(0) = 3\) and \(y(0) = 1\). Tell me what you can about the solution with that
initial condition.

Adam: \(x(0) = 3\) and starting from here [gestures on the graph a curving function
moving toward the origin] should have that kind of shape [Repeats
gesture]…Start off going towards \((0,0)\) and then it would turn and go out
towards this point and it would come towards this line.

... Adam: This was the solution for starting at point \((3,5)\) so it goes something like
right here [ marks on the paper] and this was the solution for starting at \((-2,5)\) [marks on the paper] uh and to get the solution for this [points to the
third point and draws a little line] and there’s a way to multiply one or
both of these numbers by a number and add them together to get this
vector and what you have to do , you have to multiply and add these
functions using the same method that you did to get these vectors to the
solutions and you’ll get the solution from this point.

Karen: And that would tell you the solution curve that you physically showed
before.

Adam: [nods yes]

In the above discussion, Adam was able to verbalize his understanding of one solution to
systems of differential equations that is not on the phase plane with a graphical
perspective and then explain reluctantly how it can be found algebraically.

Adam’s responses to the last two interview tasks provide evidence that he has
developed conceptualizations during instruction on solutions to systems of differential
equations. The Starburst Task (see Figure 5.10) is one that was unlike any that Adam has
seen before and he was able to find a reasonable solution using his understanding of
straight line solutions and superposition. Portions of the interview are below.
Without technology, sketch several different graphs of solutions in the \(x\)-\(y\) plane and support your claims and conclusions for why the graphs look the way they do.

\[
\frac{dx}{dt} = 2x \\
\frac{dy}{dt} = 2y
\]

Figure 5.10. Starburst Task

Adam: Okay. [He sketches the \(x\) axis and \(y\)-axis] There, several. Actually because its 2 [points to 2\(y\)] it should be [sketches a diagonal line through the origin with slope 1 and then with slope -1] when \(y\) is 0, I think, because if \(y\) is 0 then \(x(t) = e^{2t}\) [writes on the paper] [writes \(y(t) = e^{2t}\) on the paper] So along these lines [redraws all four of the lines] yeah what I said before, that will work.

In the above statement, Adam’s first comment was evidence of the way he thinks about solutions. He drew only two lines, but says “There, several.” This is indicative of the fact that he understood there are an infinite number of solutions represented by one line on the phase plane. The second part of his statement indicates that he knew that solutions to straight line solutions are exponential and he later goes on to discuss the fact that he is using the fact that the solution to \(dx/dt = kx\) is \(x(t) = e^{kt}\). I leave out much of the dialogue between Karen and Adam that follows as Adam elaborated on this thinking. Finally, after Adam described his thinking on why \(y=x\) and \(y=-x\) are representing straight line solutions, he is encouraged to discuss his reasoning. As he verbalized his reasoning, he drew the correct conclusion that every line is represented by a straight line solution. The following are selected comments by Adam that illustrate his reasoning and conclusion.

Adam: When \(x\) is 0, then \(y\) is just .. our system of differential equations just falls back on to just a differential equation [he means that \(dy/dt = 2y\) is left with \(dx/dt = 0\) from other differential equation] and you do the whole e thing and … Its just going to be this [shows with his hand a perpendicular plane}
to the $y$-axis] and if $y$ is 0, then you just do it again and $x$ is.. but what I was thinking, the reason I drew these lines [points to diagonal] is because… [writes $y=x$ and $y=-x$] and the reason I wrote that was, uh, hold up a second, I’ll get to it. [here he is reasoning to come to the desired conclusion] See I’ve never seen this many lines before, it doesn’t make sense, what I’m thinking is, the system of solutions from [writes $x(t) = e^{2t}$ and $y(t) = 0$, then $x(t) = 0$ and $y(t) = e^{-2t}$]. It’s like I’m putting those together. This one is on this line this one is on this line, and I’m adding them together to get this line and I’m adding the negative of one to get this line. [points to $y=x$ and $y=-x$]

Karen: So you are using what you know about straight line solutions and spanning ideas

Adam: Yeah…

Karen: So there are more than just those?

Adam: I think the whole thing is a line. I think the whole plane is just a big line : …Maybe a blur of lines I’m thinking because you just have to multiply one by a number to get the others. There’s like lines on the whole thing.[Adam sketches a few more line]…What I did to get the lines of $y=x$ and $y=-x$ was I took this and this and added together to get this then I took this and this added together to get this. So any other line, say $y=3x$ [writes it] I’m going to multiply this times three, and just add this to this again..[points to the $y(t)$ and $x(t)$ equations]

This last quote from Adam is his explanation for why the “whole plane is just a big line.”

He uses the idea of linear combinations of solutions, in that he reasons that he can multiply and add the solutions that are on the $x$ and $y$ axis together and get to any line on the plane, and those solutions are always exponential, so they are all straight line solutions This mathematical activity was based on Adam’s understanding of the principle of superposition. This is a relatively unusual way to reason about the task, but it proves effective here and is evidence of Adam’s comfort with the idea of sums and multiples of solutions creating new solutions.

To conclude this section, the other researcher asked him about the three dimensional solutions represented by the line $y = \frac{1}{2} x$ on the phase plane in the task. He told Adam that the curve would be concave down, not concave up as Adam had
Adam then justified this with the following statement, which was used in Chapter 4 as a good example of the use of fictive motion metaphor.

Adam: Oh, it is because, no, look, look, when time increases at the same amount of time [Karen, time constantly increases] yeah, but this is going to get bigger and bigger [sketches vectors on the straight line solution] so it is going to start going bigger in this direction [points to x direction] than it is in this direction [t direction] so that is why it is concave down. Otherwise it wouldn’t be a straight-line solution because if it slowed down, it would have to slow down to a point and then reverse the other direction so it would run into a different equilibrium point! … Otherwise it wouldn’t be a straight line solution…

This was the last statement in the interview, and Adam’s reasoning about a solution moving, and his understanding of the shape because of the way the two differential equations direct the solution are prominent. As I stated earlier, his strengths in differential equations are his visualization and graphical reasoning skills. I also note that his understanding of three dimensional straight line solutions as exponential, creating a plane in space, and having to be the same when projected on the x-t or y-t graphs is his strongest conceptualization.

5.2.2 Brandon

I next present a case study of Brandon’s ways of reasoning about solutions to systems of differential equations. Brandon was a master’s student in engineering who had returned to school after working in business for several years to work on a master’s degree while he continued to work a full time engineering job. He tended to be skeptical of the class at the beginning, and even at the time of the first interview before the
instruction on systems, he was not convinced that he was learning how to solve
differential equations. Brandon’s strength lay in his propensity to think analytically. He
was careful to think about new ideas from a viewpoint that required justification and did
not accept ideas “just because it looks that way.” There were times where his
conceptualizations emerged from his thinking of the symbolization of the ideas, but this
was less frequent. In this case study, I follow the same format as Adam and discuss
Brandon’s mathematical activity in the first interview, during his participation in whole
class and small group discussion during the instruction, and his mathematical activity in
the second interview.

5.2.2.1. Brandon’s First Interview

In the interview before systems, Brandon’s conceptualizations of solutions to
systems of differential equations were not well developed; primarily he used ideas from
calculus and first order differential equations. For example, in the Bees and Flowers task
(see Figure 5.11), he reasoned about which system was cooperative and which was
competitive by discussing the terms of the differential equation.

Brandon: I, I think that this here, with the, uh, with the plus sign would be the
cooperation model. Yeah. Without solving it for the whole situation.
Karen It's A. And, so. Do you think the other one would be the competing
model?
Brandon Competition. Because as [Refers to top equation, B.] here. As x got
bigger, the population would start, would start going down. Because
you're, you're multiplying x by y.
Bees and Flowers

Which system of rate of change equations below describes a situation where the two species compete and which system describes cooperative species? Explain your reasoning.

(A) \[
\frac{dx}{dt} = -5x + 2xy \\
\frac{dy}{dt} = -4y + 3xy
\]

(B) \[
\frac{dx}{dt} = 3x(1 - \frac{x}{3}) - \frac{1}{10}xy \\
\frac{dy}{dt} = 2y(1 - \frac{y}{10}) - \frac{1}{5}xy
\]

Figure 5.11. Bees and Flowers

Brandon correctly identified which system was for cooperating species and which was for competing species. He used the idea from calculus that if the derivative is positive then the “slope” is going up, implying that the function is increasing and if the derivative is negative the slope is going down and the function is decreasing (Artigue, 1994); this is reasoning with derivative as defined in Adam’s case study. His use of the function as having motion was clear; he was using the fictive motion metaphor to discuss the behavior of the population as if it were something that is moving along. However, in this interview, Brandon’s discourse indicated that he did not have a sense of the solution to differential equation. When Karen asks him what the solution to the cooperating system might look like, he says:

(Brandon, Interview 1, November, 2002)

Brandon: Um. The solution to a single differential equation is a curve. Or, could be.
Karen: Or?
Brandon: But, let's say there's one curve for it. Okay?
Karen: Okay.
Brandon: And, the solution to the other single differential equation is another curve. And, then, where they cross would be a specific solution for the two.
Karen: Yeah. Could you elaborate at all in that?
Brandon [Refers to activity sheet.] I'm thinking. Well. [Draws graph with upward curving slope line to right of A equations.] Let's say, uh. Okay. This, this is $t$. [Horizontal axis.] And, [After pause.] I guess you'd have to have $x$ or $y$ along here. [Vertical axis.] So. Then, you could, you could. [Draws over slope line.] You could. Let's say that this, this goes up like this. [Draws over slope line again.] And, then, this one goes up like this. [Draws second slope line, more horizontal.] So. For a specific value, [Refers to bottom equation, A.] you could. You know, if you had a specific value that you could plug in for. After you solve this equation. You know, you could find [Refers now to bottom equation.] a certain point on, on the two curves where they'd cross.

He hypothesized that the solution to “each of the differential equations” are a set of curves in two dimensions and that, given an initial condition, there is one curve for each differential equation and the solution is the point of intersection, an incorrect idea.

In contrast to that, Brandon did use parametric reasoning when discussing the skateboard problem [See Figure 5.2 from earlier] After Brandon looked at the problem, he stated the following:

(Brandon, Interview 1, November, 2002)

Brandon: Okay. Oh, Okay. I, I see what you got here. [Pause]
Brandon: Let's call this two [marks off a location on the $x$-axis]
Karen: Okay
Brandon: And, that's four. Let's call that six. Just, you know, what the heck (continues marking off value on the $x$-axis
Karen: [Laughs.]
Brandon: Um, the time period that it took to go that distance was $t$ equals, uh, three seconds. That, the length of that, is the time period.

This is the first time that Brandon introduced the idea of time and he was able to articulate uniformity of time as he discussed this situation. He continued on:

(Brandon, Interview 1, November, 2002)

Brandon: On this graph. The length. The length of it would equal. I mean, he'd be moving a certain speed.
Karen: Uh, huh.
Brandon: So. I mean, how else could I figure the time?  
Karen: Uh, huh.  
Brandon: Unless I held the stopwatch and watched him.

Here Brandon used the idea of time as equal lengths. This is not the same as Thompson and Thompson’s concept of speed length [See Chapter 2 for explanation], but it has some similarities to it. He reasoned about time moving as the graph of the trace occurs and assigned this arbitrary length to match a time value.

(Brandon, Interview 1, November, 2002)

Brandon: So. The length there is three. Which is to there. And, so. It went from one to two. So. From time zero to time three. So. At time zero, I went from one. And, here’s time three over here. One to two. I think I got that pretty close.  
Karen: Sure.  
Brandon: Now, then. Here, at times zero, I'm at four. So. One, two, three, four, five, six. And, one, two, three. So. Here. I'm going from zero to time three. Which is out here, I guess. So. That would be that graph from zero to six.  
Karen: Four to six.  
Brandon: Or, four to six. I'm sorry. And, going from time zero to time three.

Brandon constructed these two line graphs to represent $x(t)$ and $y(t)$ as seen Figure 5.11, middle row of graphs. Brandon’s reasoning was explicitly about the parameter time, so much of it has been described in Chapter 4.
These are the only two tasks I discuss in Brandon’s first interview, as the other tasks either provided evidence of parametric reasoning discussed earlier or were very confusing and I drew no conclusions. I note that this interview was before instruction, but several of the other students who were interviewed did articulate ideas that are similar. Brandon’s only responses that would prove useful were about the Bees and Flowers and the Skateboarder, where his language foreshadowed equilibrium solutions and he was able to parameterize the $x$ and $y$ with time.
5.2.2.2 Brandon’s participation in class

Brandon’s first interview provides evidence that he did not have good conceptualizations for solution to systems of differential equations. At this point, Brandon’s first interview showed that he reasoned parametrically and with the concept of derivative. He could visualize in three space at a minimal level. In this section, I describe in detail Brandon’s participation in the mathematical activity of the class. He was an active and contributing member of the class and his ideas added to the class. I focus on episodes which provide clear articulation of Brandon’s ways of reasoning. I present a brief overview of his mathematical activity in Table 5.2 and then begin the description. Those italicized in the participation table are the episodes discussed in this case study.

Table 5.2

*Summary of Brandon’s Mathematical Activity*

<table>
<thead>
<tr>
<th>Day</th>
<th>Brandon’s primary participation in class</th>
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<tbody>
<tr>
<td>Day 1</td>
<td>None</td>
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<tr>
<td>Day 2</td>
<td><em>Used conceptualization of shifting time to think about shifts of solution function in three dimensions</em></td>
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<td><em>Used reasoning about time as a variable—not dynamic</em></td>
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<td><em>Used understanding of equilibrium solution</em></td>
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<td>Day 3</td>
<td>Used reasoning about positive/negative derivative</td>
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<td></td>
<td><em>Struggled with phase plane ideas</em></td>
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<tr>
<td>Day 4</td>
<td><em>Used understanding of physics to discuss position-velocity graphs</em></td>
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<td>Used fusion of spring-mass and phase plane representations of position-velocity</td>
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<td>Day 5</td>
<td>None</td>
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<tr>
<td>Day 6</td>
<td><em>Used parametric reasoning to understand solutions in three dimensions</em></td>
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<td><em>Confused and then reasoned correctly about equilibrium vs straight line solutions</em></td>
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<td>Reasoned about the plane formed by the set of all straight line solutions for a given straight line in the phase plane</td>
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<tr>
<td>Day 7</td>
<td><em>Used algebraic reasoning to find linear combinations of solutions</em></td>
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<tr>
<td>Day 8</td>
<td>Used algebraic reasoning to understand linear homogeneous systems</td>
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<td>Day 9</td>
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Day 1. In the first day of instruction on solutions to systems of differential equations, however, Brandon participated in a minimal manner in the class. The whole class’s mathematical activity provided evidence that they made sense of the airplane task (See section 5.2.2) and he answered questions as asked but did not add unique thoughts to the class.

Day 2. On Day 2, the students worked in the computer lab and used the DE Explorer, a Java Applet. The students created a three dimensional graph that represented the solution to a homogeneous non-linear system of differential equations. The solution appeared as a spiral curve that spirals around an equilibrium line in three dimensions. Figure 5.13 shows the system and one of the curves generated by the DE Explorer. Mathematically, each point on the curve represents a time, Fox value and Rabbit value that would be a possible value lying on the solution. The DEExplorer uses a numerical method to generate the points and then plots them to create the graphical representation of the solution that is analytically represented by two functions of time $R(t)$ and $F(t)$ . In the task that Brandon is discussing, the students were asked to find an initial condition different than the one given that would generate a curve that appears as a “shift over time” compared to the one already generated. Brandon uses his understanding of time and its symbolization on the graph to discuss how they made sense of this “shift.” Another student had presented his idea that to do this, you could choose one of the points that were generated by the Applet (see Figure 5.13) and make that the initial condition. Then the time is now $t = 0$ instead of $t = “another value”$ so the whole graph appears the same with a different beginning value and thus appears “shifted.” Brandon explains his understanding of this below:
Brandon: Okay, when we first started out, we tried just changing the time as the initial conditions changes and that shifted it. That’s essentially what you are doing here by just picking a different $x$ and $y$. You are just shifting the time, only you are saying its not, but in a sense that’s the same thing.

![DEExplorer Java Applet with Two Solutions](image)

I interpret Brandon’s words to mean that he thinks of time as something that is shiftable in the graphical representation. Parametrically, he is not thinking of time as “always going” as he does in his interview, but as simply an additional variable. Once the graph is created (in this case on the screen), he sees the graph as a static object that can be manipulated. His next question lends support to this interpretation of his thinking.

Brandon: Okay, so here’s a question: I’m looking at this so I can see $x$ and $t$. Now, I can see where these two curves cross. Is there a relation between where they cross and different initial conditions? Where they cross, if you are

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**Figure 5.13.** DEExplorer Java Applet with Two Solutions

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looking directly at that, will that move back and forth were I to pick different initial conditions.

Here Brandon is not thinking of time as a dynamic variable, which creates movement, but as these curves as static. He is asking a question about the intersection of two different curves created at different times. He was thinking of the graphs as entities on their own. No one ever responded to this question, so I was unable to learn more at this point about how he coordinated his understanding of these graphs dynamically and statically as well as what the meaning is for him in terms of the solution to the system of the differential equations.

Brandon makes only one more statement on Day 2. When asked about an equilibrium solution to the differential equation and what it might look like in three dimensions, he stated “But zero rate of change is just going to proceed as a straight line.” Here he used his understanding of an equilibrium solution function from first order differential equations; the concept of equilibrium solution is emphasized during the first half of the semester. Again he shifted back to discussing the graph using the metaphor of motion; “proceed” indicates it is moving.

To summarize, during the second day of instruction, I provided evidence that Brandon’s parametric reasoning was useful in his understanding of the new idea of the three dimensional curve as a representation of the solution. He oscillated between thinking of a graph as a static and moving object. He used conceptualizations from the first order differential equation instructional sequence to reason about equilibrium solutions.
**Day 3.** In the second half of the class, the students are asked to study the Bees and Flowers instructional task. The students discussed the task extensively in groups. Brandon spoke twice about this task in the whole class discussion. In both cases, he made statements that indicate he understood the idea of how the two differential equations in a system are in relation to each other. His language was evidence that he was conflating function and rate of change. For example, here is part of a statement he makes in Day 3:

Brandon: That’s why they help each other, cause when one starts going down, the other is able to help pick that one up, so that’s why you can say this is a cooperating relationship, this is the one that helps the other.

His reasoning about the cooperating species system was not well developed. He used number sense and the understanding that negative values of a differential equation make the function decreasing and positive values will make the function be increasing. This reasoning involved using the algebraic representations of the two differential equations. At this point, the reasoning in the class is not any more sophisticated than the reasoning I documented for the rabbits and foxes scenario.

**Day 4.** At this point in the instructional sequence, Brandon’s contribution to whole class discussion had been sporadic did not show new understandings of solutions to systems to differential equations. In Day 4, the class was working on the spring mass situation, and the students began by constructing graphs of velocity and position when a horizontal spring with a mass is activated. In the instructional materials, this is an introduction to the development of a system of differential equations that models the Figure 5.13 Brandon’s work on velocity-distance graphs.
Brandon was an engineer for several years, and he was able to use his understanding of Physics to reason about this situation. When speaking to the class about what the graph would look like, he stated:

(Class Observation, November 18, 2002)

Brandon: I changed my graph just as you were doing that- drawing yours and it looks like that too [the graph is a circle around the origin] I was thinking you would have this one because if the original resting position right there
is the origin, that means that when you pull the string, its going to have to
go back before the origin. If there is no spring to it at the origin, it would
come back there and stop. It has to go beyond the origin to compress to
build up some more energy to bounce back this way. And if you call the
origin its initial position or this initial position [gestures two points]

A few minutes later, the professor asked Brandon to illustrate his ideas on the blackboard.

Brandon: [Draws a sketch on the board of a 3-D representation of the spring mass—
draws a spiral] Okay, this is P, it’s spiraling into position 0 [another
student says should start at origin] [erases first piece and redraws] there,
cause you are starting at velocity zero. [See Figure 5.14 for sketch]

Brandon was reasoning in this situation using his understanding of physics and the
spring-mass situation. He and the rest of the class illustrated how time as a parameter is
implicitly used in the phase plane situations at this point before it is a phase plane. When
Brandon and others drew the curve and discuss their ideas, the curve and the motion of
the spring were spoken about simultaneously. In their language, the motion of the spring-
mass and the curve is fused (Nemirovsky, Tierney, & Wright, 1998) and their discussion
analyzes the behavior of the spring mass and the graph of its velocity and distance
simultaneously. Time was integrated into the spring-mass situation. As the students imagined the motion of the spring, they thought about the curve on the \(x-v\) plane. Brandon’s and others utterances suggested that three conceptualizations are present simultaneously. First, the horizontal movement of the curve as time passed was conceptualized as the curve needed to be created from left to right and back with decreasing frequency. Second, the vertical movement of the curve as time passes was conceptualized as the curve needed to be created from up to down and back with decreasing frequency as time passes. Third, the velocity and position were considered as varying together and how they are related was considered (see Chapter 2 for Carlson’s covariation framework). Brandon and others’ conceptualizations of time played an important role [See Chapter 4].

**Day 6.** Brandon’s way of reasoning about eigensolutions became more sophisticated during the mathematical activity in Day 5 and Day 6. About half way into the sixth day of class, Brandon asked the following question, which showed that even though he had previously explained and demonstrated a straight line solution as he conceptualized it in space, he was still confused about straight line and equilibrium solutions for systems of differential equations.

(Class Observation, November 25, 2002)

Brandon: An equilibrium solution where you have one equation is a straight line where the slope is 0. Now what we are saying here is, an equilibrium solution is a plane. Is that what we are saying for a system? [Other students: No, No, No] The reason I’m asking: so now we saying that we start with an initial condition of \((1,0)\) that would lie in this plane, but if we graph that, its not going to be a straight line, its going to come out. [Adam interrupts: can I go up there for a second?] because the only place I would see a straight line is \((0,0)\)
Brandon’s question showed that he was confusing equilibrium solution and straight line solution for a system of differential equations. An equilibrium solution for both single autonomous differential equations and systems of autonomous differential equations is represented by a straight line in two space or three space and is defined to be a function (or pair of functions) where the differential equation is constantly 0. The plane that Brandon was talking about in this situation is the union of all the solutions to the system of differential equations that appear to lie on a straight line in the $x$-$y$ plane. These two concepts are different, and Adam and others wanted to discuss this. This was one of Brandon’s difficulties during the entire instructional sequence—new terms when they are introduced were not easily accommodated into his thinking. However, after further discussion, in a few more minutes, his statement showed that he understood.

(Class Observation, November 25, 2002)

Brandon: That is not the same as saying it is an equilibrium solution. The only equilibrium solution is (0,0). Okay, fine, I just wanted to get that clear. There is one equilibrium solution and the rest are on these two planes. What do we call this, the $y=5/2 x$ versus $t$ plane? Sounds good to me.

