Simulation of Stochastic Activity Networks Using Path Control Variates

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This article details several procedures for using path control variates to improve the accuracy of simulation-based point and confidence-interval estimators of the mean completion time of a stochastic activity network (SAN). Because each path control variate is the duration of the corresponding directed path in the network from the source to the sink, the vector of selected path controls has both a known mean and a known covariance matrix. This information is incorporated into estimation procedures for both normal and nonnormal responses. To evaluate the performance of these procedures experimentally, we examine the bias, variance, and mean square error of the controlled point estimators as well as the average half-length and coverage probability of the corresponding confidence-interval estimators for a set of SANs in which the following characteristics are systematically varied: (a) the size of the network (number of nodes and arcs); (b) the topology of the network; (c) the percentage of activities with exponentially distributed durations; and (d) the relative dominance of the critical path. The experimental results show that although large improvements in accuracy can be achieved with some of these procedures, the confidence-interval estimators for normal responses may suffer serious loss of coverage probability in some applications.

1. INTRODUCTION

Stochastic activity networks (SANs) are widely used in the scheduling and management of large projects. However, the analysis of such networks is greatly complicated by stochastic dependencies among network components that arise, for example, when some activities are common to several paths or when several activity durations are correlated. Conventional analysis techniques are based on restrictive assumptions about the probability distributions of the activity durations or about the topology of the network [12, 14, 18, 20, 21, 27]; and these
assumptions generally yield approximations of unknown accuracy. Because of its ability to represent faithfully the dependencies among the components of a stochastic activity network and to yield estimates of desired performance measures with controllable accuracy, Monte Carlo simulation is frequently the method of choice for the analysis of such networks.

In the simulation of a SAN, the usual objective is to obtain point and confidence-interval estimators for the mean completion time $\theta$ of the network. Let the random variable $Y$ denote the completion time of a given SAN. Direct simulation simply computes the sample mean response $\bar{Y}$ from $n$ independent replications of the network to yield an unbiased estimator of $\theta$ with $\text{var}(\bar{Y}) = \text{var}(Y)/n$. Since the variance of $\bar{Y}$ declines as the inverse of the sample size, a large number of replications will usually be required to achieve acceptable precision (for example, see Table 2 below). Computing costs can then become prohibitive, and we naturally seek to derive an alternative estimator $\hat{\theta}$ with $E(\hat{\theta}) = \theta$ and $E((\hat{\theta} - \theta)^2) < \text{var}(\bar{Y})$.

Several variance reduction techniques have been proposed for improving the efficiency of activity network simulations, including conditional Monte Carlo [6, 9, 15, 28], stratified sampling [7, 13, 19], antithetic sampling [7, 19, 29], control variates [7, 13], and combinations of these techniques [7, 13, 19]. Some of our recent work [2, 3, 31] has led us to the conclusion that in comparison to the other commonly used variance reduction techniques, the method of control variates is more easily adapted to a wide variety of network configurations and has greater potential to yield large efficiency increases in general applications. To estimate the target parameter $\theta$ using the method of control variates, we identify a set of auxiliary variables $C = (C_1, \ldots, C_d)'$ that are generated by the same stochastic system, have a known expectation $\mu_C$, and are strongly correlated with the response $Y$. We then try to predict and counteract the unknown deviation $Y - \theta$ by subtracting from $Y$ an appropriate linear function of the known deviation $C - \mu_C$. The objective then is to determine a vector of control coefficients $b = (b_1, \ldots, b_d)'$ that will minimize the variance of the controlled estimator $Y(b) = Y - b'(C - \mu_C)$. Lavenberg, Moeller, and Welch [16] presented a comprehensive analysis of the control variate method for univariate responses. Porta Nova and Wilson [25] developed the more general case in which control variates are applied to the estimation of a multivariate simulation metamodel—that is, a linear model for an output vector of simulation performance measures expressed in terms of an input vector of design variables for the target system.

This article is organized as follows. In Section 2 we summarize the necessary statistical framework for the application of the control variate method in the following situations: (a) the covariance matrix of the controls is unknown and is estimated from simulation-generated data; and (b) the covariance matrix of the controls is known prior to construction of the simulation and thus can be incorporated into an estimation scheme. Procedures are given for both normal and non-normal simulation outputs. In Section 3 we derive the required moment structure for path control variables in activity network simulation. Section 4 details the results of an extensive experimental investigation of the performance of the various estimation procedures based on path control variates. In Section 5 we discuss the significance of the experimental results, and we summarize the
main findings of this research in Section 6. This article is partially based on results that were originally presented in [2, 3, 31].

