Note on ‘Influences of Resource Limitations and Transmission Costs on Epidemic Simulations and Critical Thresholds in Scale-Free Networks’

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Abstract
In a recent paper entitled ‘Influences of Resource Limitations and Transmission Costs on Epidemic Simulations and Critical Thresholds in Scale-Free Networks’ by Huang et al., the authors attempted to establish a key characteristic of epidemic dynamics in a scale-free network when properly accounting for the cost of transmitting the infection at each node and the resources available for transmission to that node. The main input parameter is the effective rate of spreading the infection, i.e. the instantaneous rate at which the infection is spread to an uninfected node via a single link to an infected node. The primary result is the existence of a positive critical threshold for the infection-spreading rate at or below which the epidemic dies out and above which the epidemic is spread through the network and ultimately reaches a steady-state non-vanishing condition. Some flaws in the authors’ proof of this result are discussed, and an alternative derivation is provided that sheds additional light on the transient and steady-state behavior of the system. The alternative derivation may be adapted to the analysis of other scale-free networks with different features.

Keywords
scale-free networks, epidemic simulation

1. Introduction
This note is a follow-up to the recent work of Huang et al. on the spread of infections through scale-free networks. The authors attempt to establish a key characteristic of epidemic dynamics in a scale-free network when taking account of the following: (i) the cost $c$ of transmitting the infection across a link (arc, edge, connection) from an infected node to an uninfected node during a single time step; and (ii) the amount $R$ of resources available at the uninfected node for transmission of the infection during each time step. As in Huang et al., the assumption that $R/c \geq 1$ holds throughout this note so that at each time step, every uninfected node possesses adequate resources to become infected via even a single link to an infected node. In addition to the independent variable representing the current time $t$, the main input parameter $\lambda$ is the effective rate of spreading the infection, i.e. the instantaneous rate at which the infection is spread to an uninfected node via a single link to an infected node; see, for example, the fourth paragraph on p. 3200 of Pastor-Satorras and Vespignani. Another perspective on the meaning of the infection spreading rate is based on the following probabilistic interpretation: if at time $t$ a given node $V$ is uninfected but has links to $k$ infected nodes, then

$$\Pr \left\{ \text{Node } V \text{ is infected at time } t + \Delta t \right\} \quad \text{At time } t, \text{ node } V \text{ is uninfected but has links to } k \text{ infected nodes}$$

$$= \min\{R/c,k\lambda\Delta t + o(\Delta t) \} \quad \text{for } k = 0, 1, \ldots$$

where the ‘little-oh’ notation $o(h)$ denotes a quantity that tends to zero faster than $h$ so that...
lim_{h \to 0} o(h)/h = 0. Thus, if an uninfected node has a single link to an infected node \((k = 1)\) and the basic unit of time is taken to be sufficiently small, then in a one-unit time step \((\Delta t = 1)\) the infection spreading rate \(\lambda\) (expressed in number of infections per time unit) is approximately equal to the probability that the uninfected node will become infected. Of course, if an uninfected node has no links to infected nodes \((k = 0)\) and the size \(\Delta t\) of the time step is taken to be sufficiently small in an absolute sense, then the probability is approximately zero that the uninfected node will become infected in a time step of size \(\Delta t\), regardless of the basic unit of time used in the analysis. From Equation (1) it is also clear that in each time step, the maximum rate at which the infection can be transmitted to an uninfected node is \(|R/C|\lambda\).

The primary result of Huang et al.\(^1\) is the existence of a critical threshold \(\lambda_C > 0\) for the infection—spreading rate \(\lambda\) that they expressed as a function of the network structure as well as \(c\) and \(R\) defined above and that has the following properties:

- if \(0 \leq \lambda \leq \lambda_C\), then the infection ultimately dies out; and
- if \(\lambda > \lambda_C\), then the epidemic is spread through the network and ultimately reaches a steady-state non-vanishing condition.

The following state variable precisely defines the status of the network at time \(t \geq 0\):

\[
\theta(t; \lambda) = \text{the probability that at time } t, \text{ a randomly selected link in the network is incident on (connects to) an infected node, given that the infection—spreading rate is } \lambda
\]

and in terms of this state variable, the main result of Huang et al.\(^1\) is

\[
\theta(\lambda) \equiv \lim_{t \to \infty} \theta(t; \lambda) = \begin{cases} 
0, & \text{if } 0 \leq \lambda \leq \lambda_C, \\
0, & \text{if } \lambda > \lambda_C,
\end{cases}
\]

where \(\lambda_C > 0\) (3)

and the specific value of \(\lambda_C\) is given by Equation (22) as elaborated below. This technical note examines some flaws in the analysis given in Huang et al.;\(^1\) moreover, this note contains an alternative proof of (3) and (22) which is intended to provide additional insights into the transient and steady-state behavior of this system.