Mathematically, the students in this class had developed the idea of an invariant manifold, which is a plane that divides space and that creates a boundary through which solutions in different parts of space do not cross. In his discourse, Brandon called this plane the $y=5/2 x$ plane, using the line where the plane intersects the phase plane to name the plane. This conceptualization was part of the development of the understanding of straight line solutions and became the primary conceptualization that the class accepted together and used in their work. Brandon further confirmed his own understanding and
Adam’s acceptance of this notion a short time later. He stated (and Adam also contributed);

Brandon: That’s why it stays on that plane. That plane coming out, it belongs to....
Adam: Because this number will always cancel out, that’ why it’s always on this line that plane coming out, it always belongs to... [Brandon gestures a vertical plane with flat hand]

At this point, the class, including Brandon and Adam, used this plane idea as they further developed understanding of eigensolutions.

**Day 7 and 8.** On the last two days that Brandon participated in the whole class discussion, he used reasoning that is related to linear algebra concepts, which he had learned in an earlier class. The class was working on the concept of superposition, which can be stated as “the solutions to systems of differential equations are always the linear combinations of other solutions.” To address this issue, the professor suggested to the class that the solution to the system

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -2x - 3y
\]

with initial condition at (-4,6) can be found by adding the solutions from initial conditions (-2,2) and (-2,4), both of which the class has found in a previous task as they lie on the straight lines, and thus are eigensolutions. Brandon participated at a minimal level, this is evidence that leads me to know that he had some algebraic understanding of the solutions that do not lie on one of the straight lines, but his weakness in reasoning graphically and assimilating new definitions constrained his growth in understanding.

Brandon’s understanding of vector algebra provided him with ways to think about the linear combination that is developed at the very end of the sequence. He uses linear algebra ideas to solve and then presented to the class a method to find the linear
combination of two vectors to form a given third vector. Using algebraic algorithms is one of Brandon’s strong areas, and he was confident as he presented this “method” to the class. His algebraic skills were useful in this situation. His last statement in the whole class part of the class addresses the issue of whether adding a squared term to the system of differential equations would affect the concept of superposition.

Brandon: I think my idea works here, that $x_1^2 = z$. The way I said it was an easy way to see it, if you call $x_1^2 = z$, so if you substitute them call $x_1^2 = z$, then plus $z_2$ gives you $z_3$ on the other side you still have $x_1 + z_3$, so it’s an inequality.

I notice here that he used algebraic thinking to think about the solution set. He was able to reason using the symbolic algebra in a way he was not able to do using graphical techniques at other times.

5.2.2.3 Brandon’s Second Interview

I finally discuss Brandon’s mathematical activity as he responded to the question on the second interview, which took place immediately after the last class. Generally, Brandon’s ways of reasoning not grounded primarily in visualization and graphical reasoning. He relies more on his memory of mathematical activity in which he participated and content that he knows from earlier classes. I will describe the most relevant discourse in this interview, interpret how his ways of reasoning have become more sophisticated, and identify conceptualizations that he built during the nine days of instruction. The tasks in this interview were the same as those that Adam responded to and so I will not repeat the task, but reference the figures provided earlier in this chapter.
The first task asked if a solution to a system of differential equations is a pair of functions, a curve in three space or a curve in two space. Any of those answers is correct and reasonable depending on the situation, and Brandon responded in that way.

(Second Interview, December, 2002)

Brandon: Can I answer yes to all three. [Karen: you can] We have said at the beginning of this course that the solution to a differential equation is a function and a set of differential equations is a system so a set of two functions would be a solution to a system of differential equations. And it is a curve in space and a plane is in space so it can be a curve in a plane as well. What was that term that was used, a manifold? An invariant manifold, that was a curve on a plane, but then the other curves around it that weren’t on that plane went through three dimensions of space and they were all solutions so that is what I think.

Karen: If I hadn’t given those to start with, and I asked you what the solution to a System of differential equations is, which one would you have said or would you have said something different from that?

Brandon: I probably would have just right away thought of the first one, two functions, and then maybe talking, as I talk, I start thinking of answer, then I would have gotten more into the geometry of it.

Earlier I mentioned that Brandon’s algebraic reasoning is stronger than some of his other types of reasoning and the comment above provides evidence of this. He was willing to think of solutions using different representations but his preference was for the analytic solution. He then discussed how one could express a solution as a curve in a plane. The intent of this question was to see if the phase plane as an object to reason about and use to think about solutions had become a viable option for students. Brandon was able to articulate his ideas about this in the following comment.

Brandon: We used the phase plane view… You can have a solution that is a curve in the phase plane, and, like I said, I think that is where all your solutions that I remember were some sort of curve in the phase plane, whether they curve around and eventually go to the origin or some other point, or whether they curve around and go out to infinity, or whether they just curve around in just some sort of weird shape around something. They are curves in the phase plane.
Brandon’s verbalization of his understanding of curves in the phase plane supports the notion that students think of solutions as representations in the phase plane after instruction in systems. I then asked more about the $x(t)$ and $y(t)$ functions and their relationship to the curve in three dimensions but Brandon did not communicate his ideas in a way that indicate he has a conceptualization. His understanding was dependent here upon the mathematical activity he experienced in the classroom, both with the airplane problem and the reasoning of the students in the class on the phase plane solution conception.

I continue on to another task in the interview that shows more evidence of Brandon’s conceptualizations about solutions, specifically equilibrium solutions. I asked him how you can tell if $x=2$ and $y=3$ is an equilibrium solution for a system of differential equations.

(Second Interview, December, 2002)

Brandon:  Ok, yes, I agree, because if I plug those numbers in here I end up with 0 ..
Karen:    Why are you setting them equal to 0?
Brandon:  Because an equilibrium solution, I always think of as 0 so that’s why I set it to 0. Let’s see. An equilibrium solution for a single differential equation, is a curve that’s straight, that is going to be parallel to the $t$ axis, so its going to be running parallel to that, so it has 0 slope
Karen:    What does an equilibrium solution look like in space, if it looks like a line parallel to the $t$ axis for a regular differential equation..
Brandon:  It would be a straight line and it would be parallel to $t$, if you have $x$, $y$ here and you have $t$ out there. And it would be somewhere. And if you looked at the phase plane, it would be a point,[sketches a picture of these two concepts—equilibrium solution in a plane and in space.] 
Karen:    What would be the two functions, because you said a solution is two functions, what are the two functions for this equilibrium solution?
Brandon was unable to answer this question, and this was surprising as he
drew the solution in space correctly. Instead, he attempted to talk about a general solution
to the system of differential equations and became confused. In this situation, I conclude
that he was either answering a question with more sophisticated reasoning because I had
actually given him the functions $x=2$ and $y=3$ and so he thought it should be something
different or more complicated, or he does not think of equilibrium solutions as functions.
However, he was able to reproduce ideas from classroom mathematical activity to
actually sketch equilibrium solutions in both the phase plane and space.

The next task moves from the concept of equilibrium solution to the concept
of straight line solutions and Brandon’s response indicated that he was not able to clearly
articulate his understanding. The problem asked him to discuss the straight line solutions
for a system given these solutions.

(Second Interview, December, 2002)

Karen: What do you know about these solutions here? Do you know anything
about it?

Brandon: Ok, this solution. I think that could be a linear equation here [Karen: they
are both linear] so each one has a straight line solution [Karen: each one,
like the $dx/dt$ and $dy/dt$] yes, each one has a straight line. And I think that
gives me a slope which now we are looking at the phase plane [draws an
axis] and 3 and 5 and we have a straight line solution running through the
origin approximately 3/5. And this is –2 and 5, so this is going to have a
solution something like that. And then the e is telling me that when I add a
solution from here and a solution from here its going to give me a third
solution, and I start adding solutions and that’s where I start getting my
curve.

This statement is somewhat convoluted and Brandon was incorrect in the first thing he
articulated. There is not one straight line solution for each differential equation in a
system. However, he did look at the vectors that can be derived from the solution and
sketch the correct lines on the phase plane. Finally, he talked about e and how he was finding a third solution, which was not connected to the question and irrelevant. In the next piece of this task response, I asked him questions to clarify his first unclear response.

(Second Interview, December, 2002)

Karen: Okay, what does a solution that starts at that point look like in the phase plane and what does it look like in 3-space? And how could you represent it.

Brandon: Okay, well, if we had as the $t$ axis [draws a $t$ axis on his earlier phase plane] and this would stay on this, let’s draw this plane here. [Karen; what plane are you drawing?] I’m drawing a plane with this, this straight line. And its this solution. I don’t know from this if it is uniqueness theorem or not, just from that, it looks like it probably is, let’s say it is, it would come down along that plane, along that invariant manifold, that’s how that solution would look in 3-space. In 2-space it would just look like this line, coming down here.

Here Brandon drew a sketch in three dimensions by drawing a plane perpendicular to the phase plane and through one of the straight lines. He also articulated that the straight line solution in the phase plane is a straight line coming to the origin. The conceptualization of a plane perpendicular to the phase plane through the line where the straight line solutions project was evident here in the same way it was present in Adam’s thinking. However, Brandon talked about the uniqueness theorem and there is no evidence of the value here. He just mentioned it without reason at this point, showing that he used ideas that he has learned without always having a full conceptualization of the idea.

I next asked Brandon to draw a three dimensional picture of a solution and he drew an appropriate graph. When I asked him how he decided if it goes up or down,
meaning does the curve start near the \( t \) axis and move exponentially out or start away and move exponentially in, he appropriately answered with an idea that was developed in class and went on to describe why the three dimensional curves acted as they did.

Brandon: [pause] here because it is a negative exponent, as \( t \) gets larger, the value is going to get smaller and smaller, so it is going to get closer to 0. Here because it is a positive exponent, as \( t \) gets larger, this is going to get larger so this side is going to go away and that side is going to go down.

Karen: As it goes up, how is it going up. Is it going as a straight line? Let’s say, here’s time

Brandon: It’s curving out because of \( e \) and that one is curving in. {Gestures with finger}

Karen: Do curving in again?

Brandon: [Shows the shape with his fingers] It is not a straight line, it is going out like that

Karen: One more question: If I take all of these straight line solutions, how are they related to each other?

Brandon: I don’t have the right term: All the curves are parallel, they are all similar, they all curve with the same, let’s see with the same \( y \) value, they would have the same curve. Cause at different \( t \)’s they would have different \( y \) values, but for the same \( y \) value, they would have equal slope, which is what we were talking about early in the class when we just had one DE and we would have a group of curves but the would all be the same but the would be stretched along the \( y \) axis

Brandon’s discussion and gesture illustration of straight line solutions indicated that he knew that straight line solutions are exponential, one of the important conceptualizations developed in the instructional sequence. He then replied to the question about the relationship between solutions by reasoning about them in terms of shifts of solutions that appear in solution spaces of first order differential equations. His discourse was evidence that he had a conceptualization of the shifting of the solutions that are represented by the same straight line. His conceptualization of straight line solutions was appropriate and helpful in his work in differential equations.
5 3. **Summary and Comparison of Cases**

Adam’s responses to the tasks in the first interview revealed he had ways to reason mathematically before instruction on systems. He made time explicit in his thinking and used it to reason in the skateboard problem where he drew $x(t)$ and $y(t)$ straight line graphs after considering a line drawn on an $x$-$y$ axis. He used discrete and continuous images of time to reason about this task. He did not reason correctly for some of the tasks, particularly those that included the introduction of a third axis into the situation. He also did not have intuition to reason about the system of linear differential equations and the straight line on a phase plane. However, he did provide good explanations grounded in his understandings from calculus and first order differential equations. This can be seen by his discussion of the small predator/large prey and large predator/small prey task.

Adam participated in the classroom and small group discussions in significant ways. His ways of reasoning about the three dimensional representations of solutions to systems of differential equations were especially important to the class as a whole. For example, he contributed reasoning about what all the straight line solutions would be together. He reasoned about the $x(t)$ and $y(t)$ representations of the solution, specifically expressing his idea that the functions $x(t)$ and $y(t)$ would both be exponential functions with the same exponent. Generally, his reasoning about tasks that were directly related to physical situations was not as strong, and other students in the class contributed more in those situations. However, by the end of the sequence, he was able to use the algorithms for solving systems of equations and reason about those solutions with fluency.
In his second interview, Adam showed reasonably good understanding of solutions depicted in three dimensions, although he still could not explain appropriately reasons for two of the problems. His reasoning about the task whose goal was to analyze the differential equations $dx/dt = 2x$ and $dy/dt = 2y$ indicated a good understanding of superposition as a concept.

Brandon also was able to provide good explanations for the tasks in the first interview. He used the calculus conceptions of positive slope meaning increasing and negative slope being decreasing. He also used parametric reasoning in his work on the skateboard problem, using the characterizations that time never stops, making time explicit, and reasoning qualitatively and quantitatively. He foreshadowed some of the ideas he expressed during the instruction on solutions to systems of differential equations by discussing equilibrium solutions, as one of the graphs in the skateboard problem might be interpreted as an equilibrium solution.

Brandon did not participate in the whole class discussions as frequently as Adam. Nevertheless, his mathematical activity contributed to the whole class in a few ways. He often asked questions that provided opportunities for students to formalize their thinking about definitions and meanings. He asked for justifications and provided justification for reasoning occasionally. On Day 6 he went to the board and supported his thinking by using gestures to describe how solutions are represented in three dimensions, using the parametric characterization of imagining the motion. His strengths lie in his algebraic reasoning skills. He used conceptualizations about vectors that he learned in linear algebra to contribute to the development of straight line solutions and superposition.
In the second interview, Brandon’s responses were frequently garbled and unclear, and he confused ideas that he had learned in the instructional sequence. Because the interviews were very conceptual and graphically based, I was not able to further confirm the earlier analysis that Brandon’s strength lay in algebraic reasoning. However, he was able to answer correctly to several of the questions, even when his reasoning was not clear.

In general, Adam and Brandon were both significant participants in this differential equations class. Adam’s discourse in the class was much more extensive, as can be seen by the difference in the lengths of the descriptions for each of the students. He also was a leader in the mathematical activity in the classroom. The students often looked to Adam for ideas and he often added new conceptualizations. The reasoning used by the two men had some differences and similarities. Adam relied more heavily on reasoning that was intuitive and/or graphical in focus. As an example, he used the words “the e thing” when he was talking about solving a differential equation of the form $dx/dt=kx$ to get $e^{kt}$. This was referring to separation of variables, but he had internalized this for his own use and shared it with others in the class. He also articulated ideas about 3-D solutions more often than Brandon and used three dimensional solutions in his reasoning.

In contrast to that, Brandon was much less confident in his reasoning about graphical conceptualizations. He used parametric reasoning on a regular basis but he did not feel comfortable with his conclusions at times. He was more likely to rely on previous algebraic understandings, specifically in the content area vector algebra. He was more
careful in his reasoning than Adam and occasionally provided reasoning that solidified others conceptualizations as well.

In the second interview, his reasoning as described in the analysis supported Adam’s conceptualizations. His reasoning was more visually based than Brandon’s and he is able to reason about the starburst problem when Brandon was not. Generally, the evidence supports that he had a more robust understanding of general solutions and straight line solutions to systems of differential equations.

5.4 Ways of Reasoning in the Case Studies

To conclude the case studies of Adam and Brandon, I focus on the ways of reasoning that emerged as bigger themes in the development of new conceptualizations of systems of differential equations, their representations, and their solutions. These are not the only ways of reasoning, but they were common in the case studies. Also, many of the illustrations can not, in practice, be identified as exactly one way of reasoning and it is somewhat artificial to attempt to place reasoning in categories. However, in order to develop a “terrain” of the different forms of reasoning about systems of differential equations, it is useful to discuss these categories separately. I present four themes for ways of reasoning that Brandon and Adam used: reasoning with prior knowledge, graphical reasoning and dynamic visualization, reasoning with the context, and algebraic reasoning.
5.4.1 Reasoning with Prior Knowledge

As Adam and Brandon participated in the class discussions, there were many instances that demonstrate reasoning about solutions using prior knowledge. The knowledge they used came primarily from three areas of mathematics: first order differential equations, rate of change understanding, and linear (or vector) algebra.

Because of the design of the instructional sequence in this inquiry oriented differential equations class, it is not surprising that the students significantly reason with ideas from the first half of the semester. Specifically, the understanding that solutions to autonomous differential equations are shifts along the \( t \) axis and that there is a structure of the solution space that is used extensively in the development of understanding of solutions to autonomous systems. This notion of shift led to one of the primary conceptions in the class, that of the plane of straight line solutions, where the plane is made up of all the straight line solutions that lie on one of the straight lines in the phase plane. Also, equilibrium solution functions are conceptualized for systems based on equilibrium solution functions for single differential equations. Students grow to understand that the equilibrium solution is represented by a line in space with equations \( x(t)=a \) and \( y(t)=b \), where \( a \) and \( b \) are constants. This understanding expands on first order equilibrium solutions, which are horizontal lines in a plane with equations \( y(t)= \) “constant.” Finally, the knowledge that \( \frac{dx}{dt}=kx \) has a solution of the form \( x(t)=ae^{kt} \) is used to understand straight line solutions of systems in space after algebraically manipulating the system to create differential equations that can be solved by separation of variables.
Students’ understanding of the concept of rate of change is used before, during, and after instruction on systems. Brandon and Adam both used rate of change, both in thinking about structure of solutions, and in reasoning with derivative. Several times in the case studies, Adam talked about concavity of solutions and how solutions behave. He used his understanding of how rate (or speed, as he sometimes stated) is changing to discuss these solutions. Rate of change is one of the primary conceptions that the instruction sequence is built on, and so using the differential equations as an expression of rate is important as well. Also, derivative is instantaneous rate of change and several times the students reasoned qualitatively using derivative. The relationship between an increasing and decreasing function and the sign of the derivative is especially important in their thinking.

Finally both in the development of the phase plane as a reasoning tool, and the development of solutions to linear systems, Brandon and Adam used their understanding from linear (vector) algebra. In the phase plane reinvention, the students’ vector conceptualization both supported and constrained new understandings. Vector addition and the ideas of vector magnitude and angle are not helpful, but components of vectors and their projections onto different planes or lines were important in developing ideas about the phase plane.

5.4.2 Graphical Reasoning and Dynamic Visualization

The second theme of student reasoning is graphical reasoning and dynamic visualization. As was elaborated primarily in Adam’s case study, the conceptualizations of solutions are first based on students’ developing an understanding that a solution to a
system is best considered as a curve in space. The mathematical activity promotes this idea, specifically the airplane task [See Figure 5.2] and the computer explorations. Adam and Brandon both developed the idea that this curve in space can be thought of as the “mother curve,” as in, it is the curve from which the other representations can be generated. The x(t), y(t) and phase plane representations are visualized both by imagining through expansion of the airplane visualization and by investigating these solutions using a computer program.

Adam and Brandon continued to reason graphically as they developed an understanding of the phase plane. Lines and vectors on a plane can be visualized and graphical reasoning provided support for them to understand the phase plane and then representations of solutions on the phase plane. As the nine classes progressed, Adam then expanded this visualization into straight line solutions in space. His visualization of exponential functions in space particularly supported his and the class’s understanding of these solutions and the development of the space of solutions.

5.4.3 Reasoning with the Real World Scenario

Because of the instructional design theory, Realistic Mathematics Education, that was used as inspiration to develop the instructional materials, students used the real world scenarios in the materials to understand and use solutions to systems of differential equations. There were two primary real world scenarios used in the sequence, population growth, and the spring-mass situation. Both of these provided knowledge that Brandon and Adam used to grow in their understanding.
Population contexts are integrated into the entire differential equations class, but in the five week unit on systems, students were able to reason with a cooperative and competitive model for two populations, and the predator-prey model for two populations. Both Adam and Brandon used knowledge of these situations in the first interview, and continued to do so during instruction. The reasoning specifically supports sense making in that when they use prior knowledge about derivative or graphical reasoning, for example, knowledge about the real world scenario is at times used to check and make sure that their results are reasonable. Also, they sometimes began with the real world scenario to conjecture what should happen and then verified or refuted these conjectures with one of the other three forms of reasoning characterized in this section.

The spring-mass situation, which involves a mass on a spring being pulled or pushed along a linear path and, can be modeled with a system of differential equations. Brandon used his engineering experience to reason about the system model and its solutions extensively through his understanding of the physical situation. He discussed the system model for the spring mass in whole class discussion at its introduction into the class activity and refers back to the situation during continued growth. Both students also reasoned with the spring mass scenario when straight line solutions were introduced. The contradiction between what they believed the behavior of the mass to be and the phase plane representation of the behavior (a solution to the system) supported the reconceptualization of the spring mass behavior and then this reconceptualization supports new understanding about solutions to systems of differential equations.
5.4.4 Algebraic Reasoning

The final theme for reasoning in the case studies is algebraic reasoning.

Algebraic reasoning has been elaborated extensively in the mathematics education literature (e.g., Arcavi, 1994; Blanton & Kaput, 2003; Chazan, 2000; Kieren & Sfard, 1999; Smith & Thompson, in press) but there is no one agreed upon formal definition of what constitutes algebraic reasoning. For the purpose of this analysis, I characterize algebraic reasoning as developing, manipulating, understanding, and interpreting symbolic representations. Students use algebraic reasoning throughout the five weeks of instruction, but this reasoning is used more at the end of the sequence.

Brandon and Adam used algebra in the middle of the instruction to find straight line solutions. However, in this case, the algebraic reasoning did not support new understandings and tended to constrain growth in conceptualizations. After the students understood why straight line solutions can be found by setting slopes of lines and vectors equal to each other, then they used algebraic reasoning to find and interpret the solutions and then to find the general solutions using linear combinations. Brandon was especially comfortable with this reasoning and he contributed significant algebraic ideas in the last three days where the focus was on finding the general solutions and understanding the symbolic representations.

Algebraic reasoning also complemented some of the reasoning with earlier knowledge, specifically linear algebra. It is impossible to separate use of prior knowledge about linear algebra and algebraic reasoning, but manipulation of symbols and interpretation of their meaning played a role in vector understanding and use.
The case studies of Adam and Brandon provide a significant characterization of the reasoning used and the participation of these two students in the differential equations class during instruction on systems of differential equations. This chapter offered a detailed description of their mathematical participation, the development of new and expanded conceptualizations, and the reasoning used. These results may be extended to other undergraduate and K-12 classes, as the themes discussed can be investigated at any mathematical level.
CHAPTER 6 CONCLUSION

In the final chapter of this dissertation, I present an analysis of mathematics practices as they emerged in whole class activity. These practices were first introduced in the paper Advancing Mathematical Activity: A Practice-Oriented View of Advanced Mathematical Thinking (Rasmussen, Zandieh, King, & Teppo, 2005) and described in Chapter 1. Then I summarize the characterization of parametric reasoning and the results from the case studies of Adam and Brandon’s mathematical activity as they participated in interviews and the differential equations class. Finally, I consider the limitations of the research, instructional implications of the results and offer possible future directions for research.