2. STATISTICAL FRAMEWORK

The following notation is used throughout this article. Let $\sigma^2_Y = E[(Y - \theta)^2]$ denote the variance of the completion time $Y$ for the target network, let $\sigma_{CY} = E[(C - \mu_C)(Y - \theta)]$ denote the $q \times 1$ vector of covariances between the control vector $C$ and the response $Y$, and let $\Sigma_C = E[(C - \mu_C)(C - \mu_C)']$ denote the $q \times q$ covariance matrix of the controls. The variance of the controlled response

$$\text{var}[Y(b)] = \sigma^2_Y - 2b'\sigma_{CY} + b'\Sigma_C b$$

is minimized by the optimal vector of control coefficients

$$\beta = \Sigma_C^{-1}\sigma_{CY},$$

yielding the minimum variance

$$\text{var}[Y(\beta)] = \sigma^2_Y(1 - R_{YC}),$$

where $R_{YC}$ is the coefficient of multiple correlation between $Y$ and $C$. In practice, $\beta$ must be estimated because at least $\sigma_{CY}$ is generally unknown; and in many applications $\Sigma_C$ is also unknown so that both terms on the right-hand side of (1) must be estimated from simulation-generated data. Estimation of $\beta$ results in some loss of precision for the controlled point estimators of $\theta$ described below.

2.1. Analysis Techniques for Normal Outputs

First we summarize the conventional method for applying control variates to the estimation of $\theta$. Let $\{[Y_u, C_u': u = 1, \ldots, n\}$ denote the results observed on $n$ independent replications of a simulation of the target network. Let $\bar{Y}$ and $\bar{C}$ respectively denote the sample means of the response and the control vector computed over all $n$ replications; and let $S_{\bar{Y}}$, $S_{CY}$, and $S_C$ respectively denote the sample analogs of $\sigma^2_Y$, $\sigma_{CY}$, and $\Sigma_C$. Specifically, we compute

$$S_C = (n - 1)^{-1} \sum_{u=1}^{n} (C_u - \bar{C})(C_u - \bar{C})'$$

and

$$S_{CY} = (n - 1)^{-1} \sum_{u=1}^{n} (C_u - \bar{C})(Y_u - \bar{Y}),$$

so that the sample analog of (1) is

$$\hat{\beta} = S_C^{-1}S_{CY},$$
and the conventional controlled point estimator of \( \theta \) is

\[
\bar{Y}(\hat{\theta}) = \bar{Y} - \hat{\theta}(\bar{C} - \mu_c).
\]  

(4)

In general, \( \bar{Y}(\hat{\theta}) \) is a biased estimator of \( \theta \) because \( \hat{\theta} \) and \( \bar{C} \) are dependent so that \( E[\hat{\theta}'(\bar{C} - \mu_c)] \neq 0 \). However, in many large-scale simulation experiments, the response and the controls are (approximately) jointly normal because these statistics are simultaneously accumulated over the duration of each run and thus are subject to a central-limit effect (see [8]). Moreover, the analysis in the following sections reveals that such an effect frequently occurs in the simulation of complex SANS. Thus it is often reasonable to assume that \( Y \) and \( C \) have a joint multivariate normal distribution

\[
\begin{bmatrix}
Y \\
C
\end{bmatrix} \sim N_{q+1} \left( \begin{bmatrix}
\theta \\
\mu_c
\end{bmatrix}, \begin{bmatrix}
\sigma_Y^2 & \sigma_{CY} \\
\sigma_{CY} & \Sigma_c
\end{bmatrix} \right).
\]  

(5)

If (5) holds, then \( \bar{Y}(\hat{\theta}) \) is an unbiased estimator of \( \theta \); and an exact \( 100(1 - \alpha)\% \) confidence interval for \( \theta \) is given by

\[
\bar{Y}(\hat{\theta}) \pm t_{1-\alpha/2}(n - q - 1)DS_{YC}.
\]  

(6)

where

\[
D^2 = n^{-1} + (n - 1)^{-1}(\bar{C} - \mu_c)'S_c^{-1}(\bar{C} - \mu_c),
\]

\[
S_{YC} = (n - q - 1)^{-1}(n - 1)(S_Y^2 - S_{CY}S_c^{-1}S_{CY}),
\]

and \( t_{1-\alpha/2}(n - q - 1) \) is the \( 1 - \alpha/2 \) quantile for Student's \( t \)-distribution with \( n - q - 1 \) degrees of freedom (see [16]).