2. Flaws in the analysis of Huang et al.

Following Pastor-Satorras and Vespignani,\(^2\) for \(k = 1, 2, \ldots\), Huang et al.\(^1\) let \(P(k)\) (or \(p_k\)) denote the fraction of nodes (vertices) in the network that have \(k\) links to other nodes (i.e. degree \(k\)); and they let

\[
\langle k \rangle = \sum_{k=1}^{\infty} kP(k)
\]

denote the average vertex degree in the network. The authors also introduce the following state variables defining the status of the system at time \(t\):

\[
\rho_k(t) = \text{the relative density of infected nodes that have } k \text{ links for } k = 1, 2, \ldots
\]

The authors’ definition (4) is unclear, because the term ‘relative density’ is not standard statistical terminology; see, for example, Dodge.\(^3\) From the sentence immediately preceding Equation (1) of Pastor-Satorras and Vespignani,\(^2\) the definition of this quantity in standard statistical terminology is

\[
\rho_k(t; \lambda) = \text{the probability that a node with } k \text{ links is infected at time } t
\]

The notation in Equation (5) has been augmented to emphasize the dependence of this state variable on not only the time \(t\) but also the primary input parameter \(\lambda\); and in light of the analysis given below, it is necessary to take careful account of the way in which all system status variables of interest depend on both of these independent variables.

Next Huang et al.\(^1\) define the main system status variable of interest,

\[
\theta(\rho_k(t)) = \text{the probability that any given individual will link to an infected individual (with } \theta \text{ assumed to be a function of the partial densities of infected individuals } (\rho_k(t))
\]

Although technically it might be said that \(\theta(\cdot)\) depends on the \(\{\rho_k(\cdot)\}\), this is misleading because the \(\{\rho_k(\cdot)\}\) are in fact system-status variables that depend on \(t\) and \(\lambda\), which are ultimately the only independent variables in this analysis. Therefore, the transient and steady-state formulations of the quantity \(\theta(\cdot)\) are properly defined by Equations (2) and (3), respectively. For more on this issue, see the sentence immediately following Equation (3) of Pastor-Satorras and Vespignani\(^2\) and the sentence containing Equation (8) of Pastor-Satorras and Vespignani.\(^4\)

For each uninfected node, Huang et al.\(^1\) let \(R\) denote the resources available for transmitting the infection to that node during each time step, and they let \(c\) denote the cost of transmitting the infection to the uninfected node via a link to an infected node during each time step. The authors also assume that each infected node recovers (becomes ‘healthy’) at a rate of 1 recovery per
time unit. With this nomenclature, the authors’ Equation (5) quantifying the way in which an infection is spread through the network over time should be corrected to read

\[ \frac{\partial}{\partial t} \rho_k(t; \lambda) = -\rho_k(t; \lambda) + \lambda S_k [1 - \rho_k(t; \lambda)] \theta(t; \lambda), \]

where \( S_k = \min \{R/c, k\} \), for \( k = 1, 2, \ldots \) (6)

The basis for the key relation (6) follows immediately from the corrected definition (5) of \( \rho_k(t; \lambda) \), the authors’ assumption that each infected node has a recovery rate of 1, the probabilistic interpretation (1) of the infection-spreading rate \( \lambda \), and the definition (2) of \( \theta(t; \lambda) \). (For further discussion of (6), see the sentence immediately following Equation (1) of Pastor-Satorras and Vespignani. Moreover, the authors’ Equation (6), the steady-state solution of (6), should be revised to read

\[ \rho_k(\lambda) \equiv \lim_{t \to \infty} \rho_k(t; \lambda) = \frac{\lambda S_k \theta(\lambda)}{1 + \lambda S_k \theta(\lambda)} \]

for \( k = 1, 2, \ldots \) and for \( \lambda \geq 0 \) (7)

and it follows that

\[ \theta(\lambda) = \frac{1}{(k)} \sum_{k=1}^{\infty} k P(k) \rho_k(\lambda) \quad \text{for} \quad \lambda \geq 0 \] (8)