In Chapter 1, I delineated four mathematics practices that I would identify and describe as they occurred in the whole class discussions during instruction about solutions to systems of differential equations in this classroom teaching experiment. To review, Rasmussen et al. (2005) offer an alternative to the perspective introduced by Tall of advanced mathematical thinking. They develop the perspective that mathematical learning can be studied in terms of the notion of advancing mathematical activity, where the idea of “advancing” is situated in the social context of learning and can be examined with that lens. There are four kinds of mathematical activity that the authors explicate:
symbolizing, algorithmatizing, defining, and justifying. I propose a fifth type of mathematical activity: experimenting. These five practices can be thought of as practices for members of a broader mathematical community of practice (Wenger, 1998). In this differential equations class, students participate in these practices and I posit that students in this differential equations class become more central members of the mathematical community as they participate in an inquiry oriented classroom mathematics community. In this section, I discuss students’ participation in the mathematical activity of symbolizing, algorithmatizing, justifying, and experimenting.

6.1.1 Symbolizing

The first practice I discuss is symbolizing. Symbolizing may be informal or formal and emerges from the activity of a mathematics student or a mathematician to better record ideas, provide abstraction of ideas, and shortcut notation to make it easier to reason and communicate with others. Rasmussen et al. (2005) describe symbolizing as one type of advancing mathematical activity:

In this approach, “it is while actually engaging in the activity of symbolizing that symbolizations emerge and develop meaning within the social setting of the classroom” (Gravemeijer, et. al., 2000, p. 235-236). From this point of view, the need for notation and symbolism arises in part as a means to record reasoning and serves as an impetus to further students’ mathematical development. In this way, symbolizing is less a process of detachment and more a process of creation and reinvention. Further mathematizing activity and powerful use of conventional
symbols emerge from and are grounded in students’ previous symbolizing activities.

In the differential equations classroom, students both created and reasoned with symbols regularly. The symbolizing about solutions to systems of differential equations was supported by the instructional materials and students used symbols extensively in their mathematical activity. The symbolizing that was primary in the nine days of instruction involved reasoning about the symbols that are created by technology, symbols created by the students, and some standard symbols introduced by the professor. Following is a partial list of the symbolizing activities in which the differential equations students participated. [See Appendix A for instructional materials referenced]

A. Airplane task—students created traces of a three dimensional curve created by a red dot on an airplane propeller that is rotating slowly as the plane moves forward. This involved students imagining the motion and making representations of what the red dots trace would look like from various dimensions. These symbolizations became a base for reasoning that continued in later tasks. Each of the groups created their own symbolizations but they were consistent across the groups.

B. Use of DEExplorer—students investigated representations using a computer java applet to reason about three dimensional solutions to systems of differential equations. They represented equilibrium solutions in three dimensions and reasoned about how non-equilibrium solution graphs can become shifts of one another.
C. Students were shown a phase plane, where the $x$ and $y$ components of vectors on a set of points represent the values of the differential equations. The students reinvented nullclines (lines in the phase plane where the one of the derivatives is constantly 0) and used this to help understand the structure of solutions as they appear in the phase plane, as well as the phase plane and three dimensional concept of equilibrium solution function.

D. Students created graphs depicting the relationship between position and velocity in a traditional spring-mass situation.

E. Students used a computer program’s representation of solutions in a phase plane to discover and then reason about straight line solutions (eigensolutions) to linear systems of differential equations.

F. Students integrated understandings of the representations of solutions to single autonomous differential equations (exponential solution functions) and their thinking about three dimensions to develop conceptualizations about solutions to systems of differential equations.

G. Students connected the algebraic symbolizations of solutions to systems of differential equations and the graphical symbolizations for the same to further their understanding of solutions.

6.1.2 Algorithmatizing

Mathematicians have been creating and using algorithms in their practice for thousands of years. During the last hundred years, the focus in mathematics teaching has been on providing these algorithms to students and then teaching the students to perform
the procedure in order to find some answer. However, Freudenthal’s perspective that mathematics is a human activity provides an appropriate perspective to look at advancing mathematical activity. Algorithmatizing can be considered to be more than just “how to” but the actual creation of algorithms.

As Berlinski (2000) notes, “Algorithms are human artifacts” (p. xvii), the product of human activity. Keeping this activity perspective of algorithms in the forefront suggests that instead of focusing on the acquisition of these algorithms, we can characterize learning to use and understand algorithms as participating in the practice of algorithmatizing. By examining the activity that leads to the creation and use of artifacts, as opposed to the acquisition of the artifacts, we view mathematical learning of and in reference to algorithms through a different lens. (Rasmussen et al, 2005, pg. 63)

The primary algorithm that students were to develop in this instructional sequence is the algorithm to recover the general solution to a system of two linear homogeneous equations of the form

\[
\frac{dx}{dt} = ax + by \quad \text{and} \quad \frac{dy}{dt} = cx + dy
\]

This algorithm requires finding solutions that lie on a straight line in the phase plane by first finding the lines where the slopes of the vectors in the phase plane and the slopes of the line on which they lie are the same. Students may create this algorithm on their own and then use it to aid in reasoning about the straight line solutions. Next, they graphically investigate the sum of two straight line solutions and then construct with the teacher the algorithm for creating all linear combinations of
the eigensolutions as the general solution. This algorithm is created when the eigenvalues are real. The extension of this algorithm to complex eigenvalues is the last step and guided extensively by the professor or instructional materials.

In the differential equations class where my data was collected, reasoning to create the traditional algorithm is difficult to identify and describe but the algorithm did emerge at some point. One student reasoned about slopes of vectors in the phase plane early in the instructional sequence and the instructor used that earlier reasoning to create

\[
\frac{dy}{dt} = \frac{y}{x};
\]

he then encouraged students to reason about why this equation makes sense and the students developed the algorithm to find the straight lines with the professor’s help. After gaining some fluency with finding the straight lines, Adam and his group reasoned about the solution functions by substituting \( x \) or \( y \) in the right hand side of the differential equations to create a single differential equation that could be solved with separation of variables to find a pair of exponential functions \( x(t) \) and \( y(t) \). Finally, again with significant aid from the teacher, the class reinvented the part of the algorithm where the general solution to the system is represented as all linear combinations of two eigensolutions.

This example of algorithmatizing is interesting to examine, as this algorithm is different than the traditionally used algorithm. Students in an earlier differential equations classroom introduced the idea of this change in the development of eigensolutions to the instructional designer; it is based on graphical reasoning about the phase plane, something the students have experienced extensively in the class and leads to finding eigenvectors before eigenvalues (Rasmussen & Keynes, 2003). This is a
reversal of the conventional approach of finding eigenvalues first, and then eigenvectors. Because this algorithm is different than the traditional, it especially illustrates algorithmatizing as a mathematics practice. Mathematicians practice this same kind of activity as they develop new algorithms after new conceptualizations emerge in their work.

6.1.3 Justifying

Justifying in mathematics takes many forms and occurs at different levels of mathematical sophistication. Justifying can be activity that is mathematical communication with others to provide explanations in order to convince listeners of a statement’s validity. Justifying can be activity at a more formal level where mathematicians create a proof of the justification. Proof is usually formally presented and provides verification at a sophisticated level that something is mathematically true. In this differential equations class, the justifying takes the form of the first type of activity. Students participate in justifying mathematical conclusions to the mathematical community of their peers and teacher when statements are made, symbols created or reasoned about, and algorithms are created.

Students in an inquiry oriented differential equations class (this class and others) begin early in the class to develop ways to justify “out loud” their ideas to their classmates. In this type of classroom, statements that are made require justification. Part of a student’s responsibility is to listen to others and question their reasoning about the mathematics in the situation.
The justifications in this differential equations class take different activity forms. Students may just make a statement that contains a “because” or the justification may be a long discourse by a student in the class in response to another’s question or given spontaneously. The justification may be a graph or other representation, or analytic representations on the board. All of these are valuable and can lead to more formal justification when appropriate. There are several places where justifications were provided as the instructional sequence was enacted.

One of the earliest instances of justification occurs on the first day of instruction. Students were asked to examine a predator prey situation and reason about the meaning of each of the terms in the two differential equations:

\[
\frac{dR}{dt} = 3R - 1.4RF
\]
\[
\frac{dF}{dt} = -F + 0.8RF
\]

Students worked in small groups and then presented ideas to the class, where they discussed the real life situation and justified the positive value of the \(3R\), the negative value of the \(-F\) and the positive and negative consequences for the two second terms as shown above. There was not formal proof, but reasons were offered for the conclusions and judged by the other students in the class.

When the students worked in the computer lab and were asked to find an initial condition that would result in a shift of the solution along the \(t\) axis, one student designed a successful method. The other students then developed justifications for his ideas. Again, this was informal reasoning, but it was a type of mathematical activity that is part of a mathematician’s practice. In general, students’ justifying symbolizations that are graphical in nature are more informal in this differential equations class but still valid.
As another example, students had developed the idea of a straight line solution and later in the sequence were able to use the algorithm to find the straight line solution functions at any initial condition. The students were then asked to determine what a solution might look like if the initial condition is a vector not on a straight line. One of the students hypothesized that the solution will “head” toward the origin on the phase plane and stay exactly between the two straight line solutions as seen in the phase plane. However, another student hypothesized that the solution will curve toward one of the straight lines. Justification of which one of these was correct involved the use of some analytic strategies related to the three dimensional solution curves. The solutions to the straight line solutions as curves in space were exponential. Brandon looked at the solution on a phase plane and it appeared to curve, but he was not willing to accept only the graph as sufficient justification; he encouraged the class to justify the statements using other means. Finally, Adam discussed the fact that the sum of the two solutions include an $e^t$ and $e^{-2t}$ and that as the solution approaches the origin, the $e^t$ will be larger and therefore play a bigger role in the solution, so the solution will curve towards $e^t$.

This type of mathematical justification was not a formal proof, but it was more analytical in nature and included reasoning about algebraic representations. Similar to the previous example, the most formal justification in which students participated during instruction on solutions to systems of differential equations was in the analytic realm. Students were asked to prove that if two pairs of functions are both solutions to a system of differential equations, then the sum of the solutions is also a solution. They were also
asked to prove that if the system of differential equations is not of the form

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

then the sum of the solutions will not necessarily be a solution. When the students worked on this task, they used more algebraic and formal reasoning. They used general solution representations to substitute into the original general system and see if the differential equation holds true. Although they do not justify each step as a formal proof, I interpret this as more formal justification.

As far as justification, there is an additional perspective to be discussed. The way the students participated in the mathematical activity of justification is evidence of their enculturation into the community of mathematicians. This movement from informal to formal justification is the same practice that is seen when mathematicians are developing new theorems in their chosen research areas. They find a hypothesis that appears to be true and they work on informally justifying it through whatever means are at hand. If they can informally justify their new theorem, then it is reasonable to continue on and attempt a more formal justification (proof) of the theorem.

This progression from informal to formal happened in the differential equations classroom. The students first formed a hypothesis; they informally decided on a modification of the hypothesis and informally justified it using graphical representations. Then they continued to more formal justification of an important mathematical concept in systems of differential equations, that of superposition; superposition in differential equations is the principle that all solutions to linear systems of homogeneous linear differential equations can be written as the linear combination of two straight line
solutions. This particular sequence of justification also illustrates the fifth type of mathematics practice that students participated in during instruction on solutions to systems of differential equations. Experimenting as a mathematics practice is discussed in the next section.

6.1.4 Experimenting

As mentioned earlier in this chapter, experimenting was not identified in the advancing mathematical activity paper by Rasmussen et al. (2005). However, I assert that it is a mathematics practice that overlaps with the other practices; nevertheless it is worthy of identification and description. Experimenting is the practice where mathematicians can use schemes to discover something new, apply a variety of earlier conceptions to new problems or find ways that are not traditional to justify new hypotheses. Carlson and Bloom (2005) conducted studies of mathematicians’ practice. They found that cycles of conjecturing, imagining what happens, and evaluating are part of mathematicians practice. This cycle is very similar to what I call experimenting. Some mathematicians do not make this part of their practice explicit [personal communication, Amy Cohen, February, 2006] because the research mathematician traditionally presents results “full blown,” but discussions with mathematicians support the idea that they do use experimenting on a regular basis. This experimenting might be called the “playful side” of a mathematician’s practice, where they can try new things without fear of ridicule or failure.

Traditional mathematics classes are not conducive to providing students with opportunities to enculturate into the mathematics community by participating in the practice of experimenting. In traditional classes, students are typically presented with
algorithms to complete problems, often with some justification or a proof to show why the algorithm works. In this inquiry oriented differential equations class, students are provided with the opportunity to experiment, and there are many examples from the data collected in this class that students do participate in this practice.

1.) Students were willing to use conceptualizations or procedures from earlier sessions in the class to reason about new ideas. These conceptualizations or procedures may or may not prove helpful in the new reasoning, but students in this class were comfortable with trying them. On the first day of instruction on systems of differential equations, students in Adam’s small group were discussing how to find an equilibrium solution function to a system of differential equations that model a predator-prey context. Adam suggested that the technique they learned when reasoning about bifurcation diagrams might work since there are two variables in each of the differential equations. They discussed it briefly, but one of the other students objected to the reasoning, and so they moved on to another idea. Adam had the confidence to experiment with the ideas in this case, and the small group he was in contradicted him. Together they decided on another approach to the question and developed a relatively successful way to reason about the problem.

2.) Experimenting was also a common practice when the students use the technology available in the class. When the students were in the computer lab, they began experimenting with ways to create horizontal shifts during the second class in the nine days of instruction. Four days later, they again used the computer program to discover and reason about the straight lines of vectors that go to the origin; much experimenting occurred for students in all the groups. Not only does this experimenting lead to new
conceptualizations, it also enhances student reasoning, as students are forced to find means to justify what they are seeing in the computer program.

3.) Experimenting was supported in the class when the students’ mathematical activity resulted in the reinvention of the concept that the sum of two solutions to a system of differential equations is also a solution. Students were asked to figure out the solution to

\[
\frac{dx}{dt} = v \\
\frac{dv}{dt} = -2x - 3v
\]

with initial condition (-4,6); this point was not on one of the two straight lines in the phase plane. This solution is the sum of the solutions with initial conditions (-2,2) and (-2,4) which lie on \( y = -x \) and \( y = -2x \), respectively, and have been recovered by the students earlier in the course. Students participated in experimenting to figure out the solution using different means. For example, some students used the computer program to look at the phase plane and make some conjectures. Others analytically attempted to derive a function by educated guessing. Neither of these strategies worked, but the students were not afraid to try different ideas, experimenting in several different ways.

Students had a better way to reason about the task of finding the solution after this experimentation and when the professor suggested that the sum of the solutions to the system at (-2,2) and (-2,4) was the solution at (-4,6), students again experimented with that idea. One group typed into a calculator the sum of the solutions and viewed the graph to see if the solution “appeared” correct. Others discussed what the shape of two exponential functions added together might be and reasoned about that shape by comparing it to what they already knew. Most of the experimenting with this notion was
informal and intuitive. The professor eventually led the class to pursue a more analytic experimentation and at this point, Adam suggested the method named in the first half of the semester, “If it fits, it fits.” Students tried this method and successfully completed the analytic proof. This idea then became part of their understanding; this was seen in the case studies of Adam and Brandon.

In this section, I presented the practices of symbolizing, algorithmatizing, justifying, and experimenting. When students participated in these practices, they became increasingly central members of the mathematical community. Wenger’s (1998) theory of communities of practice may be relevant to this situation. Mathematicians form a community of practice and students in this differential equations class enculturate into this community by their participation in the practices. However, investigation into how mathematicians form a community of practice and how students in differential equations enculturate into this community is a topic for future research.

6.2. Summary of Parametric Reasoning and Case Studies

In this dissertation, the following three research goals were investigated: 1) characterize parametric reasoning as it precedes and occurs in one differential equations classroom while students learn to solve systems of differential equations; 2) describe two students’ ways of reasoning as they participate in the mathematics classroom to learn about solutions to systems of differential equations; 3) elaborate on Rasmussen et al.'s (2005) framework of advancing mathematical activity as it plays out in the differential equations classroom. The first two of these objectives were the primary focus of the research while the third was researched in less depth.
Parametric reasoning was documented as part of students’ ways of reasoning before they began studying systems. More broadly, the development of powerful new technologies to use in the mathematics of physical contexts often leads to a need to better understand how student thinking about time relates to their understanding of rate and function. The characterization that I presented adds to the mathematics education research literature concerning the mathematics of change and variation. The research on rate is extensive, but the idea of time based reasoning where time is a dynamic quantity that students use to parameterize other quantities extends the mathematics of change into a new venue. I provided evidence of five kinds of mathematical activity in which students participate as they reason parametrically. These activities are:

A. Making time an explicit quantity
B. Using the intuition from real life that “time never stops”
C. Using both quantitative and qualitative reasoning
D. Using both discrete and continuous imaging of time
E. Imagining the motion, which includes using the fictive motion metaphor, spatial visualization, and fusion of context and representation

Each of these characterizations of parametric reasoning work together. Not only do they work together, they are intrinsically related. Indeed, many of the examples tendered may be identified as more than one of these types of activity depending on the focus of the analysis. This blending of mathematical activities provides a way of reasoning for students that builds on their earlier mathematical and real life experiences. I offer this newly defined type of reasoning as one that complements other types of reasoning presented in mathematics education research, such as quantitative reasoning.
(Thompson, 1990), graphical reasoning (Kaput, 2002), and algebraic reasoning (Blanton & Kaput, 2003). In this dissertation I document the mathematical activity of parametric reasoning and relate it to the students’ activity before and as they learn to solve systems of differential equations.

However, parametric reasoning is not limited to differential equations. Students in earlier mathematics such as calculus and the mathematics of rate of change studied in middle school and high school will also use parametric reasoning. For example, parametric equations are part of the calculus curriculum. These equations traditionally are named $x(t)$ and $y(t)$ and are graphed by picking $t$ values and then finding $x$ and $y$ values and plotting points, or using technology to plot the points more quickly. Another common procedure used with parametric equations in calculus is finding the relationship between $x$ and $y$ by eliminating the $t$. There has been little research on how students may reason about these equations and how time based reasoning may contribute to students understanding of these kinds of equations. Further research is warranted to look at parametric reasoning in the calculus and algebra classroom.

Parametric reasoning as it is characterized in this dissertation is essential mathematical activity while students are learning about systems of differential equations. My second research question prompted the development of two case studies of students as they developed conceptualizations of solutions to systems. These case studies are situated in the social context of the classroom and focused on two students, Adam and Brandon, and their mathematical participation in the class. Also, I used Freudenthal’s philosophy that mathematics is a human activity and so learning was discussed in terms of the mathematical activity of the students in the classroom and in interviews. In the
analysis, I documented student learning by analyzing their mathematical activity as seen through their discourse and work. The case studies revealed that students have many ways to reason about solutions, but also may find it difficult to develop more traditional algorithms.

Some of the overarching conclusions that can be drawn from the case studies follow.

1.) Technology is a very useful tool for help in reasoning and visualizing solutions to differential equations. Students used special technology that was designed to support students’ investigation and reinvention of ideas. This technology played an important role in student understanding of solutions as three dimensional curves, equilibrium solutions as lines in space, straight line solutions and their contextual meaning when appropriate, phase planes as effective representations of solutions to systems of differential equations, and straight line solutions as pairs of exponential functions that can be represented as curves in space.

2.) Students’ understanding of derivative and solutions to first order differential equations are important to their understanding of solutions to systems of differential equations. Students rely on the mathematical activity from the first half of the class to venture into the expanded mathematical area of systems. They use the idea of rate of change (derivative) to reason and then expand on that idea to reason at a more sophisticated level about systems. They also use the conceptualizations of equilibrium solutions extensively as they reason about equilibrium solutions for systems of differential equations. Finally, but not exhaustively, they use the conceptualizations for autonomous differential equations as an important reasoning tool to study autonomous systems. This includes the
3). In this particular class, student reasoning about straight line solutions resulted in a solid conceptualization of a manifold, which was conceptualized as a plane where all the straight line solutions for a specific straight line lay. This reasoning resulted from activity that was about one straight line on the phase plane representing an infinite number of exponential functions that are shifts of each other. The students then reasoned about this plane to divide the space into sections and note that solutions that had initial conditions in different sections must stay in that section.

4.) Students understanding of vectors can be both an obstacle and a useful tool as students learn about the phase plane. Using prior knowledge about vectors can be an obstacle to student reasoning because they understand vectors as units with magnitude and direction. In systems of differential equations, when straight line solutions emerge, the notion of vector having a slope is vital and different than the magnitude and direction conceptualization and students may not use that notion as they think about the straight line solutions. However, using vector addition, scalar multiplication, and the concepts of linear combination and spanning in some of the tasks proved to be valuable. When the concept of superposition was introduced, students were able to understand the concept with reasoning about vectors.

5.) Parametric reasoning is important in the early mathematical activity of the instructional sequence. When students are thinking about graphs and solutions as three dimensional curves, they use parametric reasoning to make sense of the situation. When the instructional sequence moves to analytical solutions, however, parametric reasoning
is less obvious, although this analysis did not yield the knowledge if it is actually gone, or just absorbed into other forms of reasoning.

6.3. Limitations of the Study, Instructional Implications and Future Directions

This research study was conducted in one differential equations class and focused primarily on two students’ discourse and work in interviews and class discussions. There are two primary limitations because of this narrow focus.

First, the question of generalization to other groups of students needs to be addressed. These students were participating in a small class taught by a professor who was teaching for the first time. More research in other differential equations classes is required to see if the parametric reasoning and the other ways of reasoning are evidenced there as well. Now that these characterizations are identified, then research can continue in manner to check the original results.

Secondly, the change in focus from classroom mathematics as defined by the interpretive framework to the use of advancing mathematical activity may have resulted in less complete analysis of the four constructs of symbolizing, justifying, algorithmatizing, and experimenting. Data may need to be reviewed again for more confirmation of these constructs. More research is again called for in this situation.

The literature review of mathematics education literature in the area of rate of change and differential equations, as well as the emergence of mathematics classes using more reformed types of instruction confirms that changing instructional techniques is an ongoing and future direction of collegiate mathematics. The instructional sequence for solutions to systems of differential equations that was designed for use in this class
proves to be effective in supporting students’ mathematical growth and reinvention of concepts as they participate in the mathematical activity of the class. The use of technology is essential to their visualization and its integration into the materials proves to be particularly helpful. The instructional sequence emphasizes students’ development of the concepts of phase plane as a tool for reasoning, understanding of solutions as shifts along time, and equilibrium solutions. It also supports student understanding of eigenvectors, eigenvalues, and eigensolutions, and the design cycle for this has refined the sequence after observing students participation in differential equations class. (Rasmussen& Keynes, 2003).