The use of \( \hat{\theta} \) in the standard controlled estimator of \( \theta \) implies that the minimum variance in (2) is not achieved. To measure the efficiency loss due to estimation of the control coefficients, Lavenberg, Moeller, and Welch [16] derived the loss factor

\[
\frac{\text{var}[\bar{Y}(\hat{\theta})]}{\text{var}[\bar{Y}(\theta)]} = \frac{n - 2}{n - q - 2}.
\]  

(7)

Combining (2) and (7), we have the following expression for the net efficiency of the standard version of the control variate technique:

\[
\text{var}[\bar{Y}(\hat{\theta})] = \text{var}(\bar{Y})(1 - R_{YC}^2) \left( \frac{n - 2}{n - q - 2} \right).
\]

As the basis for a similar statistical-estimation procedure that exploits the known covariance matrix of the control vector, we propose the following alternative estimator for the control coefficient vector

\[
\hat{\beta} = \Sigma_c^{-1}S_{CY};
\]  

(8)
and this leads to a controlled point estimator for $\theta$ with the form

$$\overline{Y}(\hat{\beta}) = \overline{Y} - \hat{\beta}'(C - \mu_C).$$  \hspace{1cm} (9)$$

Under the assumption of joint multivariate normality in (5), Bauer [2] proved that $\overline{Y}(\hat{\beta})$ is an unbiased estimator of $\theta$. Furthermore, an approximate $100(1 - \alpha)\%$ confidence interval for $\theta$ is given by

$$\overline{Y}(\hat{\beta}) \pm t_{1-\alpha/2}(n - q - 1) \left[ \frac{q + 1}{n(n - 1)} S^2_y + \frac{n - 2}{n(n - 1)} S^2_{Y,C} \right]^{1/2}.$$  \hspace{1cm} (10)$$

Again the minimum variance in (2) is not achieved, and the net efficiency in using $\overline{Y}(\hat{\beta})$ as an estimator for $\theta$ is characterized by

$$\text{var}[\overline{Y}(\hat{\beta})] = \text{var}(\overline{Y})(1 - R^-_c) \left( \frac{n - 2}{n - 1} \right) + \text{var}(\overline{Y}) \left( \frac{q + 1}{n - 1} \right).$$

A comprehensive analysis of the controlled estimator $\overline{Y}(\hat{\beta})$ is given in Bauer [2].

### 2.2. Analysis Techniques for Nonnormal Outputs

To reduce the bias of $\overline{Y}(\hat{\beta})$ and to construct an asymptotically valid confidence interval for $\theta$ when the response and the controls are not jointly normal, we use the jackknife statistic [5, Sect. 2.7]. Let $\hat{\beta}_u$ and $\overline{Y}(\hat{\beta}_u)$ respectively denote the estimators computed from (3) and (4) when the $u$th observation $[Y_u, C_u]$ has been deleted from the original data set $\{[Y_w, C_w]: w = 1, \ldots, n\}$. Using the pseudovalues $J_u(\hat{\beta}) = n\overline{Y}(\hat{\beta}) - (n - 1)\overline{Y}(\hat{\beta}_u)$, $u = 1, \ldots, n$, we calculate the jackknife statistic

$$\overline{J}(\hat{\beta}) = n^{-1} \sum_{u=1}^{n} J_u(\hat{\beta}),$$  \hspace{1cm} (11)$$

and the associated sample variance

$$S^j(\hat{\beta}) = (n - 1)^{-1} \sum_{u=1}^{n} [J_u(\hat{\beta}) - \overline{J}(\hat{\beta})]^2.$$

If the joint distribution of $Y$ and $C$ satisfies certain mild regularity conditions, then the jackknifed point estimator of $\theta$ has reduced bias: $E[\overline{J}(\hat{\beta})] = \theta + O(n^{-2})$; see [10]. In this case an asymptotically valid $100(1 - \alpha)\%$ confidence interval for $\theta$ is given by

$$\overline{J}(\hat{\beta}) \pm t_{1-\alpha/2}(n - 1)[n^{-1/2} S^j(\hat{\beta})].$$  \hspace{1cm} (12)$$
To construct the corresponding point and interval estimators that incorporate the known covariance matrix \( \Sigma_c \), we use \( \tilde{\Theta} \) in place of \( \Theta \) to compute the pseudovalues \( J_u(\tilde{\Theta}) = n \tilde{Y}(\tilde{\Theta}) - (n - 1)\tilde{Y}(\tilde{\Theta}_u) \), \( u = 1, \ldots, n \). Note that \( \tilde{\Theta}_u \) and \( \tilde{Y}(\tilde{\Theta}_u) \) respectively denote the estimators computed from (8) and (9) when the \( u \)th observation \( [Y_u, C_u] \) has been deleted from the original data set \( \{[Y_w, C_w]: w = 1, \ldots, n \} \). The jackknifed point estimator of \( \Theta \) is then

\[
\tilde{J}(\tilde{\Theta}) = n^{-1} \sum_{u=1}^{n} J_u(\tilde{\Theta})
\]

with sample variance

\[
S_j(\tilde{\Theta}) = (n - 1)^{-1} \sum_{u=1}^{n} [J_u(\tilde{\Theta}) - \tilde{J}(\tilde{\Theta})]^2.
\]

An asymptotically valid 100(1 - \( \alpha \))% confidence interval for \( \Theta \) is given by

\[
\tilde{J}(\tilde{\Theta}) \pm t_{1-\alpha/2}(n - 1)[n^{-1/2}S_j(\tilde{\Theta})].
\]