Instead of (7) and (8), Huang et al. attempt to analyze the properties of the relation

\[ \theta = \left( \frac{1}{(k)} \sum_k k P(k) \frac{\lambda S_k \theta}{1 + \lambda S_k \theta} \right) \] (9)

where they treat \( \theta \) as if it were an independent variable instead of a function of the independent variable \( \lambda \). They make the following statement about Equation (9):

Note that the right-hand side of [the immediately preceding equation] is concave at about \( \theta = 0 \), and that \( \theta = 0 \) is considered a trivial solution. Since it is possible for \( \theta \) to have a non-singular solution, we derive the inequality

\[ \frac{d}{d \theta} \left( \frac{1}{(k)} \sum_{k} k P(k) \frac{\lambda S_k \theta}{1 + \lambda S_k \theta} \right) \bigg|_{\theta = 0} \geq 1. \]

By differentiating [the expression above] and replacing \( \theta \) with 0 we get

\[ \frac{1}{(k)} \sum_{k} k P(k) \lambda S_k \geq 1 \quad \text{or} \quad \lambda \leq \sum_{k} k P(k) S_k. \] (10)

The argument of Huang et al. in (10) is flawed in several respects. In the first place, the phrase ‘concave at about \( \theta \)’ is non-standard, and its precise meaning is unclear. Furthermore, the statement ‘it is possible for \( \theta \) to have a non-singular solution’ seems to be circular because it appears to be assuming what is to be proved. Finally, the authors provide no justification for the following steps: (i) performing the indicated differentiation with respect to \( \theta \); (ii) evaluating the resulting expression at \( \theta = 0 \); and (iii) asserting that the result of step (ii) must be greater than or equal to 1. Because \( \theta \) is a function of \( \lambda \), the differentiation of the right-hand side of (9) with respect to \( \theta \) and without reference to \( \lambda \) is neither meaningful nor justified. It is not even clear that a proper differentiation of the function \( \theta(\lambda) \) with respect to \( \lambda \) can be performed at the critical value \( \lambda_C \) whose existence the authors are trying to establish; in fact, Figures 3 and 4 of Huang et al. seem to portray the curve defined by the function \( \theta(\lambda) \) for \( \lambda \geq 0 \) as failing to have a tangent (and hence being non-differentiable) at the critical threshold \( \lambda = \lambda_C \). Moreover, if the authors are trying to find a unique positive value \( \lambda_C \) such that \( \theta(\lambda) = 0 \) for \( 0 \leq \lambda \leq \lambda_C \) and \( \theta(\lambda) > 0 \) for \( \lambda > \lambda_C \), then it is unclear why they are setting \( \theta = 0 \) in step (ii). Finally, it is unclear why the result of the differentiation in step (ii) should yield a value not less than one. All in all, the authors’ analysis as quoted in display (10) appears to have significant gaps and inconsistencies.

3. Alternative derivation of Equation (2)

This section contains an alternative derivation of (3) in which the main ideas are completely explained in standard terminology. First the function \( \theta(\lambda) \) is defined implicitly by combining Equations (7) and (8) to yield the functional equation

\[ \theta(\lambda) = \frac{1}{(k)} \sum_{k=1}^{\infty} k P(k) \frac{\lambda S_k \theta(\lambda)}{1 + \lambda S_k \theta(\lambda)} \quad \text{for} \quad \lambda \geq 0 \]

with \( \theta(0) = 0 \) (11)

and with the definition

\[ \theta(\lambda) \equiv 0 \quad \text{for} \quad \lambda < 0 \] (12)

Equation (11) holds for all \( \lambda \) in a neighborhood of 0. As detailed in the Appendix, the functional equation defined by (11) and (12) satisfies the hypotheses of the real analytic implicit function theorem (see p. 277 of Chaundy or p. 49 of Krantz and Parks) in a neighborhood of 0. Hence, there is a unique function \( \theta(\lambda) \) defined implicitly by Equations (11) and (12) that is real analytic in a neighborhood of 0. In particular, \( \theta(\lambda) \) is analytic at \( \lambda = 0 \); and thus by Corollary 1.1.16 of
Krantz and Parks, there exists \( \omega > 0 \) such that for all \( \lambda \in (-\omega, \omega) \), the function \( \theta(\lambda) \) is infinitely differentiable and is correctly represented by its Taylor series expansion centered at the origin.