My analysis as described in this research study provides support for the use of this instructional sequence. However, there are some tasks where, as the students worked on them, they did not reinvent the concepts and then algorithms as designed. The development of finding eigenvectors which worked well in other classes was not particularly successful in this class and the students ended up receiving direct instruction about finding the eigenvectors analytically for certain autonomous systems of differential equations. Evidence from analysis in other classrooms did not find this to be the case, but it was in this class.

Also, students were not able to reinvent the principle of superposition in this class without significant direct help. The materials for that section of the sequence did not provide enough support; this may have been true in other classrooms as well. My case study analyses are evidence for this conclusion.

In addition to revisions to the instructional sequence used in this inquiry oriented differential equations class, there are directions for research that this dissertation
suggests. This future work can be placed in three areas of research: parametric reasoning in other content areas, how instructional materials and the learning environment support student learning in differential equations, and types of advancing mathematical activity and how students’ participation in this activity will enculturate them into the mathematical community.

Parametric reasoning as an additional construct for researching time based reasoning for students in K-14 mathematics is an important place for further research. The characterization of parametric reasoning developed in the dissertation can be applied to students’ thinking in many different contexts for confirmation, modification and elaboration. The use of parametric equations in algebra and calculus is the first area that I will begin to apply this new characterization. Then I will build on some of Kaput and others’ work at the middle school and elementary school level to investigate parametric reasoning there as well. It is quite likely that modifications and new kinds of mathematical activity might emerge from further study at these early mathematical levels.

Not only is further study of parametric reasoning warranted, the use of parametric and other types of reasoning that are used at the differential equations level has great potential. Research in undergraduate mathematics is still relatively underdeveloped and ways that students reason, and particularly how learning environments and instructional materials support this reasoning may be a very fruitful arena for further research. This dissertation is part of a larger differential equations mathematics education research program where the study of student learning is continuing. This dissertation contributes to that research program, but it also provides a springboard for further work. For example, I would like to pursue the connections between parametric reasoning and
algebraic reasoning. Also, how can we improve on the design of the instructional sequences to take advantage of students’ parametric reasoning? What is the connection between the results of the case studies as individual learning experiences and the whole class mathematical activity? I had originally intended to document classroom mathematics practices as defined by Cobb and Yackel (1996), but the analysis proved impossible because the data did not support the identification of classroom mathematics practices of the type defined by them. I intend to conduct another classroom teaching experiment and document the classroom mathematics practices. This will complete the task that was started by Stephan and Rasmussen (2002) when they documented the classroom mathematics practices for first order differential equations.

Finally, the most promising future directions and the least developed in this dissertation is that of advancing mathematical activity and its potential to be a way to investigate the enculturation of differential equation students into the mathematical community. The mathematical activity of symbolizing, defining, algorithmatizing, justifying, and experimenting are all elements of a mathematician’s practice and were documented (except for defining) in the differential equations class studied in this report. I will take these ideas and develop them in other collegiate classrooms. If these constructs prove to be useful for analyzing activity in other classrooms, then I can make a case for more explicit support of advancing mathematical activity at the college and K-12 mathematics classroom.

This dissertation is significant for K-16 mathematics education. Parametric reasoning is a newly characterized form of reasoning that can be studied at other areas and in time, made explicit in learning. This is important with the advancing of dynamical
systems as one of the most important tools for studying the world, and therefore, understanding parametric reasoning better is an important contribution. The case studies of Adam and Brandon are significant for differential equations classes particularly as they document these two diverse students’ participation in the classroom. This new description of learning will prove helpful to instructional designers of differential equations. Also, it may be useful for linear algebra instruction as well, as the connections between differential equations and linear algebra are strong and my analysis proved that this connection is vital to students’ reasoning. Finally, as we continue to study mathematics learning in the social context of the classroom, finding new avenues of study and then offering ways to improve learning in that context is important. The use of the construct of advancing mathematical activity that was described above is a powerful new research tool to better understand learning in the classroom, from kindergarten to university.
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REFERENCES


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Appendix A
Instructional Materials
Rabbits and Foxes

Most species live in interaction with other species. For example, perhaps one species preys on another species, like foxes and rabbits. Below is a system of rate of change equations intended to predict future populations of rabbits and foxes over time, where \( R \) is the population (in thousands, or millions, or whatever) of rabbits at any time \( t \) and \( F \) is the population of foxes at any time \( t \) (in years).

\[
\frac{dR}{dt} = 3R - 1.4RF
\]
\[
\frac{dF}{dt} = -F + 0.8RF
\]

1. a) In our earlier work with the rate of change equation \( \frac{dP}{dt} = kP \) we assumed that there was only one species, that the resources were unlimited, and that the species reproduced continuously. Which, if any, of these assumptions is modified and how is this modification reflected in the above system of differential equations?

b) Interpret the meaning of each term in the rate of change equations (e.g., how do you interpret or make sense of the \(-1.4RF\) term and what are the implications of this term on the future predicted populations?).

2. A group of scientists studying rabbit-fox population in Illinois estimates that the current number of rabbits is 1 (scaled appropriately) and that the scaled number of foxes is 1. Figure out a way to adapt Euler’s method to get numerical estimates for the future number of rabbits and foxes as predicted by the differential equations. What are some different ways to graphically depict your \( (t, R, F) \) values?

<table>
<thead>
<tr>
<th>( t )</th>
<th>( R )</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Three Dimensional Visualization
3. A crop duster plane with a two blade propeller is taxiing down a runway. On the end of one of the propeller blades, which are rotating clockwise at a slow constant speed, is a noticeable red paint mark. Imagine that for the first several rotations of the propeller blades the red mark leaves a "trace" in the air as the plane makes it way down the runway.

a) Using a red cube as the paint smear, simulate this scenario and create the trace with a pipe cleaner. Sketch the ideal perspective for persons at alpha, beta, gamma, delta locations. What information, if any, is lost in the various perspectives?

Sketch your ideas for each of the following:

g) What if there was another paint mark on other end of the propellor, what, ideally, do the four observers see then? How does the trace of this mark relate to the previous trace?

h) What if there was a paint mark on the center of the propeller blade mechanism. What do the observers ideally see then?

i) How ideally would each observer see all of the above paint marks simultaneously?

j) What do the four observers ideally see if the red mark slowly drifts toward the center?

k) What do the four observers ideally see if the propeller is not rotating and the red mark drifts away from the center at a rate proportional to its distance from the center as the plane taxis down the runway?
4. a) For the same system of differential equations from problem 1,
\[ \frac{dR}{dt} = 3R - 1.4RF, \quad \frac{dF}{dt} = -F + 0.8RF, \]
use the DE Explorer Applet to generate predictions for the future number of rabbits and foxes if at time 0 we initially have 1 rabbit and 3 foxes (scaled appropriately). Verify the first three steps of Euler’s method with hand calculations and then generate and reproduce below all the different graphical depictions discussed in the airplane problem (3D plot and all three different views or projections of the 3D plot).

b) If possible, determine an initial rabbit and fox population at time 0 such that the 3D graph of the solution is a shift of the 3D graph in part a) along the t-axis. If this is not possible, explain why not. If it is possible explain how this shift is or isn’t depicted in the various views or projections of the 3D graph, why this shift is possible, and what your reasons do or do not have to do with the rate of change equations.

c) Use the DE Explorer Applet to generate predictions for the future number of rabbits and foxes if at time 0 we initially have the following different initial conditions: (i) 2 rabbits and 3 foxes, (ii) 1.5 rabbits and 4 foxes, (iii) 4 rabbits and 2 foxes. Reproduce below all the different graphical depictions. Show all three solutions simultaneously.
5. Discuss the advantages and disadvantages of the different graphical depictions from the previous problem.

6. a) Suppose the current number of rabbits is 3 and the number of foxes is 0. Without using any technology and without making any calculations, what does the system of rate of change equations (repeated below) predict for the future number of rabbits and foxes? Explain your reasoning.

\[
\frac{dR}{dt} = 3R - 1.4RF \\
\frac{dF}{dt} = -F + 0.8RF
\]

b) Use the DE Explorer applet to generate the 3D plot and all three different views or projections of the 3D plot to graphically depict the way in which the number of rabbits and the number of foxes evolve over time for the initial conditions in part a). Explain how each plots illustrates your conclusion in part a).

c) Suppose the current number of rabbits is 6 and the number of foxes is 0. What does the system of rate of change equations predict for the future number of rabbits and foxes? How and why is this prediction related to the prediction when the initial number of rabbits is 3 and the number of foxes is 0?
d) Repeat parts a) and b) if the initial number of rabbits is 0 and the number of foxes is 2.
7. a) What would it mean for this system to be in equilibrium? Are there any equilibrium solutions to this system of rate of change equations (repeated below)? If so, determine all equilibrium solutions and generate the 3D and other views for each equilibrium solution.

\[
\begin{align*}
\frac{dR}{dt} &= 3R - 1.4RF \\
\frac{dF}{dt} &= -F + 0.8RF
\end{align*}
\]

b) For single differential equations, we classified equilibrium solutions as attractors, repellors, and nodes. For each of the equilibrium solutions in the previous problem, create your own terms to classify the equilibrium solutions in part a) and briefly explain your reasons or imagery behind your choice of terms.

8. A group of scientists wants to graphically display the predictions for many different non-negative initial conditions (this includes 0 values for R and F, but not negative values) to the system of differential equations below and they want to do so using only one set of axes. What one single set of axes would you recommend that they use (R-F-t axes, t-R axes, t-F axes, or R-F axes)? Explain your reasoning and provide a sketch of several different solutions in your choice.

\[
\begin{align*}
\frac{dR}{dt} &= 3R - 1.4RF \\
\frac{dF}{dt} &= -F + 0.8RF
\end{align*}
\]
9. One view of solutions for studying solutions to systems of autonomous differential equations is the x-y plane, called the **phase plane**. The phase plane is the analog to the flow line for a single autonomous differential equation.

(a) Consider the rabbit-fox system of differential equations below and a solution graph, as viewed in the phase plane, for the initial conditions R = 2 and F = 3.

- We know from previous work that the solution, as viewed in the R-F plane, will be a closed curve. But in which direction will the solution go, clockwise or counterclockwise?
- Figure out a way to record your thinking as a vector in the R-F plane, with an accurate slope to the vector?

\[
\begin{align*}
\frac{dR}{dt} &= 3R - 1.4RF \\
\frac{dF}{dt} &= -F + 0.8RF
\end{align*}
\]

(b) Plot by hand those vectors in the R-F plane that you think would be the easiest to plot. Be sure to plot some vectors that are not on any of the given solution curves.

(c) Explain how your approach in part (b) gives you a way to graphically locate any and all equilibrium solutions.

(d) For a single differential equation we had three different types of equilibrium solutions: attractors, repellors, and nodes. What names would you give to describe the equilibrium solutions for the above system?
Bees and Flowers

In previous problems dealing with two species, one of the animals was the predator and the other was the prey. In this problem we study systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is both species are harmed by interaction) or cooperative (that is both species benefit from interaction).

1. Which system of rate of change equations below describes a situation where the two species compete and which system describes cooperative species? Explain your reasoning.

\[ \frac{dx}{dt} = -5x + 2xy \quad \frac{dy}{dt} = -4y + 3xy \]  
\[ \frac{dx}{dt} = 3x(1 - \frac{x}{3}) - \frac{1}{10}xy \quad \frac{dy}{dt} = 2y(1 - \frac{y}{10}) - \frac{1}{5}xy \]

2. For system (A), plot by hand those vectors that are either horizontal or vertical. Provide a hand sketch of your result and then use your result to determine all equilibrium solutions. Verify your equilibrium solutions algebraically.

3. Describe the long-term behavior of solutions with initial conditions anywhere in the first quadrant of the phase plane (use the result from part 2 and applet 2). For example, describe the long-term behavior of solutions if the initial condition is in such and such region of the x-y plane. Provide a sketch of your analysis in the x-y plane and write a paragraph summarizing your conclusions and any conjectures that you have about the long-term outcome for the two populations depending on the initial conditions.

4. Repeat problems 2 & 3 for the system of differential equations (B).
Predator-Prey

Consider the following systems of rate of change equations:

**System A**
\[
\begin{align*}
\frac{dx}{dt} &= 3x(1 - \frac{x}{10}) - 20xy \\
\frac{dy}{dt} &= -5y + \frac{xy}{20}
\end{align*}
\]

**System B**
\[
\begin{align*}
\frac{dx}{dt} &= 0.3x - \frac{xy}{100} \\
\frac{dy}{dt} &= 15y(1 - \frac{y}{17}) + 25xy
\end{align*}
\]

In both of these systems, \(x\) and \(y\) refer to the number of two different species at time \(t\). In particular, in one of these systems the prey are large animals and the predators are small animals, such as piranhas and humans. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey.

1. Which system represents the small prey and large predator? Which system represents the large prey and small predator? Explain your reasoning.

2. For system (A), plot by hand those vectors that are either horizontal or vertical. Provide a hand sketch of your result and then use your result to determine all equilibrium solutions. Verify your equilibrium solutions algebraically.

3. Describe the long-term behavior of solutions with initial conditions *anywhere* in the first quadrant of the phase plane (use the result from part 2 and applet 2). For example, describe the long-term behavior of solutions if the initial condition is in such and such region of the \(x-y\) plane. Provide a sketch of your analysis in the \(x-y\) plane and write a paragraph summarizing your conclusions and any conjectures that you have about the long-term outcome for the two populations depending on the initial conditions.
Spring-Mass Motion Investigation

In this problem we use Newton’s Law of motion (\sum \! F = ma) to develop a system of rate of change equations in order to be able to describe, explain, and predict the motion of a mass attached to a spring.

d) Depending on the values for parameters like the stiffness of the spring, the weight of the object attached to the spring, and the amount of friction along the surface that the object travels, different behaviors may be possible. Describe in words the different motions you might see or expect to see. For the different types of motion provide a rough sketch of what you think the position versus velocity graph would look like.

e) Use Newton’s Law of motion to develop a rate of change equation to model the motion of an object on a spring. Assume that the only forces acting on the object are the spring force and the friction force. Convert this rate of change equation into a system of two differential equations, one for position and one for velocity.

f) When the John Hancock Building in Boston, MA was first built it tended to sway back and forth so much so that people in the top floors experienced motion sickness. Similar to the spring mass system, we can model the back and forth motion of the building by using \sum \! F = ma. The model for the swaying building needs to take into account the building’s restoring force (analogous to the spring force) and the frictional force. What other forces do we need to take into consideration for the swaying building and what is the resulting system of differential equations?
A Swaying Skyscraper

As developed in class, the following system of rate of change equations is a model for helping us make predictions about the motion of a tall building. In this simplified system of rate of change equations, \( x \) stands for the amount of displacement of the building from the vertical position at any time \( t \) and \( y \) stands for the horizontal velocity of the building at any time \( t \). Use the DE Explorer applet as a tool to explore solutions as viewed in the \( xy \)-plane (i.e., the phase plane).

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -x - y + x^3
\]

1. Determine all equilibrium solutions and explain the meaning of each one in terms of the swaying skyscraper. Create any terms needed to classify new types of equilibrium solutions and briefly explain your reasons or imagery behind your choice of terms.

2. Provide a sketch of several representative curves in the phase plane and give an interpretation for the motion of the building for the different types of curves (e.g., does the building remain standing? If so, for what initial conditions? For what range of initial conditions is a disaster predicted?)
Further Spring-Mass Investigation

1. Application of Newton’s Law of Motion to the spring-mass situation in the previous problem results in the following:
\[
\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0,
\]
where \(x\) is the position of the object attached to the end of the spring, \(m\) is the mass of the object, \(b\) is the friction parameter (also called damping coefficient), and \(k\) is the spring constant.

Converting this to a system of two differential equations yields the following:
\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= -\frac{k}{m} x - \frac{b}{m} v
\end{align*}
\]

Use the vector field Java applet to investigate the motion of the object when \(m = 1\), the spring constant \(k = 2\), and the friction parameter, \(b\), is between 0 and 4. You can compare vector fields for different \(b\)-values by opening two or more windows of the applet on one computer or on different computers. Note, to generate the vector field, define the vector as
\[
\begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix} = \begin{bmatrix}
y \\
-2x - hy\end{bmatrix}.
\]

9. Vary the value of \(b\) to investigate how the motion of the mass, as predicted by the system of differential equations, differs when the friction parameter is 0 as compared to when it is, say 2.3, or 3, or 3.8, for example. Use graphs and descriptions of how to interpret those graphs to summarize the results of your investigation. Be as specific as possible. In addition, discuss changes, if any, to the type of equilibrium solution as the friction parameter changes.
10. Joey sets the friction parameter to 3 and notices that graphs of solutions in the position-velocity plane seem to get sucked into the origin via a straight line. He conjectures that along this line all vectors head directly towards the origin with a specific slope. Do you agree with Joey’s conjecture? Figure out an algebraic way to either support or refute Joey’s conjecture. If you think Joey’s conjecture is true, (i) what is the precise slope? (ii) what does the 3D graph look like? (iii) what is the corresponding physical motion of the mass? (iv) is this the only straight line of vectors?

11. Are there any solutions that, when viewed in the position-velocity plane, lie along straight lines when the friction parameter $b$ is equal to 1? What is the smallest value of $b$ for which we get solutions that, when viewed in the position-velocity plane, lie along a straight line? Algebraically support your conclusions.
Straight line Solutions for the Spring-Mass System
\[ \frac{dx}{dt} = y \]
\[ \frac{dy}{dt} = -2x - 3y \]

In our investigation of the spring-mass system we found that when the friction parameter was equal to 3, solutions with initial conditions that are either on the line \( y = -x \) or on the line \( y = -2x \) head directly toward the origin along a straight path.

1. For the initial condition \((-2, 4)\), what are the equations for \( x(t) \) and \( y(t) \)?
2. Thinking geometrically, Susan conjectures that the $x(t)$ and $y(t)$ equations for the solution with initial condition (-2, 4) are both exponential decay functions and in order to create a straight line in the phase plane these equations must have the same exponent. Do you agree with her conjecture? Why or why not?

3. a) What are the $x(t)$ and $y(t)$ equations for the solution with initial condition (-1, 2)?

b) If you multiplied $x(t)$ and $y(t)$ equations from problem 1 by some number, say $-3$ for example, is the result also a solution to the system of differential equations? Prove that your conclusion is correct.

c) What are the $x(t)$ and $y(t)$ equations for any solution with initial condition along the line $y = -2x$?

4. For the initial condition (-2, 2), what are the equations for $x(t)$ and $y(t)$? What are the $x(t)$ and $y(t)$ equations for any solution with initial condition along the line $y = -x$?
5. a) Suppose you were to start with an initial condition somewhere in the second quadrant between the two straight line solutions, say at (-4, 6). Sketch what you think the solution as viewed in the phase plane looks like.

b) Figure out what the $x(t)$ and $y(t)$ equations are for the solution with initial condition (-4, 6). According to these equations, what should the solution in the phase plane look? Explain.

c) What are the $x(t)$ and $y(t)$ equations are for the solution with initial condition (1, 1)? According to these equations, what should the solution in the phase plane look? Explain.

d) How can you get the $x(t)$ and $y(t)$ equations for any initial condition?
6. a) For the initial condition (-2, 4), what does the 3D graph look like? What do the \( x(t) \) and \( y(t) \) graphs look like?

b) What does the 3D graph with initial condition (-1, 2) look like and how does this graph relate to the 3D graph with initial condition (-2, 4)? Explain.

c) Joaquin argues that along the line \( y = -2x \) in the phase plane there are an infinite number of solutions. Joey argues that along the line \( y = -2x \) in the phase plane there are three different types of solutions (one that heads towards the origin from the second quadrant, one that heads toward the origin from the fourth quadrant, and the origin). Explain how and why both statements make sense.
Straight line Solutions for Systems of the Form

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

1. In the previous problem we had a specific physical context for the system of differential equations under study. In this problem we examine solutions for a more general system of differential equations. For each of the system of differential equations below, address the following questions:

- How many equilibrium solutions are there and what are they?
- Are there solutions that, when viewed in the phase plane (i.e., the \(x\)-\(y\) plane), lie along a straight line? If so, algebraically figure out the exact slope of the straight line(s).
- For those systems that do have solutions that, when viewed in the phase plane, lie along a straight line, figure out the exact \(x(t)\) and \(y(t)\) equations for any solution with initial condition on the straight line(s).
- For those systems that have straight line solutions, write down the general solution.
- How would you classify the equilibrium solution? Create terms if needed to classify any new types of equilibrium solutions and explain the meaning of your terms.
- For those systems of differential equations that do have solutions that, when viewed in the phase plane, lie along straight lines, what do these straight lines look like in 3D? Provide your best 3D sketch.

(a) \[
\begin{align*}
\frac{dx}{dt} &= -3x + 2y \\
\frac{dy}{dt} &= 6x + y
\end{align*}
\]
(b) \[
\begin{align*}
\frac{dx}{dt} &= x + y \\
\frac{dy}{dt} &= -x + y
\end{align*}
\]
(c) \[
\begin{align*}
\frac{dx}{dt} &= -2x - 2y \\
\frac{dy}{dt} &= -x - 3y
\end{align*}
\]
(d) \[
\begin{align*}
\frac{dx}{dt} &= 2x + 2y \\
\frac{dy}{dt} &= x + 3y
\end{align*}
\]
(e) \[
\begin{align*}
\frac{dx}{dt} &= 4x + 2y \\
\frac{dx}{dt} &= 3x - y
\end{align*}
\]
As we figured out from our analysis on the previous problems, sometimes we have solutions in the phase plane that lie along a straight line headed directly towards or away from the equilibrium solution at the origin and sometimes we don’t.

Explain in words how you figure out whether there are any straight line solutions in the phase plane and if so, what the slopes of this line or lines are. Demonstrate how your approach works in general for linear systems of the form

\[ \frac{dx}{dt} = ax + by \]

\[ \frac{dy}{dt} = cx + dy \]

Explain in words how you figure out the \( x(t) \) and \( y(t) \) equations for any and all straight line solutions in the phase plane. Demonstrate how your approach works in general for linear systems of the form

\[ \frac{dx}{dt} = ax + by \]

\[ \frac{dy}{dt} = cx + dy \]

Explain in words how you would use two different straight line solutions for finding the \( x(t) \) and \( y(t) \) equations for any initial condition.
Below is a vector field for the system of differential equations \( \frac{dx}{dt} = 2x + 3y \) and \( \frac{dy}{dt} = -4y \). Straight line solutions lie along the line \( y = 0 \) (with positive exponent in the \( x(t) \) and \( y(t) \) equations) and along the line \( y = -2x \) (with negative exponent in the \( x(t) \) and \( y(t) \) equations).