3. ESTIMATION WITH PATH CONTROLS

The graph-theoretic structure of a given SAN is described by the pair \((\Psi, \Delta)\), where the set of all nodes (vertices) in the network is \( \Psi = \{1, 2, \ldots, \psi\} \), and the set of all activities (directed lines) in the network is \( \Delta = \{(u_i, v_i): \text{activity } i \text{ has start node } u_i \in \Psi \text{ and end node } v_i \in \Psi, i = 1, \ldots, \delta\} \). We assume that the network is acyclic with a single source node and a single sink node. The probabilistic structure of the network is described by the given joint distribution function \( F(a_1, \ldots, a_\delta) \) of the random vector \((A_1, \ldots, A_\delta)\) whose \( i \)th element \( A_i \) is the duration of the \( i \)th activity \((u_i, v_i) \in \Delta \). Thus for \( i = 1, 2, \ldots, \delta \), the activity duration \( A_i \) has a known marginal distribution \( F(a_i) \) whose mean \( \mu_i \) and variance \( \sigma_i^2 \) can at least be evaluated numerically. Moreover, for \( h, i = 1, \ldots, \delta \), the covariance \( \sigma_{hi} \) between the activity durations \( A_h \) and \( A_i \) is also known or can be evaluated numerically. (In many SANs the activity durations are assumed to be stochastically independent so that \( \sigma_{hi} = 0 \) for \( h \neq i \).)

The path controls and the overall network completion time depend on the path structure of the network as follows. Let \( \zeta \) denote the number of directed paths from the source to the sink. Corresponding to the \( j \)th directed path \( \pi_j \) is the index set of component arcs \( I(j) = \{i: \text{activity } (u_i, v_i) \text{ is on path } \pi_j\} \) for \( j = 1, \ldots, \zeta \). The duration of path \( \pi_j \) is the random variable \( P_j = \Sigma_{i \in I(j)} A_i \) with mean and variance

\[
E(P_j) = \sum_{i \in I(j)} \mu_i \text{ and } \text{var}(P_j) = \sum_{i \in I(j)} \sigma_i^2 + \sum_{h \in I(j)} \sum_{h \neq i} \sigma_{hi},
\]

\[
\sum_{i \in I(j)} \mu_i \text{ and } \text{var}(P_j) = \sum_{i \in I(j)} \sigma_i^2 + \sum_{h \in I(j)} \sum_{h \neq i} \sigma_{hi},
\]
respectively. Note also that for \( j, l = 1, \ldots, \xi \), the covariance between the path durations \( P_j \) and \( P_l \) is

\[
\text{Cov}(P_j, P_l) = \sum_{i \in I(j) \cap I(l)} \sigma_i^2 + \sum_{h \in I(j)} \sum_{l \in I(l)} \sigma_{hl}.
\]

The overall project completion time is \( Y = \max\{ P_1, \ldots, P_\xi \} \), and the desired estimand is \( \theta = E(Y) \). Note that if all activity durations are mutually independent and if all paths are disjoint, then in principle we can compute the distribution function of the path duration \( P_j \) as the convolution of \( F_i \) for each component activity \( i \in I(j) \); and then \( \Pr\{ Y \leq y \} = \prod_{j=1}^{\xi} \Pr\{ P_j \leq y \} \) so that \( \theta \) can be obtained numerically in this case. However, if there are dependencies among the \( \{P_i\} \), then analytical and numerical techniques for obtaining \( \theta \) generally fail. These considerations motivated the control-variates estimation procedures proposed in this article.

From the set \( \{P_j; j = 1, \ldots, \xi\} \) of all path durations for the stochastic activity network \((\Psi, \Delta)\), we must select the control vector \( C \) that is to be applied to point estimators of \( \theta \) with the form (4), (9), (11), or (13). For simplicity in the experimental evaluation of all of these controlled estimators, we employed the following control-variates selection rule. Ranking the expected path durations \( \{E(P_j); j = 1, \ldots, \xi\} \) in ascending order so that we chose the last \( q \) paths in this list to build the control vector

\[
C = [P_{(\xi-q+1)}, P_{(\xi-q+2)}, \ldots, P_{(\xi-1)}, P_{(\xi)}]'.
\]

[In all of the experimentation described in the next two sections, we took \( q = 3 \) to ensure a reasonable bound on the loss factor (7).] The mean vector \( \mu_c \) and the dispersion matrix \( \Sigma_c \) of the resulting \( q \)-dimensional control vector were then computed from (15) and (16). Of course in general applications the user would need some guidance in determining \( q \) and in selecting the appropriate set of \( q \) controls from the set of available path controls, but this is a separate issue that is not addressed in this article. For controlled estimators of the form (4) and (9), Bauer and Wilson [4] have devised some control-variate selection criteria that are based on minimizing the mean square volume of the delivered confidence region and that appear to be effective when the normality assumption (5) holds; however, a more extensive Monte Carlo study is required to support any general conclusions about the performance of these selection procedures.