The critical threshold \( \lambda_C \) is defined as follows:

\[
\lambda_C \equiv \sup \{ \lambda \in \mathbb{R} : \theta(\lambda) = 0 \} \tag{13}
\]

where \( \mathbb{R} \) denotes the real line. Because \( \theta(\lambda) = 0 \) for \( \lambda \leq 0 \) and \( \lim_{\lambda \to \infty} \theta(\lambda) = 1 \), it follows that \( \lambda_C \) in (13) is well defined and \( 0 \leq \lambda_C < \infty \). The next step is to prove that \( \lambda_C > 0 \).

A Taylor series expansion of the function \( \theta(\cdot) \) is expressed in terms of its derivatives

\[
\theta^{(\ell)}(\lambda) = \frac{d^{\ell} \theta(\lambda)}{d\lambda^\ell} \quad \text{for} \quad \lambda \in (-\omega, \omega) \quad \text{and} \quad \ell = 1, 2, \ldots
\]

For \( \lambda \in (-\omega, \omega) \), the first derivative of \( \theta(\cdot) \) at the point \( \lambda \) has the form

\[
\theta^{(1)}(\lambda) = \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k P(k) S_k \left[ \frac{\theta(\lambda) + \lambda \theta^{(1)}(\lambda)}{1 + \lambda S_k(\lambda)} \right] - \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k P(k) S_k(\lambda) \left[ \frac{\theta(\lambda) + \lambda S_k \theta^{(1)}(\lambda)}{[1 + \lambda S_k(\lambda)]^2} \right] \tag{14}
\]

where the interchange of the summation and differentiation operators in (14) is justified because by (11), \( \theta(\lambda) \) is a finite weighted average of the functions \( \{\lambda S_k(\lambda) / [1 + \lambda S_k(\lambda)]; k = 1, \ldots, \lfloor R/c \rfloor\} \); for more on this point, see Equation (25) in the Appendix. It is straightforward to show that for \( \ell = 1, 2, \ldots \), the order-\( \ell \) derivative of \( \theta(\lambda) \) is continuous at every \( \lambda \in (-\omega, \omega) \) and satisfies the condition

\[
\theta^{(\ell)}(\lambda) = \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k P(k) S_k \lambda \theta^{(\ell)}(\lambda) \left[ 1 + \lambda S_k(\lambda) \right] \left[ \text{additional terms, each with the (multiplicative) factor } \theta(\lambda), \text{ or at least one of the factors } \{\theta^{(u)}(\lambda) : u = 1, \ldots, \ell - 1\} \right] \tag{15}
\]

The condition \( \theta(0) = 0 \) and Equation (14) imply that \( \theta^{(1)}(0) = 0 \); and then Equation (15) and an induction argument show that

\[
\theta^{(\ell)}(0) = 0 \quad \text{for} \quad \ell = 1, 2, \ldots \tag{16}
\]

Taking a Taylor series centered at 0 for the function \( \theta(\lambda) \), where \( \lambda \in (-\omega, \omega) \), and exploiting (16), we see that

\[
\theta(\lambda) = \sum_{\ell=0}^{\infty} \frac{\theta^{(\ell)}(0)}{\ell!} \lambda^\ell \quad \text{for all} \quad \lambda \in (-\omega, \omega) \tag{17}
\]

and thus by (17) and the definition of \( \lambda_C \), it follows that

\[
0 < \omega \leq \lambda_C < \infty
\]

The next step is to observe that if \( \theta(\lambda) > 0 \), then from Equations (12) and (17) it follows that

\[
\lambda \geq \omega > 0 \quad \text{and} \quad 1 + \lambda S_k \theta(\lambda) > 1 \quad \text{for} \quad k = 1, 2, \ldots
\]

and dividing Equation (11) by \( \theta(\lambda) \), we have

\[
1 = \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k P(k) \frac{\lambda S_k}{1 + \lambda S_k \theta(\lambda)} < \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k P(k) \lambda S_k
\]

by Equation (18). From Equation (19) it follows that

\[
\lambda > \frac{\langle k \rangle}{\sum_{k=1}^{\infty} k P(k) S_k} \tag{20}
\]

The analysis of Equations (18) through (20) is summarized as follows:

\[
\text{If } \theta(\lambda) > 0, \text{ then } \lambda > \frac{\langle k \rangle}{\sum_{k=1}^{\infty} k P(k) S_k} \tag{21}
\]

By the definition (13) of \( \lambda_C \), it follows that \( \lambda_C \) is the greatest lower bound of the set \( \{\lambda \in \mathbb{R} : \theta(\lambda) > 0\} \); and therefore (21) immediately implies that

\[
\lambda_C = \frac{\langle k \rangle}{\sum_{k=1}^{\infty} k P(k) S_k} \tag{22}
\]

Moreover,

\[
\theta(\lambda_C) = 0
\]

because the assumption that \( \theta(\lambda_C) > 0 \) immediately leads to the contradiction that \( \lambda_C > \lambda_C \) by Equations (21) and (22). This completes the derivation of the key results (3) and (22).