1. Use two different cubes to simultaneously simulate two different solutions, one with initial condition at the point \((1, 0)\) and one with initial condition at the point \((3, 0)\). Determine, with reasons, what happens to these two cubes as time progresses and sketch the 3D curves.

2. Repeat problem 1 for the initial conditions \((-1, 2)\) and \((-3, 6)\).

3. Use a third cube to simulate the solution that is made by adding the solution with initial condition at \((3, 0)\) to the solution with initial condition \((-3, 6)\). Explain how the flow of the solution with initial condition \((0, 6)\) can be thought of in terms of the addition of vectors from the solution with initial condition \((3, 0)\) and \((-3, 6)\).
Find a value or a range of values for the parameter $n$ between $-4$ and $4$ (including non-integer values) in the system of differential equations
\[
\frac{dx}{dt} = -3x + ny, \\
\frac{dy}{dt} = 6x + y
\]
so that when you view solutions in the x-y plane there are

a) exactly two different straight line solutions
l) no straight line solutions
m) exactly one straight line solution
n) an infinite number of straight line solutions

Without using technology, sketch many different solutions in the phase plane for each of the following systems of differential equations.

\[
\begin{align*}
\text{a) } \frac{dx}{dt} &= 2x, & \frac{dx}{dt} &= -3x - \frac{1}{2}y \\
\frac{dy}{dt} &= 2y, & \frac{dy}{dt} &= 6x + y \\
\text{b) } \frac{dx}{dt} &= -3x - \frac{1}{2}y
\end{align*}
\]
In an earlier problem we applied Newton’s law of motion for a spring mass system and obtained the second order differential equation $\frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$, where $x$ is the position of the object attached to the end of the spring, $m$ is the mass of the object, $b$ is the damping coefficient, and $k$ is the spring constant. Using the fact that velocity is the derivative of position and choosing the mass $m = 1$ and the spring constant $k = 2$, we converted this to the following system of two differential equations:

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -2x - by
\]

We were able to figure out the $x(t)$ and $y(t)$ equations when the value of the friction parameter was such that there were straight line solutions in the phase plane. Such a situation is typically referred to as over-damped. The situation is called damped when the differential equations predict that the mass will oscillate about the 0 position and undamped when there is no friction. In the following problems we figure out the $x(t)$ and $y(t)$ equations for the damped and undamped situations.

The vector field for the case when $b = 2$ is shown below. Based on this vector field, it appears that the differential equations predict that the mass will oscillate back and forth. Even though there are not any straight line solutions, we can still use the same algebraic approach as before to get the $x(t)$ and $y(t)$ equations for any initial condition, but we will have to deal complex numbers. Problems 1-7 outline a way to do this.
1. For the system of differential equations
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -2x - 2y
\]
use the same algebraic approach as before to verify that the slopes of the “straight line” solutions are \(-1 \pm i\).

2. For solutions with “straight line” slope \(y = (-1 + i)x\), find the \(x(t)\) and \(y(t)\) equations (in terms of complex numbers) for the solution along this “straight line” with initial condition \((1, -1+i)\).

3. For solutions with “straight line” slope \(y = (-1 - i)x\), find the \(x(t)\) and \(y(t)\) equations (in terms of complex numbers) for the solution along this “straight line” with initial condition \((1, -1-i)\).

4. Use Euler’s formula \(e^{a+ib} = e^a e^{ib} = e^a (\cos b + i \sin b)\) to rewrite the \(x(t)\) and \(y(t)\) equations from problem 2 (call these \(x_1(t)\) and \(y_1(t)\)) and then again from problem 3 (call these \(x_2(t)\) and \(y_2(t)\)).

5. Denise suggests that if you add \(\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}\) to \(\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}\) the resulting pair of equations is (i) real valued and (ii) a solution to the same system of differential equations. Verify that this is true.

6. Verify that if you subtract \(\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}\) from \(\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}\) and multiply the result by the complex number \(i\), then the resulting pair of equations will be a real and a solution to the same system of differential equations.

7. Form the general solution to the system of differential equations
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -2x - 2y
\]
8. Which part(s) of your general solution accounts for the fact that the differential equations predict that the mass will oscillate about the zero position? Which part(s) of your general solution accounts for the fact that the amplitude of the oscillations decreases over time?

9. Suppose that for a different system of differential equations you got the exact same general solution except instead of \( e^{-t} \) you got \( e^t \). How would this change graphs of solutions in the phase plane? Explain.

10. Find the general solution to the spring mass problem when there is no friction. Sketch these solution in the phase plane and explain how this general solution fits with your expectation for the behavior of the mass over time. Note: when there is no friction \( (b = 0) \) and the spring constant \( k = 2 \), we get

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -2x
\]
Does Adding Solutions Always Result in Another Solution?

In deriving the general solution to the spring mass problem, two solutions were added to get another solution. This worked for the particular equations at hand, but does adding two solutions to a system of differential equations of the form \( \frac{dx}{dt} = ax + by \) \( \frac{dy}{dt} = cx + dy \) always result in another solution to the same system of differential equations? Below is a proof that this in fact is true.

Claim:

If \( \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \) and \( \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \) are solutions (not necessarily straight line solutions) to a system of differential equations of the form \( \frac{dx}{dt} = ax + by \) \( \frac{dy}{dt} = cx + dy \), then the sum of these two solutions is also a solution. That is, if we call the sum of these two solutions \( \begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} \) where

\[
\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix},
\]

then \( \begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} \) is also a solution to the same system of differential equations.

Proof:

In order to show that \( \begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} \) is a solution, we need to verify it satisfies the system of differential equations. This is, we need to show that

\[
\frac{d}{dt}x_3(t) = ax_3(t) + by_3(t) \quad \frac{d}{dt}y_3(t) = cx_3(t) + dy_3(t)
\]

Since \( \begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix} \), we know that

\[
\frac{d}{dt}x_3(t) = \frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t) \quad \frac{d}{dt}y_3(t) = \frac{d}{dt}y_1(t) + \frac{d}{dt}y_2(t)
\]
Because \( \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \) is a solution, it satisfies the system of differential equations. That is,
\[
\frac{d}{dt} x_1(t) = ax_1(t) + by_1(t) \tag{2}
\]

Similarly, since \( \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \) is a solution, 
\[
\frac{d}{dt} x_2(t) = ax_2(t) + by_2(t) \tag{3}
\]

Substituting (2) and (3) into (1) yields
\[
\frac{d}{dt} x_3(t) \frac{d}{dt} y_3(t) = ax_1(t) + by_1(t) + ax_2(t) + by_2(t).
\]

Rearranging terms yields
\[
\frac{d}{dt} x_3(t) = ax_1(t) + ax_2(t) + by_1(t) + by_2(t) = cx_1(t) + cy_1(t) + cx_2(t) + cy_2(t).
\]

Finally, using the fact that \( \begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix} \) yields 
\[
\frac{d}{dt} x_3(t) = ax_3(t) + by_3(t)
\]

which is what we set out to show. Therefore \( \begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} \) is also a solution to the system of differential equations.

1. Suppose that \( \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \) and \( \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \) are solutions to the system of differential equations
\[
\frac{dx}{dt} = ax + by + 1, \quad \frac{dy}{dt} = cx + dy + 2
\]

where \( a, b, c, \) and \( d \) are constants. Josh claims that the sum of these two solutions is also a solution to the same system of differential equations. Do you agree with his claim? Either develop a similar proof as above to support this claim or point to where (and why) the above proof fails.

2. Suppose that \( \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \) and \( \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \) are solutions to the system of differential equations
\[
\frac{dx}{dt} = ax^2 + by, \quad \frac{dy}{dt} = cx + dy
\]

where \( a, b, c, \) and \( d \) are constants. Aaron claims that the sum of these two solutions is also a solution to the same system of differential equations. Do you
agree with his claim? Either develop a similar proof as above to support this claim or point to where (and why) the above proof fails.
Swaying Skyscraper Revisited

Finding the \( x(t) \) and \( y(t) \) equations for systems of differential equations that are non-linear, like those that model the swaying skyscraper

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -x - y + x^3
\]

is typically very difficult and often impossible. Moreover, as the previous problem showed, even if you could find two solutions to non-linear systems there is no guarantee that adding them together will result in another solution.

One approach for dealing with non-linear systems of differential equations is to approximate the non-linear system with a system of linear differential equations of the form

\[
\frac{dx}{dt} = ax + by \\
\frac{dy}{dt} = cx + dy
\]

at each equilibrium solution. Although these approximations are only good in a region nearby each equilibrium, one can often piece together these “local” pictures in the phase plane to be informative for the entire phase plane.

For the above system of differential equations that model the swaying skyscraper, there are three equilibrium solutions \((0, 0)\), \((1, 0)\), and \((-1, 0)\). When the \((x, y)\) values are close to \((0, 0)\), the term \(x^3\) is much much smaller that the other terms. Therefore the non-linear system can be approximated by the linear system

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -x - y
\]

1. Explain why the linearized system \( \frac{dx}{dt} = y \) is NOT a good approximation for values of \( (x, y) \) far away from \((0, 0)\).

2. Based on the general solution for the linearized system of differential equations about the equilibrium solution \((0, 0)\), what type of equilibrium solution is \((0, 0)\)?
3. Figure out the linearized system of differential equations about the equilibrium solution (1, 0) and then determine, based on the general solution for this linearized system, what is the type of equilibrium solution at (1, 0).

4. Repeat problem 3 for the equilibrium solution at (-1, 0).

5. As you probably figured out in problems 3 and 4, the equilibrium solutions at (1, 0) and (-1, 0) are both saddles. The local straight line solutions (called separatrix) that enter each equilibrium are especially important because if you follow them backwards in time they determine a region of initial conditions in the phase plane for which the differential equations predict the building will not fall down. Ask your professor if technology is available to graph backwards in time the straight line solutions that head into each equilibrium and provide a sketch of the result. Shade in the region of initial conditions for which the building will fall down.
Equilibrium Solutions for Linear Systems

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

1. For each part below, use two different ways (one algebraic and one geometric using nullclines) to figure out the number and location of equilibrium solutions.

   a) \[\frac{dx}{dt} = 3x + 2y \quad \frac{dy}{dt} = -2y\]

   b) \[\frac{dx}{dt} = 4x - 2y \quad \frac{dy}{dt} = -2x + y\]

2. Is it possible to find values of \(a, b, c, d\) such that the system of differential equations \[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\] has exactly two equilibrium solutions? Explain why or why not.

3. Develop criteria (in terms of the parameters \(a, b, c,\) and \(d\)) that tell us about the number and location of equilibrium solutions for systems of differential equations of the form

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]
Matrix Notation and Equilibrium Solutions for Linear Systems

\[ \frac{dx}{dt} = ax + by \]
\[ \frac{dy}{dt} = cx + dy \]

One way to approach problem 3 is to think about there being an infinite number of equilibrium solutions when the two nullclines coincide. That is, when the equations \( ax + by = 0 \) and \( cx + dy = 0 \) determine the same set of points. Put another way, the equations are dependent when \( \frac{y}{b} = \frac{-a}{-c} \) are the same equation. Thus, \( \frac{-a}{b} = \frac{-c}{d} \), which says that \(-ad = -bc\).

Rewriting this yields \( ad – bc = 0 \).

As shown next, another way to arrive at this result is to use matrix notation and the fact that two equations are dependent when the determinant of the matrix is zero.

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Thus, the equations \( ax + by = 0 \) and \( cx + dy = 0 \) are dependent when the determinant of the coefficient matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is zero. That is, when \( ad – bc = 0 \).

Next, we develop an approach for finding the general solution to a system of differential equations of the form \( \frac{dx}{dt} = ax + by \) by first finding the value of the exponent (that is, the eigenvalue) associated with any straight line solution before finding the slope of the straight line solutions (typically called eigensolutions). Note that in your previous work you first found the slope of straight line solutions and then found the exponent. Some students have referred to this as the “slope first” method. In the pages that follow, an alternative approach is developed – the “eigenvalue first” method.

We develop this alternative method for four reasons:

- The eigenvalue first method can be used for systems of three or more differential equations whereas the slope first method cannot.
• Often times just knowing the eigenvalues is sufficient for understanding the overall picture of solutions in the phase plane and so therefore this method is more efficient
• The eigenvalue first approach makes important connections with linear algebra
• The eigenvalue first approach is algebraically more efficient
Eigenvalue First Method

For linear systems of the form
\[
\begin{align*}
\frac{dx}{dt} &= ax + by, \\
\frac{dy}{dt} &= cx + dy,
\end{align*}
\]

one way to determine the exponent (i.e. \(\lambda\), the eigenvalue value) for possible straight line solutions (or eigensolutions) is to use the fact that if eigensolutions exist in the phase plane, then \(\frac{dx}{dt} = \lambda x\) and \(\frac{dy}{dt} = \lambda y\). (Can you explain why this has to be true?)

Combining the fact that \(\frac{dx}{dt} = ax + by\) with the fact that for straight line solutions \(\frac{dx}{dt} = \lambda x\) and \(\frac{dy}{dt} = \lambda y\) along the straight line, we can set up the following two equations:

\[
\begin{align*}
ax + by &= \lambda x, \\
\lambda y &= cx + dy.
\end{align*}
\]

Rearranging these equations we get \((a - \lambda)x + by = 0\) and \(cx + (d - \lambda)y = 0\). Note that although these equations look similar to the nullcline equations, the coefficients are different.

In order to get straight line solutions, the equations \((a - \lambda)x + by = 0\) and \(cx + (d - \lambda)y = 0\) need to be dependent. (Can you explain why this has to be true?)

Rewriting these dependent equations in slope form yields \(y = \frac{-a - \lambda}{b} x\) and \(y = \frac{-c}{d - \lambda} x\). Rearranging this last equation we get the following:

\[
(a - \lambda)(d - \lambda) - bc = 0 \Rightarrow \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \Rightarrow \\
\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}
\]

We can more efficiently obtain this same result using matrix notation and the fact that two equations are dependent when the determinant of the coefficient matrix is zero as follows:

\[
\begin{align*}
(a - \lambda)x + by &= 0, \\
(cx + (d - \lambda)y &= 0, \\
\begin{pmatrix}
a & b \\
c & d - \lambda
\end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\]
Thus, the equations \((a - \lambda)x + by = 0\) are dependent when the determinant of the coefficient matrix \(\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}\) is zero. That is, when \((a - \lambda)(d - \lambda) - bc = 0\).
EXAMPLE

Determine the general solution for the system of differential equations
\[
\frac{dx}{dt} = 4x + 2y \quad \text{using} \quad \frac{dy}{dt} = x + 3y
\]

the “eigenvalue first” approach.

In order to get eigensolutions, we need to have
\[
\begin{align*}
4x + 2y &= \lambda x \\
x + 3y &= \lambda y
\end{align*}
\Rightarrow
\begin{align*}
(4 - \lambda)x + 2y &= 0 \\
2x + (3 - \lambda)y &= 0
\end{align*}
\Rightarrow
\begin{pmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\Rightarrow (4 - \lambda)(3 - \lambda) - 2 = 0
\Rightarrow \lambda^2 - 7\lambda + 10 = 0
\Rightarrow (\lambda - 5)(\lambda - 2) = 0
\Rightarrow \lambda = 2, \lambda = 5
\]

For \( \lambda = 2 \)

Since these two equations
\[
\begin{align*}
4x + 2y &= 2x \\
x + 3y &= 2y
\end{align*}
\]
are dependent, we can use either one to determine the straight line of vectors (called eigenvectors) in the phase plane. In this case, straight line solutions are found along the line \( y = -x \).

Any solution along this line can therefore be written as
\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

For \( \lambda = 5 \)

Since these two equations
\[
\begin{align*}
4x + 2y &= 5x \\
x + 3y &= 5y
\end{align*}
\]
are dependent, we can use either one to determine the straight line of vectors (called eigenvectors) in the phase plane. In this case, straight line solutions are found along the line \( y = \frac{1}{2} x \).

Any solution along this line can therefore be written as
\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

The general solution is therefore
\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]
1. In the previous example the general solution was determined to be
\[
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} = k_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]
What is the specific solution for the initial condition \((-3, -2)\)? Without using technology, sketch the graph of this solution in the phase plane (for \(t \to \infty\) and as \(t \to -\infty\)) and explain how you figured out what the graph looks like based on the equations for the solution.

2. If the general solution for a system of differential equations of the form
\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]
is \[
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} = k_1 e^{-2t} \begin{pmatrix} -1 \\ 4 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix},
\]
what do solutions in phase plane look like? What do solutions that are not straight lines look like? Do they curve a particular way? Figure out a way to use the general solution (without technology) to decide. Explain and graph your ideas.

3. Repeat problem 2 for the general solution
\[
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} = k_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]

4. Repeat problem 2 for the general solution
\[
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} -3 \\ 2 \end{pmatrix}
\]

5. For each of the following systems of differential equations, find the general solution and then sketch the phase portrait (i.e. graphs of solutions viewed in the phase plane) without using technology.

(a) \[
\begin{align*}
\frac{dx}{dt} &= 2x + y \\
\frac{dy}{dt} &= x + y
\end{align*}
\]
(b) \[
\begin{align*}
\frac{dx}{dt} &= -4x - 2y \\
\frac{dy}{dt} &= -x - 3y
\end{align*}
\]
(c) \[
\begin{align*}
\frac{dx}{dt} &= 4x + 2y \\
\frac{dy}{dt} &= x + 3y
\end{align*}
\]
(d) \[
\begin{align*}
\frac{dx}{dt} &= 4x - 2y \\
\frac{dy}{dt} &= -2x + y
\end{align*}
\]
(e) \[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -4x - y
\end{align*}
\]
(f) \[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= 2x - y
\end{align*}
\]
(g) \[
\begin{align*}
\frac{dx}{dt} &= 2x \\
\frac{dy}{dt} &= 2y
\end{align*}
\]
(h) \[
\begin{align*}
\frac{dx}{dt} &= -3x - \frac{1}{2}y \\
\frac{dy}{dt} &= 6x + y
\end{align*}
\]

g) For the system of differential equations \( \frac{dx}{dt} = rx - 2y \) and \( \frac{dy}{dt} = 3x + ry \), figure out all the possible types of equilibrium solutions for different values of \( r \), where \( r \) is some real number. Show all work to support your conclusions.

h) Denise claims that all solutions (except the equilibrium solution) to the system of differential equations \( \frac{dx}{dt} = ax + by \) and \( \frac{dy}{dt} = cx + dy \) will spiral if \( (ad-bc) \) is negative. Do you agree with Denise’s claim? If yes, justify your response. If not, explain why not.
**Summarizing Your Results**

The purpose of this assignment is for you to reflect on and organize what you've figured out about the phase portrait for systems of linear differential equations based on knowing just the eigenvalues. Without using your notes, try and complete as much of the table below as you can.

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Typical phase portrait</th>
<th>Basic format of the general solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 &lt; 0 &lt; \lambda_2$</td>
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<tr>
<td>$\lambda_1 &lt; \lambda_2 &lt; 0$</td>
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<tr>
<td>$\lambda_1 &gt; \lambda_2 &gt; 0$</td>
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<tr>
<td>$\lambda_1 = 0$</td>
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<tr>
<td>$\lambda_2 &gt; 0$</td>
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<tr>
<td>Eigenvalues</td>
<td>Basic format of the general solution</td>
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<td>$\lambda_1 = 0$</td>
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<tr>
<td>$\lambda = a \pm ib$</td>
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<td>$a &lt; 0$</td>
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<tr>
<td>$\lambda = \pm ib$</td>
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In the Swing of Things

A pendulum is attached to a wall in such a way that it is free to rotate around in a complete circle. Without provocation, Debra takes a baseball bat and hits it, giving it an initial velocity and setting it in motion.

\[ \theta = 0 \]

i) If we call \( \theta \) the angular position of the pendulum (where \( \theta = 0 \) corresponds to when the pendulum is hanging straight down) and we call the velocity of the pendulum \( v \), what would angular position versus velocity graphs look like for a variety of different initial velocities due to Debra’s hit? Provide a brief description of the motion of the pendulum for your graphs.

\[ \nu \]

\[ \theta \]

j) How many equilibrium solutions are there, where are they, and how would you classify them?
Applying Newton's 2nd Law of motion (where \( \theta = 0 \) corresponds to the downward vertical position and counterclockwise corresponds to positive angles \( \theta \)) yields the differential equation

\[
\frac{d^2 \theta}{dt^2} + \frac{b}{m} \frac{d\theta}{dt} + \frac{g}{l} \sin \theta = 0,
\]

where \( b \) is the coefficient of damping, \( m \) is the mass of the pendulum, \( g \) is the gravity constant, and \( l \) is the length of the pendulum. Estimating the parameter values for the pendulum that Debra hits and changing this second order differential equation to a system of

\[
\frac{d\theta}{dt} = v
\]

\[
\frac{dv}{dt} = -0.2v - \sin \theta
\]

k) How many equilibrium solutions does this system of differential equations have, where are they, and how would you classify them?

l) For what range of initial velocities will the pendulum make exactly one complete rotation before eventually coming to rest? [Hint: Sketch part of the phase portrait by finding the slopes of the straight line solutions entering and leaving the equilibrium solution at \((\pi, 0)\) by linearizing the system about the equilibrium solution \((\pi, 0)\) and then, if available, using technology to go backward in time along these solutions.]
Bees and Flowers II

In an earlier problem we studied systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is both species are harmed by interaction) or cooperative (that is both species benefit from interaction).

\[
\begin{align*}
\frac{dx}{dt} &= -5x + 2xy & \frac{dx}{dt} &= 3x(1 - \frac{x}{3}) - \frac{1}{10} xy \\
\frac{dy}{dt} &= -4y + 3xy & \frac{dy}{dt} &= 2y(1 - \frac{y}{10}) - \frac{1}{5} xy
\end{align*}
\]

4. Explain why the second system of rate of change equations describes a situation where the two species are competitive.

5. Verify that the equilibrium solutions for system (B) are \((0,0), (3, 0), (0, 10),\) and \((\frac{2}{9}, \frac{7}{5})\).

6. Determine the linearized system of differential equations about each equilibrium solution and use the information you gain about the solutions near each of these equilibrium solutions to sketch the phase portrait.