To illustrate the construction of path control variables, we consider the network shown in Figure 1. We have \( \Psi = \{1, 2, 3, 4\} \), and \( \Delta = \{(u_1, v_1), (1, 2); (u_2, v_2) = (1, 3); (u_4, v_4) = (1, 3); (u_4, v_4) = (3, 4); (u_5, v_5) = (2, 4)\} \). Thus there are \( \xi = 3 \) directed paths from the source (node 1) to the sink (node 4). If we define paths \( \pi_1 \), \( \pi_2 \), and \( \pi_3 \), respectively, in terms of the activity index sets \( I(1) = \{1, 5\}, I(2) = \{1, 3, 4\}, I(3) = \{1, 3\} \), then we have the corresponding path durations \( P_1 = A_1 + A_5 \), \( P_2 = A_1 + A_3 + A_4 \), and \( P_3 = A_2 + A_4 \). It
follows that \( E(P_1) = \mu_1 + \mu_5, E(P_2) = \mu_1 + \mu_5 + \mu_4, \) and \( E(P_3) = \mu_2 + \mu_4. \) We assume that the \( \{A_i\} \) are mutually independent; and for the sake of concreteness, we assume \( \mu_1 = \mu_2 = 10, \mu_3 = \mu_4 = 50, \) and \( \mu_5 = 110. \) In this case we have \( E(P_3) < E(P_2) < E(P_1). \) If we take \( q = 3 \) as in the following sections, then \( P_{(1)} = P_3, P_{(2)} = P_2, \) and \( P_{(3)} = P_1; \) and the control vector \( C = [P_3, P_2, P_1]^T \) has the known moment structure

\[
\mu_C = \begin{bmatrix} \mu_2 + \mu_4 \\ \mu_1 + \mu_3 + \mu_4 \\ \mu_1 + \mu_5 \end{bmatrix} \quad \text{and} \quad \Sigma_C = \begin{bmatrix} \sigma_1^2 + \sigma_4^2 & \sigma_4^2 & 0 \\ \sigma_4^2 & \sigma_2^2 + \sigma_4^2 & \sigma_3^2 \\ 0 & \sigma_3^2 & \sigma_1^2 + \sigma_3^2 \end{bmatrix}.
\]

This procedure for computing \( \mu_C \) and \( \Sigma_C \) is utilized in forming the estimators \( \overline{Y}(\hat{\beta}) \) and \( \overline{J}(\hat{\beta}) \) in Section 2.

### 4. EXPERIMENTAL EVALUATION

We conducted an extensive Monte Carlo study to evaluate the performance of the following controlled estimators for \( \theta; \overline{Y}(\hat{\beta}) \) [estimator 1, also denoted \( \hat{\theta}(1)\)]; \( \overline{Y}(\hat{\beta}) \) [estimator 2, also denoted \( \hat{\theta}(2)\)]; \( \overline{J}(\hat{\beta}) \) [estimator 3, also denoted \( \hat{\theta}(3)\)]; and \( \overline{J}(\hat{\beta}) \) [estimator 4, also denoted \( \hat{\theta}(4)\)]. This study involved the simulation of a set of three SANS in which the following characteristics were systematically varied: (a) the size of the network (the number of nodes and activities); (b) the topology of the network; (c) the percentage of activities with

<table>
<thead>
<tr>
<th>Network</th>
<th>Nodes</th>
<th>Activities</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>13</td>
<td>[11], p. 275</td>
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<td>24</td>
<td>42</td>
<td>[1], p. 190</td>
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<td>3</td>
<td>51</td>
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<td>[22], p. 324</td>
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</table>
exponentially distributed durations; and (d) the relative dominance (criticality index) of the critical path. All efficiency gains were reported relative to the direct simulation estimator $\hat{Y}$ [estimator 0, also denoted $\hat{\theta}(0)$]. Table 1 shows the range in the number of nodes and activities for the three networks used in the study. Figure 2 depicts the first network.

For each activity duration $A_i$ in a selected network, the associated distribution $F(A_i)$ was taken to be either (a) a normal distribution with the specified mean $\mu_i$ and standard deviation $\sigma_i = \mu_i/4$ whose tail was truncated below the cutoff value zero; or (b) an exponential distribution with mean $\mu_i$. We chose the exponential distribution as the nonnormal alternative because it has a higher coefficient of variation (equal to 1) than the beta and triangular distributions commonly used in the simulation of SANS [19], and this property partially counteracts the central-limit effect described in Section 2. Sullivan, Hayya, and Schaul [29] used a similar approach in their experimental study. For each of the three networks, we varied the percentage of exponentially distributed activity durations over the five levels {0%, 25%, 50%, 75%, 100%}. For each network and for each specified percentage of exponentially distributed activities, we assigned appropriate exponential distributions to the activities in the network according to a series of independent Bernoulli trials with success probability equal to the specified percentage of exponential activities; moreover, this assignment was made prior to performing any simulations of the network.