In the course of establishing (3) and (22), we showed that \( \theta(\lambda) \) is an infinitely differentiable function of \( \lambda \) for \( \lambda \in [0, \omega) \); and using a minor modification of the analysis detailed in the Appendix, we can establish a similar result for all \( \lambda \geq \omega \). In particular this means that \( \theta(\lambda) \) is differentiable at the critical threshold \( \lambda = \lambda_C \). This conclusion is inconsistent with Figures 3 and 4 of Huang et al., which seem to depict the function \( \theta(\lambda) \) as being non-differentiable at the point \( \lambda = \lambda_C \). On the other hand, as mentioned in the last paragraph of Section 2, it appears that the existence of the derivative \( \theta^{(1)}(\lambda_C) \) is an essential but not explicitly recognized part of the analysis (10) of Huang et al.
4. Conclusion
This note details the reasons why the analysis of Huang et al.\(^1\) fails to provide a rigorous justification of the key results (3) and (22); moreover, the note provides an alternative proof of these results. The approach outlined in Section 3 may be easily adapted to the analysis of epidemic dynamics in other scale-free networks with different features.

References

Appendix: Verification of the hypotheses of the real analytic implicit function theorem
Associated with the functional equation (11) is the following function of two real variables,

\[
F(x, y) = y - \frac{1}{(k)} \sum_{k=1}^{\infty} kP(k)x^2 \frac{x^2}{y} \quad \text{for} \quad (x, y) \in \mathbb{R}^2
\]

(23)

where \(x\) and \(y\) in Equation (23) are associated with \(\lambda\) and \(\theta(\lambda)\), respectively, in Equation (11); and \(\mathbb{R}^2\) denotes two-dimensional Euclidean space. Observe that

\[
F(0, 0) = 0
\]

(24)

and the first step is to establish that \(F(x, y)\) is real analytic in a neighborhood of the point \((0, 0)\) in \(\mathbb{R}^2\). It follows immediately from Proposition 2.2.2 of Krantz and Parks\(^6\) (concerning conditions under which a sum, product, or ratio of two real analytic functions is also a real analytic function) that for \(k = 1, 2, \ldots, \) there exists \(\eta > 0\) such that

\[
f_k(x, y) = \frac{x^2}{y} \quad \text{is real analytic at each} \quad (x, y)
\]

with \(\max\{|x|, |y|\} < \eta_k\)

With the definitions

\[
K^* = [R/c] \quad \text{and} \quad \eta^* = \min\{\eta_k: k = 1, 2, \ldots, K^*\}
\]

we can again apply Proposition 2.2.2 of Krantz and Parks\(^6\) to the finite set \(\{f_k(x, y): k = 1, \ldots, K^*\}\) of functions that are all real analytic in the neighborhood \((x, y) \in \mathbb{R}^2: \max\{|x|, |y|\} < \eta^*\) of the point \((0, 0)\) to conclude that

\[
F(x, y) = y - \frac{1}{k} \sum_{k=1}^{K^*} kP(k)f_k(x, y)
\]

\[
F(x, y) = y - \frac{1}{k} \sum_{k=1}^{K^*} kP(k)\left[\frac{1}{(k)} \sum_{k=1}^{\infty} kP(k)\right]
\]

(25)

The final observation is that

\[
\frac{\partial}{\partial y} F(x, y) \bigg|_{(x,y)=0(0,0)} = 1 \neq 0
\]

(26)

and it follows immediately from (24), (25), (26), and the real analytic implicit function theorem\(^6\) that there exists a positive number \(\eta \leq \eta^*\) and a unique real-valued function \(\varphi(x)\) such that

\[
\varphi(x) \text{ is real analytic and } F(x, \varphi(x)) = 0 \text{ for } |x| < \eta
\]

(27)

With the identification \(x \leftrightarrow \lambda\) and \(\varphi(x) \leftrightarrow \varphi(\lambda)\), it follows from Equations (23) and (27) that there is a unique solution \(\varphi(\lambda)\) to the functional equation (11) that is real analytic in the neighborhood \((-\eta, \eta)\) of the origin.

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