Putting it all Together

Without using technology, use the tools of linearization and nullclines to sketch the phase portrait for the nonlinear system

\[
\begin{align*}
\frac{dx}{dt} &= \cos(y) \\
\frac{dy}{dt} &= y - x
\end{align*}
\]

Be as accurate as possible and show all supporting work.
Adam’s progress on systems
17 statements coded for using first order differential equations (F)
16 statements coded for context (C)
7 statements coded for visualization (V)
7 statements coded for differential equation (D)
8 statements coded for graphical interpretation (GI)
6 statements coded for graphical visualization (GI)
8 statements coded for Prior discussion (PD)
19 statements coded for Phase Plane reasoning (PP)
13 statements coded for Algebra Concept (AC) or (A)
3 statements coded for fictive motion metaphor (FMM)
5 statements coded for exponential (E)
11 statements coded for Vector Algebra (VA) or Linear Algebra (LA)
8 statements coded for Physics (Ph)

<table>
<thead>
<tr>
<th>Date and time</th>
<th>Statement</th>
<th>Analysis/ Coding</th>
<th>Comments</th>
<th>Relation to Classroom</th>
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</thead>
</table>
| 11/6 4:45 or so | -Just that there are two species that interact  
-They are constantly reproducing and there is no limit to resources | Brings discussion from first order and applies to system F | Looking at rabbit/fox system for the first time-question about changes to assumptions- no reconceptualizations | Small Group Answering question but not using deep thinking |
<table>
<thead>
<tr>
<th>Time</th>
<th>Event Description</th>
<th>Notes</th>
<th>Group Type</th>
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<tbody>
<tr>
<td>11/6</td>
<td>7:00 or so - probably the way we can look at that is that in this one up here, the more foxes you have in this equation, the smaller it gets. In this one, the more rabbits you have, the larger it gets so unless rabbits are eating foxes, then R is rabbits and F is foxes.</td>
<td>Reasons about the differential equation- can’t tell if we are talking about the population or rate in “the smaller it gets”. Uses understanding from real life about rabbits and foxes and relates the two C + D related</td>
<td>Small Group Reasoning in small groups</td>
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<td>Connections here-rate understanding and context understanding-which one is prevalent?</td>
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<tr>
<td>14:40</td>
<td>-then they are all going to die</td>
<td>C</td>
<td>Whole Class</td>
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<tr>
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<td>No one really expresses here an understanding of what the solution functions might look like or whether or not they might die off using the rate of change equations- he’s just guessing/brainstorming here</td>
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<tr>
<td>17:50</td>
<td>- I don’t like the oscillation - I just don’t “Wei” you just went them to go all the way to zero? - no “josh” there is probably a point where there is an equilibrium - that’s what I am saying</td>
<td>C+I He is just using whatever ideas he has about predator prey populations—doesn’t have a way to express, so may just be using intuition from science background</td>
<td>Whole Class</td>
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<td>In small class earlier, Adam’s group had talked about equilibrium with no firm results—check this?</td>
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<td>22:00</td>
<td>the problem with this, you know the fish something we just did, where h was changing. Well, R is changing in each one of these, so there is not going to be one E. S. for each graph, it is going to be changing, its going to be moving around, its going to be doing different things as F change or as R changes, so we can’t think about it in the we did for that. Is there any way we can think about it in the way h vs. P?</td>
<td>Off base here for this problem, but shows how Adam looks to classroom learning from earlier to identify equilibrium solutions</td>
<td>Whole class—they pretty much ignore this idea, however</td>
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<tr>
<td>32:00</td>
<td>We were thinking the same as h versus P. We have two different things that are equal to zero no, not just one. “Chris” so you were looking for equilibrium solutions - Yea. But I think his idea, the two different lines, the equilibrium solutions for the two different. - draws sketch</td>
<td>Adam still has not expressed ideas about equilibrium that make sense at this point—he is now rejecting idea of h vs. P to think about different lines</td>
<td>Small group</td>
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<td>11/6 TCB 13:50</td>
<td>(about airplane problem) -this guy sees a circle, he’s always going to see a circle. Are we going to say this guy in the blue is keeping up with the airplane -he’s going to first see a zig zag, then its going to get in front and he’ll see a spiral</td>
<td>V+ C- Adam’s visualization skills are strong. He uses the pipe cleaner, but can “see” in his head immediately- He is imagining the movement and “sees” the trace</td>
<td>Adam is not challenged much by this task How is “animating” something important to time based reasoning – where does “animating” come into play</td>
</tr>
<tr>
<td>23:20</td>
<td>-Well, it would start depending where you start, if the plane was way back there, you’d see a spiral (in answer to what about beta?) -the curve—and the guy from back here you’d see the curve</td>
<td>V- see above</td>
<td>This is an interesting question as it is about straight line solutions that will be discussed later—however, no evidence that this ever connects to that idea</td>
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<td>37:50</td>
<td>-that’s the start of the condition (talking about the equilibrium values of 1/8 and 3/1.4 that they had found earlier -no, you have a curve….do they? They were numbers. One was a straight line in one direction, one was a straight line in the other direction. (refers to picture drawn earlier)</td>
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<tr>
<td>D+ F-</td>
<td>referring to having set the DEs equal to zero and finding numbers. He isn’t thinking of the DEs at roc equations, I don’t think. He is interested in these two numbers found—since ES are lines in two space, he thinks about two lines for the two numbers in 3-Space</td>
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<td>Actually, Adam’s ideas here are not wrong at all. They are related to nullclines and the intersection of nullclines as ES.</td>
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<td>Small group - No one ever develops this and it never comes back—or is it used? Check this out</td>
<td></td>
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<tr>
<td>11/11 TCA- in computer lab 4:30</td>
<td>(Adam uses the curser to follow the solution as he had watched it be created) -that looks something like a pipe cleaner -no look at this, it gets smaller in the back -so this thing does have perspective?</td>
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<td>PD-</td>
<td>(pipe cleaner comment) (perspective issues) He is watching the animation that the computer does and reanimating for himself—Physical experience</td>
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<td>What role does the physical experience of reanimating play in helping him to think about time and its integration into the problem</td>
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<td>Small Group—different than in class</td>
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<td>8:50</td>
<td>Adam changes the initial condition -is it a shift? -the amplitudes get larger and the initial conditions get larger -no, each one changed the amplitude in each different direction, like if were just 10… -alright, I changed my rabbits to be small and I got a bunch of foxes I’m looking at the rabbit population right now</td>
<td></td>
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<tr>
<td>V</td>
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<td>GI</td>
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<tr>
<td>13:00</td>
<td>-these are your rabbits, no put it the other way- -now those are your foxes, its just easier to flip it up and down</td>
<td>PD- This is connected to the ideas of mother curve that Wei mentioned. Does Adam see it that way?</td>
<td>Small group</td>
</tr>
<tr>
<td>18:00</td>
<td>The reason we were, … at least I was just plugging in random values and we were getting totally different curves and the reason why were getting totally different curves is we were starting off with initial populations that didn’t lie on that curve… and if we plug in any of those numbers that lie, that were on the curve, then you would have the same curve [gestures a spiral in air- any other number would give you a different curve besides the numbers that are x and y pairs there.</td>
<td>F- RC Here is the first evidence I see of major reconceptualization about the 3-d curve—earlier, he admits to trying arbitrary numbers, here he explains why it didn’t work, that you have to be on the curve to get a shift He doesn’t talk about why this is, but I think he understands it</td>
<td>Whole class- Instructionally, this is a switch that works well- instead of “notice a shift” we create a place to “create a shift’- if I recall, I helped one group with this idea and then it “spread”</td>
</tr>
<tr>
<td>25:20</td>
<td>(answering question about R-F System if F is 0) -like one of our original it would go up forever -but if you look at the differential equations, if this is 0, then this is 0 so you are not going to go anywhere [points to dF/dt equation] - so rate of change of foxes can’t ever leave 0, if foxes is 0 in this one, your rate of change is going to be 3R</td>
<td>F- uses exponential function understanding from first order (using dR/dt = 3R FMM Think about the connection between equilibrium solution in first order and now—the idea of movement (equilibrium point, versus solution function) Adam still discuss this as a static value here</td>
<td>Not difficult here</td>
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<tr>
<td>Time</td>
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<td>Small Group</td>
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<td>28:00</td>
<td>“Joseph” We should have gone to like 2 (notices how large b values get and art our of control) -that’s really small, buddy, because it is growing really really fast - well, look at it, its growing really really fast, at one minute -how come it didn’t grow? [Adam redoes and changes parameters] -aah</td>
<td>GI- what exponential functions do and the movement of the function Adam has an image of exponential functions that comes from earlier—before First Order, I think</td>
<td></td>
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<tr>
<td>29:00</td>
<td>-it should be just a line if I could ever find it, Is that it? (speaking of equilibrium)</td>
<td>F</td>
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</table>
37:30  - put 0 over here then find out? I don’t know what that means “Joseph”- what I’m going to try and do is put into another spot R= .8 and R=1/.8 -so why do we have R= .8 and R=1/.8. No you divide “Chris” what do you expect to see? -really not sure yet. That’s not right (when they see an exponential curve on the R-t plane) we set them equal to 0. On equilibrium solutions, the rate of change is 0.

-There you go (sees a straight line when looking at any view)

“Joseph” it’s a dot 9 (looking in R-F plane)

F  GI
D
Adam is struggling here with the idea of equilibrium solutions- he is thinking of two lines still and their intersection

He knows however, that when he sees an exponential curve, that can’t be

I’m wondering about this dance between when you know the computer is wrong and when you trust your computer, how do you know when?

Small group

43:30  “Wei” What is an equilibrium solution? -its when the number of foxes and the number of rabbits stay the same for all time

F  GV
equilibrium solution understanding gets transferred to thinking in systems- here is a reconceptualization

The computer’s graph connected to Adam’s thoughts from first order allow a reconceptualization of what ES is for a system

Whole Class
<table>
<thead>
<tr>
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<th>Page</th>
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<tbody>
<tr>
<td>8:20</td>
<td>(they have determined there are two ES and wondering what kind they are—as related to attractor, node, repeller) -I think he was just saying to go through. We had it only going through—we needed to see it over her going down. Cause here it looks like a node going one way and then a node going the other way I don’t see much here—no new ideas-just talking</td>
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<tr>
<td>11:00</td>
<td>-so does that mean it’s a node going both ways or switching off and on? -why would we have, that’s not target, it goes up and down. A target when we were doing it before was just like on forever, boom. “Wei” looks like everything is a closed circle. So if we just look at foxes and rabbits space an solutions around—not really a circle, but a closed curve What would you call it? -oscillating. F- GV FMM Adam realizes that the at the (2.14,1.2) equilibrium solution.—it goes around and around—it doesn’t get closer here but goes around and around—this is a reconceptualization as he knows that it is a different type of E.S. than before. The computer plays a major role in the FMM and the way Adam talks about movement—he watches the animation and then animates himself-Check the drivjers dissertation</td>
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<tr>
<td>20:50</td>
<td>-ok, so if we plug in the very center, the value is going to be 0, right? Connecting D and GV-Adam is starting to understanding what a point on the phase plane looks like Here is where we start seeing a lot of phase plane analysis—all are struggling with that</td>
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<tr>
<td>22:00</td>
<td>-so what the heck is that all about, you’ll get a different one each time, like this one is going to be in this direction- GET PAPER</td>
<td>PP FMM, Adam is participating in the class here and talking about the curves as represented in the phase plane—they are “moving for him” however no thoughts about the 3-D piece—just talking about following the curve</td>
</tr>
<tr>
<td>23:45</td>
<td>-So you combine the two -the vector together would be</td>
<td>VA- here is a new place for connections- Adam thinks of vectors as addition because we teach them this way- he keeps thinking about the vector, not its slope</td>
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<tr>
<td>Date</td>
<td>Event</td>
<td>Notes</td>
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<td>11/13</td>
<td>Students work on the idea of vectors in the phase plane all day—null clines are developed—Wei pushes for component idea</td>
<td>C + V&lt;br&gt;Ph Group is really just floundering around here trying to think about position/velocity—Adam uses some of the ideas about springs—like “you can go farther away in this direction, but we still don’t know what he is thinking here.</td>
</tr>
<tr>
<td>11/18 2:30</td>
<td>Wei introduced spring mass problem and asks them to make a sketch in a position vs. velocity plane</td>
<td>Small Group</td>
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<td></td>
<td>-position right here, when it is at rest its x=0</td>
<td>C + V&lt;br&gt;Ph Group is really just floundering around here trying to think about position/velocity—Adam uses some of the ideas about springs—like “you can go farther away in this direction, but we still don’t know what he is thinking here.</td>
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<td>-OK, and as it gets farther away from x, it gets less increasing</td>
<td>Small Group</td>
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<td>-or if you push it all the way in, [“Joseph” you’d still have the farthest from its rest of its 0 velocity]</td>
<td>Small Group</td>
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<td>-it can go farther in this direction than it can this way.</td>
<td>Small Group</td>
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<td>CHECK THIS in the CD!</td>
<td>Small Group</td>
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<tr>
<td>4:20</td>
<td>-and so is it at rest at 0?</td>
<td>Small Group</td>
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<td>-you pull it to say</td>
<td>Ph C g&lt;br&gt;Adam pulls from his understanding of physics to sketch a graph that makes some sense – I’d need to see the work here</td>
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<td>-velocity is going to go up? Then it's going to start going down, then it's going to go up [he’s drawing as he says this] then it’s going to go down. What the heck [get work to see this—he draws something on his ]</td>
<td>Small Group</td>
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<td>6:00</td>
<td>-wait, no that’s going faster, buddy, velocity is going up. Velocity should be highest when you… -so it just goes around and around and around -no air resistance -we are pulling it and letting it go -the top one is positive velocity and the bottom is negative velocity -yeah, if you pull it or push it, but then you are going around it this…. If you are going around it this way [indicates counterclockwise]</td>
<td>C Ph Reconceptualization here of the idea of a perfect spring—gets the idea of movement as it can be represented on the position/velocity plane I think it makes sense the circle idea for spring mass/ or pendulum idea—but here there is nothing more than just the idea that both oscillate in an offsetting position Check on physics literature about kinematics</td>
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<td>23:30</td>
<td>answering the question is the sign of kv and cx positive or negative] -it needs to be opposite the what? Pause. It needs to be working against the [“Jeff” spring] -and kx, shouldn’t that be negative when x is positive? --so it is –kx+cv</td>
<td>Ph C Connecting understanding of negative and positive to physics ideas of friction. Adam starts with this understanding==doesn’t have any new conceptualization-but this is the process of developing a feel for the phase plane Has ideas for spring constant At this point, we see Adam thinking about the spring as pushing in the opposite direction as the side of the stop spot—hasn’t dealt with velocity yet, but I wouldn’t say “reconceptualization”</td>
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| 24:00 | that’s static friction, can’t it?  
- Joseph, yea, cause if it is a negative x, then it becomes positive  
- well its positive, its +cv  
- no once you hit zero, it would be on the other side  
- so its going to be opposite velocity  
- but your friction, is going, you are calling this negative velocity, right and friction is  
- well, if our position going in the negative direction, we are calling that negative velocity, right?  
- yea it makes more sense because velocity is negative if it is going toward a negative number, I don’t know | Ph, C  
Adam struggling in this whole set with the sign of velocity—he is pretty sure about the kx sign, but mixed up and saying the velocity—he expresses again and again the idea of velocity sign, but doesn’t use that to think through the sign of cv. | Small Group |
| 28:00 | we thought that the kx, it should be a negative kx because um, when we start off at a positive position, your velocity is going to be negative because you are going back towards a negative position  
- that’s because it is pushing it there. The spring is pushing it once it passes 0.  
- But you’re still negative there so the spring is helping velocity ??? | Ph, C | Repeats what they talked about in small group—nothing new | Whole class |
| 36:20 | -here, do it… got to remember it is going too [all are drawing- CHECK PAPER]
- ok, explain to me again why cv has to be negative
- this is velocity over here, right, you want it to be anti-velocity?
- why the same? Friction works against the velocity, you are right, friction would slow down he velocity, that’s what friction does right? So when velocity is positive, then friction damn well better be negative. Seeing we are plugging velocity into this friction thing, I think this should be negative out front? Does that make sense? |
<p>| C, Ph- Adam reconceptualizes the cv term at this point. – he listens to others and then proceeds to explain himself—I am not thinking that parametric reasoning is involved here—still purely context/physics |
| Adam never shares this with the group, and only says it once, is that enough? |
| Small Group |</p>
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<td>11/18 5:00</td>
<td>(the students are looking at the second order DE and the system and figuring why they work - so those two differential equations. Did you look at the other equation, did you look at it, You liked it? Alright so this is our original which one are you calling the second one? Dx/dt? The lowest one “Wei” says yes. You just said. That’s what the change in position over the change in time is just velocity. - Oh,!!! Yeah, change in position over change in time!</td>
<td>I don’t see any new ideas coming out here—he is just talking about the three equations—One possibility is that saying something out loud helps you see something—when he said change in positive over change in time, then the dx/dt= v made sense after he said it out loud</td>
<td>Small Group</td>
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<td>10:50</td>
<td>Discussing the sky scraper problems and what forces on there - the wind doesn’t come into play in your equation - you r initial push whatever you want to start somewhere - no no no… we were saying can it be different gusts, like first its 5 mph then second its 10 mph or is it always you start with it then you stop “Wei” yes.OK so wind shouldn’t come into your equation at al</td>
<td>C + D Adam has no problem with this, although the other students in class do</td>
<td>Whole class</td>
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| 18:00  | Wei asked if $x^3$ term is positive or negative  
-yeah, it’s going to want to keep on going. If that’s positive, then it means that the number is going to be positive too [waves hand]  
right and greater  
-you get a negative $x$, $-x^3$ is negative so C and algebra understanding of positive and negative== again, no challenge to Adam |                                                          | Small Group |
| 11/20  | Wei asked them to think about how the phase plane varies with the friction parameter  
-when n is zero, it’s a circle going back and forth because.. it just goes from a to – a back and forth because it is not slowing down at all [Adam gestures back and forth] then as n gets larger, it goes to 0 faster, its kind of like a spiral  
-the larger n gets, it doesn’t even complete a cycle.  
PP + Ph + C  
I am analyzing that Adam is using his physics understanding about friction to say when n is zero, it’s a circle but then he starts using his Phase Plane ideas to connect to physics stuff when he talks about the spiral idea by looking at the phase plane. He is talking about the mass as an animate object by looking at this phase plane and creating the animation on his own |                                                          | Small Group |
| 4:30   | When n gets to four it seems like a line, if it is in this its just going to go straight to 0, if its on this side, its going to go straight to zero no matter what your initial velocity is.  
-What do you think that line is, like why? Definitely weird about that line.  
-let’s stop this where there’s a line? All right in the second quadrant, the line goes straight down to 0.  
PP  
This is what we might call a conceptual conflict—his physics ideas don’t match with the straight line he sees—he actually expresses this perturbation  
Adam asks exactly the question we had hoped |                                                          | Small Group—The materials create exactly the intellectual curiosity they are meant to do- |
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<td>6:00</td>
<td>-I don’t know. The vectors over here look like they might miss it and then loop both around, like it just goes past equilibrium and then just go a little bit and come back. [Jeff that’s when I was seeing the vectors go positive, negative, positive negative, above and it and then Quadrant 3, they are always in same direction. Adam here shows his conceptualization of the phase plane is developed and firm—he sees the vectors as a road map that directs the movement depending on the “start” FFM. Note the difference in how Jeff sees this plane—he looks at the vectors by Quadrant and thinks up down, up down. Can I identify the place where this conceptualization becomes actualized? This is important piece in the parametric reasoning for systems—he sees trajectories as being created in time and making a graph on the phase plane.</td>
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<td>6:30</td>
<td>-so in this one, there is only one equilibrium solution, right? -when x and y are both 0 and its going down from this direction [ looks at all quadrants] it not exactly a saddle, I think its one we’ve never seen before -all we’ve gone over is a saddle. Think about how there is an “it” that exists for students- no mention of spring here, the it is this infinitesimally small object that moves and makes a path. Maybe we can analyze that system and find it out that way? “Joseph” says what n defines that line? How would we do that? -I don’t know, when we could plug in</td>
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<td>Again, Adam is moving toward asking the right question—how can we analyze the system. He believes that there is some way to analyze system to find the straight line!!!!</td>
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| 9:30  | -Right, it doesn’t go straight through so the slope is at –1, you know what I’m saying?  
-OK, when we plug in –1, 1 for x, y we are plugging in that point right and there we need to find out when this going to say that the point is going to have a slope of -1  
VA introductory thinking  
PP  
Adam knows they want to find a place that when we plug in values for x and y the point has a slope of –1—  
This is an important point that doesn’t get used for a long time—what does it mean that the point has a slope? Very interesting that a point can have a slope  
Is this metonymy or similar to Kevin’s idea of point with directionality? I need to think and check into this.  
Small Group  
Again—materials lead us to this—why does it take so long to get there in the whole class?/
|       |       |
| 11:30 | -Josh was saying that the first quadrant would you around but part of the second quadrant would send you around also  
-through the first quadrant into fourth and into 0  
-I’d say its still swirling (about the graph when n is at some value)  
- it is not a straight line, guys?  
PP  
Adam is being more precise—he has really studied the phase plane—his discourse indicates that not only is the phase plane a map of the flow of solutions, but it “sends” the object –more vital in its behavior  
Whole Class—  
not particularly helpful here  
Small Group  
Again—materials lead us to this—why does it take so long to get there in the whole class?/
|       |       |
| 18:10 | -the only place on this thing where both x and y are zero is at the center, it’s the only place where the vectors… no its not.. they are 0 [points to something on the graph]  
-its zero all the way this way for this vector, points to x-axis line. Then there is like zero this way when you get up here points to second quadrant out?  
It’s got to flip over something  
PP-  
Here Adam is just using his eyes to look at a phase plane- he doesn’t seem to want to prove it algebraically, but he is also not sure, just tentative  
I don’t see anything here to use in paper—I don’t know what he is thinking—or listen again  
Small group  
Again—materials lead us to this—why does it take so long to get there in the whole class?/
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<td>20:00</td>
<td>-this whole adding idea for some reason doesn’t make sense to me “Joseph” we need a slope of −1 -yeah, but when are adding you are adding? Is there another way to put them together?</td>
<td>He is right on the edge here—he knows the vector algebra idea of adding isn’t doing what he wants it to do</td>
<td>Small Group</td>
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<td>26:00</td>
<td>“josh” which is downward, right, but if you did it NO? -No, that’s just the slope of the line, Josh, if you look at the components themselves, they point in directions</td>
<td>Moving him here to knowing that using slope is different than the components—we are close</td>
<td>Whole Class</td>
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<td>32:30</td>
<td>0 so ( \frac{dy}{dt} / \frac{dx}{dt} ) is going to be –x. -which gives you the slope of the line you wanted in the direction you wanted! CHECK WORDS HERE! “Wei” ok, its true that point on the straight line does have vectors in the direction? - sounds like it</td>
<td>Assuming words were right here—he states to class that the ration is the slope of the line, which is what you want—does he understand?</td>
<td>Whole Class</td>
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35:00 We asks if there can be another straight line
- [looks at phase plane] No way
“Chris” do it algebraically
No way, hold up! Because you have that (x, -x) up there if you plug in x, if you multiply one of those x or -x by a number say multiply the first one by 2, so you have (2x, -x) then your slope is going to change when you plug it into your system. It’s not going to be a line anymore. It’s going to change… you plug that into your system and its going to change
Someone asks—is it going to be a curve?
Adam: I don’t know!