Relative dominance of a given path $\pi$, is defined as the probability $Pr\{Y = P_{\pi}\}$ that path $\pi$, has the longest duration in a single realization of the network. For each network and for each selected percentage of exponentially distributed activities, we rescaled the expected duration of each activity on the so-called “critical path” (that is, the path with the longest expected duration) to achieve a prespecified level of relative dominance for that path. In view of (17), the critical path is $\pi_{C}$ with mean duration $E[P_{C}]$, and the associated index set of activities on the critical path is denoted $I([C]) = \{i: \text{activity } (u_i, v_i) \text{ is on path } \}$.
For every \( i \in I[(\xi)] \), a common scale factor \( \phi \) was multiplied by the nominal mean \( \mu_i^\circ \) to yield the actual mean \( \mu_i = \phi \mu_i^\circ \) that was used when sampling \( A_i \). For each network to be simulated, we determined empirically values of \( \phi \) that achieved levels of relative dominance for the critical path in the following ranges: \{20%-40%, 50%-70%, 80%-100\%\}.

For a given configuration of a target activity network (that is, for a selected SAN with a given level of relative dominance and a given percentage of exponentially distributed activities), we determined the corresponding mean completion time \( \theta \) to within \( \pm 0.2\% \) of its true value by a preliminary Monte Carlo experiment involving direct simulation of the target network, and the final number of replications \( N^* \) in this preliminary experiment was determined by a relative-precision stopping rule. To estimate \( \theta \) by a \( 100(1 - \alpha)\% \) confidence interval of the form \( \bar{Y} \pm \gamma|\bar{Y}| \) that is asymptotically consistent and efficient as the percentage error tolerance \( \gamma \to 0 \), we employed a variant of Nádas’s [24] sequential confidence-interval estimation procedure that was proposed by Law, Kelton, and Koenig [17]. For a fixed number of replications \( n \) of the target network, let \( \bar{Y}(n) \) and \( S_y(n) \) denote the corresponding sample mean and standard deviation of the observed network completion times. Given prespecified values of the percentage error tolerance \( \gamma \), the confidence coefficient \( \alpha \), and the preliminary sample size \( n_0 \), we determined the final number of replications according to the stopping rule

\[
N^* = \min \left\{ n: n \geq n_0, n = 0 \mod 10, S_y(n) > 0, \right. \\
\left. \text{and } t_{1 - \alpha / 2}(n - 1) \frac{S_y(n)}{\sqrt{n}} \leq \gamma|\bar{Y}(n)| \right\}, \quad (18)
\]

and the “true” value of \( \theta \) was taken to be \( \bar{Y}(N^*) \). In the preliminary experimentation with the selected networks, we took \( n_0 = 1000, \gamma = 0.002, \) and \( \alpha = 0.01 \). Table 2 summarizes the results of the preliminary experimentation to determine \( \theta \) for each network configuration in which 50% of the activities were assigned an exponentially distributed duration. The results for other percentages of exponentially distributed activity durations are not displayed because they

<table>
<thead>
<tr>
<th>Network</th>
<th>Relative dominance of ( \pi_i(\xi) )</th>
<th>Final estimate of ( \theta )</th>
<th>Final sample size ( N^* )</th>
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<tbody>
<tr>
<td>1</td>
<td>20%-40%</td>
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<td>50%-70%</td>
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<td>2733.76</td>
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<td>141.79</td>
<td>81960</td>
</tr>
</tbody>
</table>
were not used in the final analysis of the controlled estimation procedures as explained below. To simulate each network configuration as efficiently as possible, we developed a general simulation program for stochastic activity networks that uses the discrete-event component of the SLAM II simulation language [26] and that is available from the authors on request.

Inspection of Table 2 reveals that the true value of $\theta$ for each selected network configuration has been determined to at least two significant figures; but for some configurations, the third significant figure is in question. Throughout the rest of this article, the estimates of $\theta$ in Table 2 are taken to be the “true” values of $\theta$ for each selected network configuration.

In the main experimental evaluation of the proposed estimation procedures for each selected network configuration, we conducted a metaexperiment composed of $m = 4096$ basic experiments that were executed separately and independently. A basic experiment involved $n = 32$ simulation runs (independent replications) of the target activity network; and from each basic experiment we computed point and confidence-interval estimators of $\theta$ using $q = 3$ selected path controls. To provide a fair assessment of the efficiency gains achieved in the simulation of each network, we estimated the bias, variance, and mean square error for each of the four controlled point estimators; moreover, for each of the four controlled confidence-interval estimators, we estimated the actual coverage probability of a nominal 90 percent confidence interval as well as the percentage reduction in confidence-interval half-length relative to direct simulation.

Properties of the controlled point estimators of $\theta$ were evaluated as follows. From the $n = 32$ simulation runs comprising the $w$th basic experiment ($w = 1, \ldots, m$, where $m = 4096$), we computed the $k$th point estimator $\hat{\theta}_w(k)$ ($k = 0, \ldots, 4$). Across the entire metaexperiment, we computed the grand mean $\hat{\theta}(k)$ and the sample variance $\hat{g}(k)$ of the replicates $\{\hat{\theta}_w(k): w = 1, \ldots, m\}$

$$\hat{\theta}(k) = m^{-1} \sum_{w=1}^{m} \hat{\theta}_w(k) \quad \text{and} \quad \hat{V}(k) = (m - 1)^{-1} \sum_{w=1}^{m} [\hat{\theta}_w(k) - \hat{\theta}(k)]^2.$$  \hspace{1cm} (19)

Thus for the $k$th controlled point estimator $\hat{\theta}_w(k)$ of the mean completion time $\theta$, the corresponding bias was estimated by $\hat{\theta}(k) - \theta$, the variance was estimated by $\hat{V}(k)$, and the mean square error was estimated by

$$\text{MSE}(k) = m^{-1} \sum_{w=1}^{m} [\hat{\theta}_w(k) - \theta]^2.$$  \hspace{1cm} (20)

Properties of the controlled confidence-interval estimators of $\theta$ were evaluated as follows. From the $n = 32$ simulation runs comprising the $w$th basic experiment ($w = 1, \ldots, m$, where $m = 4096$), we computed not only $\hat{\theta}_w(k)$ but also $\hat{V}_w(k)$, the estimator of $\text{var}[\hat{\theta}_w(k)]$; see (23) below. Thus in the $w$th basic experiment, the $k$th controlled confidence-interval estimator of $\theta$ has the general form

$$\hat{\theta}_w(k) = [\hat{\theta}_w(k) - \hat{H}_w(k), \hat{\theta}_w(k) + \hat{H}_w(k)]$$  \hspace{1cm} (21)
with half-length

\[ \hat{H}_n(k) = t_{1-\alpha/2}[v(k)] \cdot \hat{V}_w^{1/2}(k), \]  

(22)

where \( v(k) = n - 1 \) if \( k = 0, 3, \) or 4; \( v(k) = n - q - 1 \) if \( k = 1 \) or 2; and

\[
\begin{align*}
\hat{V}_w(k) &= \frac{S^2_{\gamma} + (n - 2)S_{\gamma,C}}{[n(n-1)]}, & \text{if } k = 2, \\
\hat{V}_w(0) &= \frac{S_{\gamma,C}}{n}, & \text{if } k = 0, \\
\hat{V}_w(1) &= \frac{S_{\gamma,C}}{n}, & \text{if } k = 1, \\
\hat{V}_w(3) &= \frac{S_{\gamma,C}}{n}, & \text{if } k = 3, \\
\hat{V}_w(4) &= \frac{S_{\gamma,C}}{n}, & \text{if } k = 4.
\end{align*}
\]

Note that for \( k = 1, 2, 3, \) and 4, display (21) specializes to the forms (6), (10), (12), and (14) respectively. The average half-length of the \( k \)th confidence-interval estimator \( \hat{\Theta}(k) \) computed across all \( m \) basic experiments is

\[ \hat{H}_n(k) = m^{-1} \sum_{w=1}^{m} \hat{H}_w(k), \]  

(23)

for \( k = 0, 1, 2, 3, 4; \) and the percentage reduction in half-length for \( \hat{\Theta}(k) \) relative to direct simulation is estimated by

\[ 100 \left[ \hat{H}(0) - \hat{H}(k) \right] / \hat{H}(0). \]

Finally, we consider the estimation of confidence-interval coverage probabilities. For the \( w \)th basic experiment, let

\[ \hat{I}_w(k) = \begin{cases} 1, & \text{if } \theta \in \hat{\Theta}_w(k), \\ 0, & \text{otherwise}, \end{cases} \]

(24)

for \( k = 0, 1, 2, 3, 4, \) and \( w = 1, \ldots, m. \) The actual coverage percentage for \( \hat{\Theta}(k) \) is then given by

\[ \hat{I}(k) = m^{-1} \sum_{w=1}^{m} \hat{I}_w(k), \]  

(25)

for \( k = 0, 1, 2, 3, 4. \)

5. EXPERIMENTAL RESULTS

To validate the normality assumption (5), we applied a multivariate extension of the Shapiro–Wilk test for normality [23] to the observations generated in each basic experiment. From Tew and Wilson [30] we see that for the significance level \( \alpha = 0.05 \) with dimension \( p = q + 1 = 4 \) and replication count \( n = 32, \) the multivariate Shapiro–Wilk statistic \( W^* \) has the lower critical value \( w_{0.05}^* (4, 32) = 0.873. \) Table 3 shows the percentage of basic experiments on network 3 that yielded \( W^* < w_{0.05}^* (4, 32), \) indicating a significant departure from normality for the random vector \([Y, C']\) generated on each run. (Note that unlike the majority of the results reported below, the entries in Table 3 were computed from metaexperiments composed of \( m = 32 \) basic experiments; and each basic experiment consisted of \( n = 32 \) independent simulation runs.) It is apparent
Table 3. Percentage of basic experiments on network 3 with $W^* < W_{0.05}^*$ (4, 32), indicating nonnormality of $[Y, C']$.