PP + VA + AC
Adam is looking at the picture and when he talks—and then uses slope ideas – he is working from the assumption that the slope has to be negative one to stay on the line and if the slope changes it veers off the line—which is why someone asks about curve, but Adam hasn’t thought about that

Whole class
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<td>37:40</td>
<td>Plug in some values? We have to explain why we only see one line, so if we plug in, it better change or do something screwy, right? I don't know because that's just, yea, it would change [using (2x, -x) all right, wouldn't it? You get 1,7, you wouldn't have x's they are all going to cancel. I don't know “Joseph” you get the slope of the tangent at that point? Who, man, you are throwing in the tangent? -see that is what I was saying. It would change for different values of x, For different values of x up this it would be stay constant. Yeah, or back up there</td>
<td>AC connecting to PP and going from there—he gets values and does a ratio and see they aren’t the same—but doesn’t exactly know what that means</td>
<td>Small Group</td>
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<td>39:00</td>
<td>When we have this one, it is x over –x, if I was to plug in my thing (2x, -x) (9x,-x) get –15 for slope - you get, never mind- that’s your slope</td>
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<td>Whole class</td>
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<td>11/20</td>
<td>-yea, but what he did right here was when x=-y that gives you the whole line, it doesn’t just give you one point, gives you arbitrary values. So is there any other form of x and y that you can plug in there that is going to give you another line? What if x=-3y?</td>
<td>Using PP and AC and idea of slope to try to think about the “whole line” not just the point—not expressed well, but when he talks about whole line, he is reconceptualizing that we can’t just look at individual points, but the slope of the whole line</td>
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| 7:30  | Still talking about if there are ways to find other Straight lines coming into the origin -guys its telling us right here [looking at the phase plane] look at this I moved it up, to 4 a little while ago, clicked on a vector and looked at the x component and y component. The x component is –1, the y component is –2  
-you plug in 4 there  
-no I plugged in x and –x… o that doesn’t make any sense.  
-That’s what we wanted it to be, but I have no clue why we wanted it to be that | Here there are moving away to an incorrect way of reasoning— they are looking at the slopes of vectors at points essentially on different lines to see what you get, but not putting it together with the slope idea from before                                                                                      | AC, SLS beginning reasoning but still isn’t showing he’s got the whole idea down                                                                                                                   | Small Group                                                                                                                                                                  |
| 12:00 | If we used that we did before when n=4, we plugged in 1,-1, that one on one side and 2 on the other, right? That’s not right. When we plugged 1,-1 and –1,1 then we had this work it out. This is going to be –1 and this is going to be –2, which isn’t cool  
-see we are going on a line and we have a constant slope of –2, when n-4. When n is 3.5 on this line, when you plug in (2,02) is should be 2 over 3. Is there a line here? When n=4, its steep down right, and then as you go to 3, it goes to a slope of 2.5? “Jeff” what if you go below 3, 2.5?  
- you get .5  | I interpret it the way Chris did” so what you are saying is at that point, you have the slope of that vector, but the slope is not matching up with the slope form at that point                                                                 |                                                                                                                                                                                     |
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<td>23:30</td>
<td>- we have stuff we were looking at but we don’t know how to use it. We were looking at the 45 line and changing values of n and seeing what that would do to your graph. What we came up with was when n=4, the slope of the tangent vectors that go along the 45 line are going to be –2. When n is 3.5, the slope of the tangent vectors is going to be 1.5, when n-3, then m is going to be 01, when n is 2.5, the slope is going to be -.5 and when n is 2, the slope is going to be 0. Ant that’s really easy to see in our bottom equation because you’ll have 1/-1 or 4/-1 or something like that times 02, minutes 1. so the equation is not supported.</td>
<td>PP, AC</td>
<td>I think that this parametric reasoning is the other kind—where the unknown is changing and students reason and generalize from that check Drijvers dissertation.</td>
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<td>30:00</td>
<td>Wei asks for solutions to differential equations -just for this particular case? Yeah, they are all along the same line, so they are all going to go to 0.</td>
<td>PD</td>
<td>Whole Class</td>
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<td>30:30</td>
<td>Why don’t we just solve the first one using separation of variables “Jeff” not separable - good reason.</td>
<td>F</td>
<td>Small Group</td>
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- so a= 0 and a=-2.5
- this is what we did in class
- rise over run, we are talking about the phase plane here right is it’s the.. we are trying to figure out exactly what that boxed equation means and how we are applying it because we are obviously applying it and makes sense to know what it means before we apply it
- and we are trying to find a relation between the two, but why is that giving us that line?

PC - no sign of really understands why and wants to

- Anywhere else in the plane, besides on that line if you like pick another line the slopes are going to change {gesture} . the only place on that plane where we’re going to have a constant line is where the slopes don’t ever change. Does that make sense? Otherwise, you’d have a curve which would be changing constantly.

PD  
SLS + AC reasoning- Adam has listened to the other students and reconceptualized what is going on the SLS—he is reasoning that the slopes don’t change ( I assume he means of the vectors) on the line, not that the slope of the vectors is the slope of the line, only that they don’t have to change

- When you think about a, a is the slope of a line it has to go through that equilibrium Sol. Doesn’t it
- if you put that line anywhere it doesn’t work. It has to be put in a certain place.

Don’t know why he says this, or what he means to say, why this is important-

Here he is getting closer and closer to the SLS understanding— using the algebra from the earlier day
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<td>14:00</td>
<td>No, No, No, when you are looking at the phase plane you are looking from infinity down, you are seeing all of t, positive, negative everything - that line doesn’t depend on t but when you shift that line around it might not uh, I don’t know.</td>
<td>Small Class</td>
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<td>PP V This shows his visualization of the phase plane and how you see all time t, so if there were other places that it was zero, it would show up – continues to develop phase plane understanding—now he is thinking about the idea of the solutions being “smooched down” to see them as curves on the plane- maybe?</td>
<td>Small Class</td>
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<td>20:00</td>
<td>-don’t we need to take a new look at the uniqueness theorem, then? -I was talking to Jakub about the uniqueness theorem because in the worksheet 2.2, when we found the different equilibrium solutions, there was one at zero, there was one where x and y are zero, there was one where y is 0 and x is 1 and one where y is 0 and x is –1 and when you look at the one x and y are 0, it just looked like they were going to eventually get there because they were all heading towards it. - so how do we know if it is going to hit that or not? And also, the other way how do we know if it is going to hit those either because it going--</td>
<td>Whole class</td>
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<td>AC, PP, UT, F First mention of uniqueness question in answer to if the curve will ever touch zero? Adam thinks it is going to hold intuitively but his curiosity demands more.</td>
<td>Whole class</td>
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<td>Wei goes on to talk about the “image” of the 3-d curve in 2=space and seems to be accepted by several</td>
<td>Whole class</td>
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<td>21:30</td>
<td>-how do we know if it is ever going to get to one of those solutions. It looks like it never going to get to the ones where x=1 and y=-1 and its going to get to the one where x-0, but I don’t know -remember when we observed in NuCalc, I sent everybody on a wild goose chase when it actually hit it If any of them get arbitrarily large, then they cross</td>
<td>F GI PP</td>
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<td>26:40</td>
<td>That’s what got me confused, when I asked you if they are, but then I finished the uniqueness theorem the way they are -if one of them is arbitrarily large, we don’t have anything -none of them is ever going to touch -there is an equilibrium solution at (0,0)</td>
<td>F</td>
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<td>33:00</td>
<td>Discussing SLS to a system where they solved and a=-5/2 and a=0. The question is are there two SLS, one at 0? -so when x and y are both 0 -Or you can plug it into the equations right now, you can plug it into the equation there</td>
<td>AC PP DE</td>
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and dy/dt is always going to be 0 so you are not going to have any change in the y component, but you can see that if you plug in a negative x, its going to have a negative value so its going to be going positively away from 0 -its going like this on your arm

<p>| 37:00 | “Wei” it looks like we can figure out the equation of the solution, can we? We know y=0. -oh, yeah, its just what we’ve done before where it is 3x is e to the its going to be e to the 3x. No, its going to be e to the 3t, that’s the solution curve and when you take the derivate of that its going to be 3e to the 3t, which is what you have up there. -what I did was, when you have a system of equations, we haven’t figure out how to get a solution curve yet, but when we have this (points to dx/dt = 3x) we know how to get a solution out of that, and that’s all I did over there its that e thing -wow, I just used the e thing. I knew, we had developed that when you have a DE like that. |
| F, E- interesting that the idea of solving using the “e thing” becomes apparent when dx/dt = 3x + 2y when Wei mentions y=0. Adam is very aware that we are struggling to get a solution and at this point, he sees his idea as a one time solution to this particular solution (remember they are look at the one where a=0 and a = /5/2 which leads to solving the dx/dt equation |
| This “e thing idea” is really part of Adam’s conceptualizations but not part of others – however, they use it in class and it becomes TAS |
| Whole class—this needs to be clearer maybe to the whole class—using this DE might be in the teacher notes |</p>
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<td>40:00</td>
<td>In answer to question is an equilibrium solution function a plane? -But if t is right here, it is coming out. If we flip it out like this, you are going to have these things, right (draws two horizontal shifts of exponential curves - what he is saying is they are all going to lie on this plane[gestures to plane out of board] coming out here. It is like when this blackboard and rolling it on its side and they are going to stay flat. They can go up or down in either direction.</td>
<td>E PP Adam is still talking about the SLS y=0, he knows it is not an ex.’s. but class may or may not- Brandon works it out. Here he sees the SLS’s as exponential equations that are all horizontal shifts of each other and together actually create a plane</td>
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<th>Student</th>
<th>Group</th>
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<tr>
<td>7:40</td>
<td>Trying to make the solution meet I. C. e to the negative two t, when t = 0, is equal to derivative, right. So why don’t we just add 4 -never mind, this is all bad, we have to use what the other technique “Joseph” reverse product rule. -yeah, that one</td>
<td>F, A</td>
<td>Small Group</td>
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<td>11:40</td>
<td>[Joseph does his thing with the plugging in and gets e^-2t] -you are right dude!! Because that way it would go along the same line [gestures] wouldn’t it? -because they have to have like same speed of change for it to be.. the reason why it is the line there is that you are on the same path, they are going at the same rate. [responding to Wei’s question] -then I jumped in and said, seeing this and the y. OK. seeing this and this are related., They are the same thing actually. That’s why it forms that line there because they are changing at the same, like, one couldn’t because one would be going this way [gestures] you’d move and you wouldn’t have a line. I don’t know</td>
<td>GI SLS V E</td>
<td>Small Group</td>
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| 15:00 | “Jos” I substituted for y, put $y=-\frac{5}{2}x$ and you get $\frac{dx}{dt} = -2x$ and then you just solve that and you get $2e^{-2t}$ and it seems to be right because it is $e^{-2t}$  
-the reason why I think it is right on the $e^{-2t}$ is because they both have a factor of $e^{-2t}$ that means.. that’s why is forms a straight line here “Josh” that seems to make sense  
-we just have different initial. They just start off at different places | Small group            |
| 17:05 | “Joseph” that just your projections of each curve on that plane if you are looking for your x-curve that would be your x(t) projected on that plane and y(t) projected on that curve, but when you put the together somehow, they could out to that one curve  
-there’s a bunch of different curves | Small group            |
| 17:30 | [Wei asks about the common e]  
yeah, because you are on that line. Because they are both changing at the same speed, or something like that, along that thought process.  
-wait a second! Yeah, and the slope is 5/2 and the slope is 5/2 there…. Oh, I think I get some place and I just lose it.  
“Jos” if you put y(t) over x(t) you have your original $-\frac{5}{2}$  
-which is what you wanted  | SLS  
Moving toward more understanding | Small Class            |
<p>| 19:00 | - we are seeing y- goes to board -If you didn’t have these things right here, if there was [erases coefficient], if this was that and this was that, it would be that [points on graph to a y=x line because this was –2 and this was 5, it’s telling what line they are going to be on because it has these things (points to –2t in each function] the same, that means their rates of changes are going to be the same “Wei” so you are saying this ratio has to be constant -this ratio is always going to be the same because when you stick in a different time here you are going to stick in the same time, here [points to x(t), y(t)] this number will always cancel out -that’s why its always on the plane “Wei” this ratio has to be constant is the e part has to cancel completely | Not sure what he means here? Trying to deal with the I. C., I believe He is using the fact that the class seems to have accepted that there is this plane of SLS’s and that the ratio has to remain –5/2 to stay “on the plane” | Need to look at this in terms of the interaction with the class for Adam | Whole Class |</p>
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<td>30:30</td>
<td>Wei asks them to talk about -yeah, it would never go to zero because we can see from the uniqueness theorem -you can’t really have it in two dimensions “Josh” yes you can -it is not going to spiral around if it can’t go through one of those lines, so it is going to go like this [gestures] on the inside, it is just going to go like this at a slight angle. On this side its going to go up and around “Wei” interrupts the class to ask what he is talking about -and the way you can kind of look at that is those lines are planes coming out toward .. its like a solid thing of solutions. You can’t cross the solutions because of the uniqueness theorem</td>
<td>Whole Class</td>
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<td>34:00</td>
<td>-is one going to be an attractor and the other repeller (he’s talking about the two SLS’s). [Jakub- no nodes] So it is going to go like this (draws) -or maybe like that [draws} and then its going to flatten out like this</td>
<td>Small Group</td>
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<td>GV SLS UT</td>
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<td>Adam has now extended his UT ideas into the 3-D by thinking of it as a plane that solutions can’t cross or touch- He expresses that as he puts this into perspective as he considers a solution from an IC somewhere not on the plane</td>
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<td>PP Just doing visualizing now on the phase plane – not thinking of the solution in three space</td>
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<td>34:15</td>
<td>-but what does it do far out on this other side. Cause you have it coming in like this and it goes closer, closer, but what about far out. No, on the bottom, on the high line- you have it coming in like this</td>
<td>Are we thinking about this in terms of certain IC or time backward</td>
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<td>12/27/02 TCA 3:00</td>
<td>Wei asks about what happens when the IC is (-4,6) continuing conversation from the day before -could we plug in a point really close to y=-2x, say (-2, 3.9) -when we were looking at it before, Jakub and I said it should go against the, to the top one, in class we observed that it went towards the bottom one with part in middle.</td>
<td>GI, GV Not really thinking here, just observance</td>
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<td>Small Group</td>
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<td>8:40</td>
<td>Wei</td>
<td>Wei asks same question—figure out solutions if it’s off the straight lines. “Joseph” To find the solution for the curves in the between—or the general shape. “Chris” not the shape. “Joseph” the x(t) and y(t) -the way he wrote it up there, the new the new way, the are both multiplied by the same -so I don’t think it is possible to write it like he did the second time. Do you agree? “Jeff” as far as a x(t) and y(t) -times the initial conditions unless we multiply it by, there’s an e thing in the top and bottom -and the way you wrote it up there I don’t see how one could multiply it by the same e thing because it is not going to be on a straight line. You’d have to multiply the top one times an e thing and the bottom one times an e thing and they’d change so that’s why it would curve and I have no clue to finding it how, why, whatever.</td>
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<td>10:40</td>
<td>Adam</td>
<td>Adam buys into the solutions have the e thing in both of them and he doesn’t see how this new solution could be written in the same way (he already has a conceptualization of the SLS as having the same, to stay on a straight line, so if this isn’t on a straight line, it won’t have the same e)</td>
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<td>Small Group</td>
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<td>12:30</td>
<td>“Jos” what if we found this slope here, woe found this curve -the derivative of the curve “jos” and this curve would give maybe some kind of relation between these -well, yeah, that’s what they want us to do, to find that curve there. “jos” in the phase plane, no they want us to find x vs. t and y vs. t -that gives us that curve, though, cause the combo of those, uh. Dy/dt over dx/dt would give us that curve you just drew right? “Jos” time is irrelevant -it is completely “Jeff” as far as the shape.</td>
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<td>15:00</td>
<td>- so what you are saying is, or that curve we are looking at some sort of warped plane where all the solutions are going to fit that warped place. They are going to stay on that.</td>
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<td>SLS—trying to extend it to the solutions not on a phase plane—imagining it—not sure what he is really thinking about</td>
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<td>Is this place going to consist of all the solutions as shifts making up a curve, or a more general surface? I think he doesn’t, but not sure.</td>
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<td>21:00</td>
<td>Wei asks if the solution could be the sum of the solutions for (-2,2) and (-2,4) - for some reason, that doesn’t really make sense to me because you are setting up something like a ratio. And like, you should go to the middle point all the way down the line and we know that solution doesn’t do that it goes more towards one line than the other.</td>
<td>VA, GV. SLS</td>
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<td>He seems to be confusing a sum of vectors as a ration—he thinks that the sum of the two vectors would always be half way between the two lines—not thinking of the solutions—he’s thinking of vectors.</td>
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<td>25:00</td>
<td>if we could do that, then we can get any point on the graph, I mean on the phase plane. By adding different points together, like you want. this plus this, you are adding the x-values and the y-values, well, you could get any other point by picking a point down here and a point over her, you add the x-values and y-values and get a point somewhere else, so if this works, we’ve got solutions for the whole thing -yea, and I’m not sure how we go about seeing if it works. It’s something simple.</td>
<td>LA- he has intuition that you can add any two vectors to get to any point on the plane—and he sees that if this is true, it is very powerful!!!</td>
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<td>28:00</td>
<td>I don’t know, but the reason why we said it is going to go down and what I originally said is why doesn’t it go straight down, why isn’t it a straight line, but what Josh said really makes sense, this lien is $e^{-2t}$, so that means its going faster [ points to $y=-2x$ line] than this is [ points to $y=-x$ line] its $e^{-t}$ going to bend it down. “Jos” its pushing it.</td>
<td>E, SLS, PD</td>
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| 32:20  | In answer to question about how to justify sum is correct  
-what we are doing right now, we broke it up into x(t) and y(t) and wrote it out and we are just going to plot that on a regular graph  
-what we are going to do is plot that just on a regular graph on our calculator and we want you to rotate the picture you just threw up on the ceiling  
- is there any chance we can look at DEE and rotate out axis so we are looking at x vs. t or y vs. t? | GI: Adam and his group revert to graphically justifying. He is not going with the algebraic thinking—however, this is the beginning of the group justifying it algebraically, as they split up the solutions to x(t) and y(t) – this thought will be better |
| 40:00  |   | Small Group |
| 11/27/TCB O:00 | “Josh” suggests that we can take the two functions and derivatives and plug them into the DEs and get two valid equations  
-Good job, Josh  
-the first one you can, right off the book, see that it is true, you just have to bring down the –2, you bring it down in first equation…..that’s right  
-so we should do it to dy/dt, give me a piece of paper.  
-no chain rule, just substitution- Boolyah! It works. That was so easy we’re stupid.  
-we were looking at that and saying it looks like it makes sense it’s working out graphically. Well how can we prove it? All we have to do is plug these solutions into our differential equations and that will prove it… and it did. | AC, F-  
Josh thinks of this- but Adam understands it immediately and his understanding of “solution” comes back from first order | Why did it take the class and Adam so long to think of this- |
<p>|        |   | Small Group- Seems like we haven’t emphasized enough the idea that you something is a solution algebraically if you can plug it in and get a true statement |</p>
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| 10:00 | Wei asks if this would work for all points, especially (1,1)?
   - I think you can use any point that add up to that point
   - hang on, both of them are positive, so to get your x, you need to have something on the bottom, but also going to balance what you need for the y-thing. There’s got to be an easy way, right?
   - I know this is really really lame, to get that point, there’s an easy way, but we can go the hard way, but list a couple paints on each on of the lines negative and positive and we’ll look at it and find which one gives us that.
   - Try (3,-3), (2,-4) subtract the side minus—we found pints that add up to it.
   - So our new equation would be and says the solutions |
|       | VA- PD- trial and error |
| Adam believes so much that there are “easy ways” to do things – and so he knows we should have a way to find this solution 
He buys into the solution being the sum after seeing one thing work and being told |
| Small Group |
| 30:30 | Wei asks them to find solution if the IC is (3, 2.5)
   “Josh” 3 and 5 and the first is 2, right, that’s b for y, b for x is 8.5
   - where did you get that, ok, gotcha |
| Adam is still trying to understand where Josh got the original values to solutions different from trial and error and proved it |
| Small Group |
| 12/2  | Josh presents a way to find the linear combination in general x2=-b1-b2, x1 = 2b1+b2
   “Wei” how is this related to the DE or SLS
   - these are related to the solution functions on this straight lines |
<p>| Whole Class |</p>
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<td>20:50</td>
<td>-OK, what do you guys think, what just happened here? Joseph, what did they do. “Jos” just solved the equation -but what equation did they solve? -but why do those 8.5 and –5.5 work for the x(t) and y(t) -and those e^-7, and e^-2t, those, what happens to those at this point right here? -so they just tag along no matter what - so what is our goal here. Our goal in words, OK, (.1,1) is a vector point in this direction [gestures} (-1,2) is a vector point in this direction right here [gestures} and we are using different combinations of those to get somewhere else and we just tack on the e thing and that will give us the solution at that point.</td>
<td>LA, VA, AC? Adam is accepting that this is how you do it, but doesn’t really understand why or how it works—I think he understands what to do, and the part about the linear combination of vectors, but no idea yet how that connects to the general solution of a linear system.</td>
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<td>25:30</td>
<td>-all we need is in the direction and the reason we chose (-1,1) and (-1,2) because those were the smallest unit vectors, not unit vectors, smaller with units - now I see where linear algebra and this subject really mesh, I didn’t until now</td>
<td>VA, LA</td>
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<td>32:40</td>
<td>Wei asks them to justify or refute that the sum of two solutions is always a solution -if one of the solutions is for the equilibrium point ((0,0)). Then you have a problem. If you add one to the other the you are going to get “Jeff” the other one still -Did you copy down what he had up there? OK I was thinking is maybe we can plug in one of the points, like the point we got and add it to one of our originals and see if we can get back the other. Subtract or whatever and see if it works for at least one. That’s not proving it is going to work for all,s but seeing if it will work for one, that make sense? -all right, we added one along the line and another along the line to get one that wasn’t on the line. OK, add the one that wasn’t on the line and subtract one that’s on the line to get back to one that is on the line.</td>
<td>No clue- Adam is attempting to come up with something, but not necessarily know why—it is interested that he comes up with this—it has to do with his LA understanding, but trying to find a way to extend since he has no idea of what to do</td>
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<td>12/2 TCB 0:00</td>
<td>We’ve done this before, this is easy. Could you give us a prod in the right direction? -what does that mean: it is a solution? -well, what we did for that system, we found the straight-line solutions and then we used that to find the solution function along that straight line. Wei asks what Josh said On both of the. “Josh” if it fits, it fits! - it is what Josh says</td>
<td>F- Adam is really trying to know how to do this?</td>
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<td>Time</td>
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<td>6:00</td>
<td>-the truth is, you can use an idea and find out it is true for that idea but that doesn’t prove it, it might be a lucky guess. If it works for one, that doesn’t mean it is going to work for all of this. You can find out if its true and it not be true for others. What we thought of doing is taking the solution that we found for the other point, point (3,2.5) and subtracting from it one of the solutions from the line and you would get back the solution on the other line. And you could do that for any other point. Josh came up with using ( x_1 = b_1 - b_2 ), you could pick any solution using that method and then reverse the whole thing and get back another one on the line. I guess we could do that same thing find two solutions and do that same thing. Why don’t we do that?</td>
<td>Just reexpressing his earlier ideas, but isn’t helpful, although he seems to know that</td>
</tr>
<tr>
<td>11:00</td>
<td>Wei asks how we proved Kathy’s idea is true (to add two solutions) - if I’m not mistaken, we found the solutions, added them up and plugged them into the differential equations to see if they fit.</td>
<td>Adam says this—but still does click until Wei keeps pushing</td>
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<td>13:00</td>
<td>So we are going to add these together and we want to get ( x_3(t) ), ( y_3(t) ), and if all of these … these is ridiculous, somebody’s going to have to baby me… this is ( x ) and this is ( y ), for this one its going to be ( dx_1(t)/dt ), you plug in this one -hey, that’s what I was doing you just added the two differential equations so it is -I don’t understand I looks -we are doing the same thing over and over again, it’s not -I don’t know</td>
<td>Still struggling</td>
</tr>
<tr>
<td>23:10</td>
<td>I was just saying how ( x_1(t) + y_2(t) ), instead of having the derivative of ( x_1 ) plus derivative of ( x_2 ) - have the derivative of ( x_3 ) -we are saying that it is because when you plug that in, the two sides are equal and if ( x_1 + x_2 = x_2 ), then when you plug it in, you end up with… “Josh” finishes and says which is exactly what we are looking for -which is what we have been saying the whole time.</td>
<td>Participating in this conversation, he is really getting lost in the algebra, and doesn’t have any intuition to rely on, but just keeps hanging in there</td>
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</table>
25:00 | Wei asks if the idea works if you add +1 to one of the equations?  
-no, cause you get a 2 out there, when you add them together  
-it wouldn’t fit the differential equations because you would get a 2 out front once you added. This is the system of differential equations (points to the general version) and this is for that. You’d have a 1 here and a 1 here which is the same form. 

AC | Adam is thinking about the algebra here and seems to understand 
Whole class-good question by Wei. 

29:00 | Well, if that is x₁ and x₂, and that’s = x₃, When you square both sides of the equation….  
- which is not = x₁ squared plus x₂ squared 

AC | This is all Wei does to prove that this only works for linear systems—vague understanding 
Small Group 

12/4 TCA 4:40 | Wei asks groups to sketch a solution if n is less than root 8.  
-wouldn’t it just be curving in?  
- it would be a spiral sure, that is going down 

C, PP, ? | Poorly worded question 
Small Group 

15:15 | Working on solving with complex slopes, particularly -1+i, -1-i.  
“Wei”, would you expect them to have different components/  
-I would hope they are different because if they are the same, they lie on the same line.  
-use Euler’s formula y=e^(-1-I)x  
- you knew what x is so you can say y=(-1+I)x or (-1+I)e^-t 

Thinking about the algebra==no intuition involved | Need to think about this more 
Small Group
<p>| 29:15 | We asks how to find solutions with these complex, can we find real solutions - you have two solutions, we added the ones before to get another solution - when we were doing it before, we had lines, if we found the solution on any lines, we can’t do it here, because we don’t have lines. What if we found another solution along that curve? | PD | Why did Adam think to utilize here, when he didn’t before | Whole Class |</p>
<table>
<thead>
<tr>
<th>Date and Time</th>
<th>Statement</th>
<th>Analysis/Coding</th>
<th>comments</th>
<th>Relation to Classroom</th>
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<tbody>
<tr>
<td>11/6 TCB 19:40</td>
<td>[discussing airplane problem and the pipe cleaner Wei is using on the overhead] that’s either directly in front or back. [Wei: that’s gamma?] that’s the guy standing back and side [spiral image] that’s beta and the guy in the air. This is what the guy off to side, alpha</td>
<td>visualization</td>
<td>Brandon has no problem visualizing here</td>
<td>Class is fine with this problem—few issues- although Josh had a difficult time in small group.</td>
</tr>
<tr>
<td>11/11 TCA 4:30</td>
<td>[Teacher: What did you observe?] The Euler method is getting larger and also smaller</td>
<td>No follow through—what do I know here?</td>
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<tr>
<td>22:30</td>
<td>[discussing how to get a shift of the solution on DEExp] OK, when we first started out, we tried just changing the time as the initial conditions changes and that shifted it. That’s essentially what you are doing here by just picking a different x and y. You are just shifting the time, only you are saying its not, but in a sense that’s the same thing</td>
<td>Shifting time/shifting curve—bob can think of shifting time as a reasonable idea</td>
<td>So Brandon thinks that just changing the time is same as picking another x and y on the curve and starting there—what does this tell me about his understanding of shifts of solutions? Maybe look back at his work in first order</td>
<td>In the computer lab, I don’t think anyone is listening to him—this is pretty much indication</td>
</tr>
<tr>
<td>23:00</td>
<td>OK, so here’s a question: I’m looking at this so I can see x and t. Now, I can see where these two curves cross. Is there a relation between where they cross and different initial conditions? Where they cross, if you are looking</td>
<td>knowledgeable</td>
<td>Is Brandon not thinking about this in terms of movement—interesting that he is thinking of the curves as static—point of analysis here, no FMM, or anything like that- just static</td>
<td>Doesn’t seem to connect to what is going on in the classroom.</td>
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<td>11/11</td>
<td>TCB 3:40</td>
<td>Cause zero rate of change is just going to proceed as a straight line</td>
<td>EQ understanding F. Can’t tell much except he is using what he knows from first order.</td>
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<tr>
<td>11/13</td>
<td>TCA 1:50</td>
<td>That’s the slope of the actual curve. The resultant vector. [Wei: now that’s an interesting point]. I want to ask a question. Right there the fox vector is pointing vertically. At other points… How do you know that just by looking at it that that would be the fox vector and why is only the fox vector the numerator and rabbit the denominator. [Wei: What’s the question] How do you know which vector is which, just generally, and why is one the numerator and one the denominator?</td>
<td>This questions seems pretty obvious to me. Brandon is not understanding the vectors as representing the magnitude and direction of the rate of change as related to the axis—that fox is vertical because the vertical axis is fox! Why doesn’t someone tell him.</td>
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<td>15:10</td>
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<td>At the instant it goes back, it has to be zero at some point</td>
<td>Again, not getting the vector idea= as components</td>
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<td>32:30</td>
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<td>[discussing the rabbit fox non-homogenous The key thing is that you have to look at the relationship between the two equations, you can’t look at one at one point and say this is hurting or helping. It’s the relation between the two equations and that’s what we are talking about. By isolating the one, you can</td>
<td>Brandon is thinking about the equations in an analytic way—he is seeing them two interacting equations—this is a good thing as he understands they interact together, but I don’t see any evidence here that he has a way to think in depth or with understanding</td>
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<td>Time</td>
<td>Description</td>
<td>R?</td>
<td>C?</td>
<td>Notes</td>
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<td>33:30</td>
<td>That’s why they help each other, cause when one starts going down, the other is able to help pick that one up, so that’s why you can say this is a cooperating relationship, this is the one that helps the other.</td>
<td>R?</td>
<td>C?</td>
<td>Does he use his rate understanding or is he conflating rate and function—he seems to use the rate of change as the function as he discusses its behavior.</td>
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<td>11/13</td>
<td>Well, at zero is another equilibrium point—at the origin is another equilibrium point</td>
<td>In small group, this is what they were discussing and so they had figured it out before with calculations-Brandon is just relating what he knows from the small group discussion</td>
<td>Brandon is reverting back to his equilibrium point ideas, but we don’t know how he knows what he just said.</td>
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<td>24:00</td>
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<td>In small group, this is what they were discussing and so they had figured it out before with calculations-Brandon is just relating what he knows from the small group discussion</td>
<td>Brandon is reverting back to his equilibrium point ideas, but we don’t know how he knows what he just said.</td>
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<td>36:00</td>
<td>Its just a coincidence there because in that other quadrant, in the first quadrant. Go to the third quadrant, you have to take into account all four lines so you just can’t arbitrarily say those [Wei: arbitrarily pick a point] Now you have to consider all 4 lines.</td>
<td>Brandon answers incorrectly here, he doesn’t understand the connection between the null clines and the points in between-again, I see here a</td>
<td>Wei has asked the class if it is always true that the vectors at any point are the resultant of the two vectors on the jakub lines in a line with them? It is not always true, but there is no evidence of it at this point</td>
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<td>No one corrects or talks with him about this response—the class doesn’t seem to respond at all</td>
<td>No one corrects or talks with him about this response—the class doesn’t seem to respond at all</td>
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lack of understanding about the vectors in the phase plane

[discussing the spring mass problem—what would a graph of velocity vs position look like] I changed my graph just as you were doing that—drawing yours and it looks like that too (like a unit circle) I was thinking you would have this one because if the original resting position right there is the origin, that means that when you pull the string, it's going to have to go back before the origin. If there is no spring to it at the origin, it would come back there and stop. It has to go beyond the origin to compress to build up some more energy to bounce back this way. And if you call the origin its initial position or this initial position [gestures two points]

Physics understanding—experiential understanding

There doesn’t seem to be any use of DE stuff, only physics understanding—the null cline stuff from before is not being used

[Draws a sketch on the board of a 3-D representation of the spring mass—draws a spiral] OK, this is P, it's spiraling into position 0 [Jakub says should start at origin] [erases first piece and redraws] there, cause you are starting at velocity 0

[TCB] OK, say for second order differential

Brandon is struggling with
<table>
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<td>11/18</td>
<td>2:10</td>
<td>equations. So second order equation is right there [one on top] then the two equations, how is that one different from the one above? how to convert a second order into a linear system—it isn’t well explained and no one seems to come up with the ideas—no connections to earlier stuff</td>
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<td>14:30</td>
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<td>skyscraper problem] if the wind was constant, you see a flagpole or flag go back and forth because the flag pushed it to a point where it just goes beyond the force of the wind. The tension of the building there or flagpole or whatever the overcomes that force to the wind and it starts springing back and as it comes back, it pushes again and that’s where you get the oscillation. So it doesn’t have to be a gusting wind, a constant win, you see that pattern too. Brandon is much more confident about his ideas in physics. When are they going to connect to DE?</td>
</tr>
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<td>11/20</td>
<td>14:00</td>
<td>Basically it stops it real fast</td>
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<td>11/25</td>
<td>16:33</td>
<td>It would be easier to act out rather than draw it start back there and you come out like this [gestures an exponential function moving away from Wei’s arm which is the t axis] If it starts on one side of that line, it would swing out towards infinity. It would come out and swing out towards infinity this way and if it would start on the other side, it</td>
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<td>IMPORTANT: is the parameter time as Brandon uses it perceived as linear? He says it moves along and moves along the t axis—where does thinking of time come in—here obviously—that time moves linearly— but he shows it continuously</td>
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<td>Note on 1/11/06—I don’t know what I was talking about here with the “linear” idea---what does it mean constantly moving, not changing? Now I</td>
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</table>
would swing out toward infinity the other way but coming off the board.

18:20  [Brandon goes to board] ok, it seems like if you start up here somewhere on this side of this line, it would swing towards that but coming out and then away this way, all the way moving along t, so coming out this way and then for that way, lets say it starts down here on this side of the line, its coming up and also moving along t, and then its swing out this way [Wei: Shouldn’t the image][Josh: its going toward 0][Jakub: Just show where it starts at that line] Right here. Then it’s moving along t moving towards the line. I think you also have to see some vectors along here because if it is off of the line, its going to move

21:30  Using the DE applet, instead of just going to 10 units, I went out to 100 units and its pretty small like, you now, to the negative 40, 50 something like that, but it still didn’t reach it, it was still circling around. So I don’t think it ever reaches, it ever touches. I’m just observing. I didn’t mathematically prove it. I’m just Brandon is using empirical evidence to justify his answer that it never reaches the equilibrium, but he declares that is only an observation

This is unusual for Brandon as he has not used graphs to justify very often—not sure what else it says

look back for examples of the parameter as linear in mind—and constantly moving, not changing!

see it as This static/dynamic transposition of time—check for other places in the analysis
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<th>Reasoning</th>
<th>Description</th>
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<tr>
<td>30:20</td>
<td>In response to the question will the exponential looking straight line solution ever touch? No, because you can put a second one, the derivative of $2y$, … none of them you can see. UT, algebraic reasoning</td>
<td>Here he returns to the UT as he knows it, saying that you can see all the derivatives do not get arbitrarily large.—He is much more secure with algebraic ideas of why things work.</td>
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<td>40:00</td>
<td>Let me ask a question. An equilibrium solution where you have one equation is a straight line where the slope is 0. Now what we are saying here is, an equilibrium solution is a plane. Is that what we are saying for a system? [Josh/Adam: No No No] The reason I’m asking so now we saying that we start with an initial condition of (1,0) that would like in this plane, but if we graph that, its not going to be a straight line, its going to come out. [Adam: can I go up there for a second?] because the only place I would see a straight line is (0,0). First Order reasoning, and Brandon is trying to extend it—Brandon is right here—that is the only EQS—can’t someone help him?</td>
<td>He hasn’t figured out that SLSs and EQS are different ideas—I can see why this would be a problem for students as EQ define the solution plane in single DEs and SLS define the phase plane in systems.</td>
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<td>41:00</td>
<td>I understand that, but they stay on the x-plane. That is not the same as saying its an equilibrium solution. The only equilibrium solution is (0,0). OK, fine, I just wanted to get that clear. There is one equilibrium solution and the rest is on these two planes. What do we call this, the y—5/2 x plane? Vs t plane? Brandon understands the difference in the two ideas now and sets it out for himself and all to see—it is interesting that this class developed the notion of invariant manifold without trying. They learn to think of this infinite set of straight line</td>
<td>People do address this question and try to help think about it, but I don’t know if he gets it</td>
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<tr>
<td>TCB 11/25</td>
<td>19:00</td>
<td>Sounds good to me.</td>
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<td>That’s why it stays on that plane. That plane coming out, it belongs to. [Adam: because this number will always cancel out, that’s why its always on this line that plane coming out, it always belongs to [Brandon gestures a vertical plane with flat hand]</td>
</tr>
<tr>
<td>11/27 TCA</td>
<td>17:00</td>
<td>We’ve just come up with –10/6 [ and Josh explains for the group that this is the initial slope they got by plugging in to dx/dt and dy/dt and creating a ratio]</td>
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<td>19:20</td>
<td>[In response to the question is the sum of the solutions equal the solution at a third point] If it was directly in the middle, it would seem like it would be the sum of those two, but what if you are more towards one side or the other. It seems like it would make sense because Brandon is thinking vector algebra here- but not what would the solution sum do—no students are doing that either.</td>
</tr>
<tr>
<td>11/27 TCB</td>
<td>21:00</td>
<td>Make a new coordinate system</td>
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<tr>
<td>12/2 TCA</td>
<td>3:00</td>
<td>[Brandon presents his work on finding the linear combination of vectors with [- LA continues</td>
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<td>solutions as forming a plane-parametrically, (but not time based) this is like the idea of +C</td>
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<td>Uses algebra- but no idea if he understands</td>
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<td>Students at this point do not have a way of thinking about adding solutions—this is never really proven either- but here Brandon’s idea is that since we started in the middle between the two values, then it stays in the middle and is just the sum, but it would be different closer to one end or the other</td>
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</tbody>
</table>
I think my idea works here, that x one squared = z. The way I said it was an easy way to see it, if you call s sub 1 squared as z, so if you substitute them call x sub one squared = z, then plus z sub 2 gives you z sub 3 on the other side you still have s sub one plus z sub 2 no a sub 3, so its an inequality.

Algebraic reasoning
VITA
VITA

Karen Sue Allen

work

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Education

Ph. D. Purdue University (expected) 2006 Mathematics
Education
M. A. Butler University 1978 Mathematics
B. S. Butler University 1975 Mathematics

Dissertation Thesis (completion date Spring, 2006)

Major Advisors, Dr. Terry Wood, Purdue University; Dr. Chris Rasmussen, San Diego
State University

Parametric Reasoning: The Case of Systems of Linear Differential Equations

In my dissertation, I characterize parametric reasoning by analyzing students’
mathematical thinking before and during a five-week instructional sequence on systems
of differential equations. In this study, parametric reasoning is defined as developing and
using conceptualizations about the dynamic quantity time that implicitly or explicitly
coordinate with other quantities (possibly functions) to understand and solve problems.
I will characterize parametric reasoning by analyzing students’ mathematical activity
from both an individual and collective perspective. This research will contribute to an
understanding of students’ mathematical ways of thinking in classrooms at the
undergraduate level, as well as contribute to K-16 mathematics education research in the
area of the mathematics of variation and change. Finally, the research will add to research
about the coordination of psychological and sociological perspectives on classroom learning.

Professional Experience

Research Associate Position

2004-2005 Project Manager, Knowledge for Algebra in Teaching, Division of Science and Mathematics, Michigan State University

University Teaching

2005 Assistant Professor, Mathematics Education, Department of Mathematics and Computer Science, Valparaiso University

1998-2004 Mathematics Instructor, Department of Mathematics and Computer Science, Valparaiso University

1999-2000 Student Teacher Supervisor, Ball State University

1998-1999 Adjunct Mathematics Instructor, Purdue University North Central

1985-1990 Mathematics Instructor, Saint Mary's College, Notre Dame, Indiana

1984 Adjunct Mathematics Instructor, Indiana University South Bend

Graduate Assistantships

2002-2003 Graduate Assistant at Purdue University Calumet- working with Dr. Chris Rasmussen in a research program to study student learning in inquiry oriented differential equations classes and other connected areas.

2000-2001 Graduate Assistant at Purdue University West Lafayette-working with Dr. Richard Lesh in the Case Studies for Kids program to develop and implement model eliciting activities as a instructional and research tool. I conducted a summer workshop for teachers, as well as spending time in classrooms of middle school teachers.
**High School Teaching**

1996-1998  Mathematics Teacher, Mississinewa High School, Gas City, Indiana

1994-1996  Mathematics Teacher, Indiana Academy for Science, Mathematics, and Humanities, Ball State University, Muncie, *(a state funded residential high school for grades 11-12)* Department Chairperson 1995-1996

1990-1993  Mathematics Teacher, North Carolina School of Science and Mathematics, Durham, NC *(This is a nationally prominent state funded residential high school for grades 11-12)*

1980-1984  Mathematics Teacher, Mishawaka Marian High School, Mishawaka, Indiana

1976-1980  Mathematics Teacher, North Central High School, Indianapolis, Indiana

**Professional and Academic Association Memberships**

- American Educational Research Association (AERA) and the AERA Special Interest Group for Research in Mathematics Education
- Mathematical Association of America (MAA) and the Special Interest Group of the MAA on Research in Undergraduate Mathematics Education
- National Council of Teachers of Mathematics (NCTM)
- North American Chapter of the International group for the Psychology of Mathematics Education (PMENA)
- Indiana Council of Teachers of Mathematics (ICTM)

**Grants and Honors**


- Purdue University Calumet, Outstanding Graduate Student, Mathematics and Computer Science, 2002.

- Valparaiso University Alumni Association Faculty Development Grant, 2003-2004. ($2,200)


Woodrow Wilson Summer Program, Geometry for Middle School Teachers, Eisenhower Program through North Carolina Department of Education. 1993 (~$30,000)

**Scholarship**
Name changed to Karen Allen in Fall 2003, previously Karen Whitehead.

**Refereed Journal Publications**


**Other Publications**


**Refereed Conference Proceedings**


Invited Talks


Presentations


Whitehead, K. (February, 2003) “All I want to say about a function, I can’t about these three points...”: Student conceptions of function when beginning differential equations. Poster Presented at the Mathematics Education and Mathematics in the 21st Century: The Roles of Outreach, Teacher Preparation, and Research on Teaching and Learning in a Research I Mathematics Department Conference, Tucson, AZ.


Books


Professional Activity

Collegiate Representative to State Board of Directors, Indiana Council of Teachers of Mathematics


Reviewer, American Education Research Association, 2004

Teacher Development Leadership Roles

2000- 2002  Developer and Lead Instructor, Building Science and Mathematics Connections Through Technology at Valparaiso University: Professional Development Program for Middle School and High School Mathematics and Science Teachers. This program consisted of two summer two week workshops and many follow-up sessions, both at Valparaiso University and in the schools to integrate technology, mathematics, and science.

1999-2001  Project Director, Mathematics in Business and Industry at Valparaiso University: Professional Development Program for Middle School and High School Mathematics Teachers. This program was designed for math teachers to connect with persons in business and industry and learn about the mathematics in those areas. The teachers then wrote problems and implemented them in the class directly connected to their experiences. There were two summer workshops, and many follow-up sessions, both at Valparaiso University and in the schools.

1991-93  Executive Director, Durham Mathematics Council. This was a half time position to administer a long running program to improve mathematics education in Durham, North Carolina. I developed short term and long term professional development programs for teacher, wrote grant proposals, and administered grants for this purpose.
Teacher Workshops and Presentations

Presenter, Grade K-5 Teachers, Workshop on Problem Solving, Northview Elementary School, November 22, 2003.
Facilitator, Grade 3-5 teachers: Twenty-Second Annual Conference for the Improvement of Mathematics Teaching, Purdue University Calumet: Using Problem Solving in the Teaching of Mathematics.
Speaker, National Council of Teachers of Mathematics
  National Convention, Indianapolis, 1994
  Regional Convention, South Bend, 1996
  ICTM State Convention, Indianapolis 1998, 2001
Workshop Organizer and Teacher, Advanced Placement Calculus for Duke University, 1994
Workshop Organizer and Teacher, Mathematica in High School Mathematics, Wawasee High School, 1995


Classes taught

Calculus I
Intuitive (Business) Calculus
Differential Equations
Linear Algebra
Mathematics for Elementary Teachers
Methods
Undergraduate Research Colloquium
Precalculus
Finite Mathematics

Calculus II
Multivariable Calculus
Elementary Statistics
Quantitative Problem Solving
Secondary Mathematics
Developmental Mathematics