<table>
<thead>
<tr>
<th>Percentage of exponential activities</th>
<th>20%-40%</th>
<th>50%-70%</th>
<th>80%-100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>46.9</td>
<td>68.8</td>
<td>93.8</td>
</tr>
<tr>
<td>25%</td>
<td>50.0</td>
<td>65.6</td>
<td>93.8</td>
</tr>
<tr>
<td>50%</td>
<td>37.5</td>
<td>56.3</td>
<td>93.8</td>
</tr>
<tr>
<td>75%</td>
<td>40.6</td>
<td>71.9</td>
<td>96.9</td>
</tr>
<tr>
<td>100%</td>
<td>37.5</td>
<td>59.4</td>
<td>96.9</td>
</tr>
</tbody>
</table>

From this table that the normality assumption was increasingly untenable at higher levels of relative dominance for the critical path $\pi_{(C)}$, where the associated control $P_{(C)}$ matched the overall completion time $Y$ with progressively larger probabilities; and thus the joint distribution of $Y$ and $C$ was more nearly singular. Since the statistic $W^*$ was designed for application to nonsingular distributions, some of the anomalous behavior in Table 3 can be explained by near-singularity of the joint distribution of $[Y, C']$ at higher levels of relative dominance for $\pi_{(C)}$. On the other hand, the percentage of exponentially distributed activities did not seem to have a systematic effect on the behavior of $W^*$. The same behavior observed for network 3 was also observed for the other networks described above. These results indicated that the confidence-interval estimation procedures (6) and (10) based on normal distribution theory were not appropriate for many of the network configurations used in this study; and the subsequent experimentation discussed below confirmed this conclusion.

Next we attempted to analyze the effect of the percentage of exponentially distributed activities on the performance of the four controlled estimation procedures. For each of these procedures as applied to network 3, Tables 4 and 5, respectively, show the percentage reduction in confidence-interval half-length and the percentage of confidence intervals that actually covered the estimand $\theta$. (Note that the results in Tables 4 and 5 were computed from metaexperiments composed of $m = 1024$ basic experiments, and each basic experiment consisted of $n = 32$ simulation runs.) Clearly the percentage of exponentially distributed activities had no effect on confidence interval coverage, but it did affect the half-length of the confidence intervals. Since the worst-case behavior was observed with the percentage of exponentially distributed activities in the range 25%-50%, we fixed this factor at the 50% level throughout the rest of this study.

Table 4. Percentage reduction in half-length of nominal 90% confidence intervals for network 3 (relative dominance of $\pi_{(C)}$: 50%-70%).

<table>
<thead>
<tr>
<th>Percentage of exponential activities</th>
<th>$\hat{H}(0)$</th>
<th>$100[\hat{H}(0) - \hat{H}(k)]/\hat{H}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$k = 1$</td>
</tr>
<tr>
<td>0%</td>
<td>1.14</td>
<td>58.4</td>
</tr>
<tr>
<td>25%</td>
<td>4.76</td>
<td>22.7</td>
</tr>
<tr>
<td>50%</td>
<td>5.65</td>
<td>32.9</td>
</tr>
<tr>
<td>75%</td>
<td>9.00</td>
<td>61.4</td>
</tr>
<tr>
<td>100%</td>
<td>11.86</td>
<td>73.4</td>
</tr>
</tbody>
</table>
Table 5. Actual coverage percentages of nominal 90% confidence intervals for network 3 (relative dominance of \( \pi_{(i)} \): 50%-70%).

<table>
<thead>
<tr>
<th>Percentage of exponential activities</th>
<th>( k = 0 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>90.9</td>
<td>82.7</td>
<td>87.8</td>
<td>88.5</td>
<td>94.4</td>
</tr>
<tr>
<td>25%</td>
<td>87.8</td>
<td>79.1</td>
<td>76.1</td>
<td>80.5</td>
<td>87.2</td>
</tr>
<tr>
<td>50%</td>
<td>88.9</td>
<td>82.7</td>
<td>78.3</td>
<td>84.0</td>
<td>91.4</td>
</tr>
<tr>
<td>75%</td>
<td>89.2</td>
<td>83.6</td>
<td>77.6</td>
<td>84.9</td>
<td>94.0</td>
</tr>
<tr>
<td>100%</td>
<td>89.3</td>
<td>82.5</td>
<td>78.2</td>
<td>85.0</td>
<td>95.0</td>
</tr>
</tbody>
</table>

For each controlled point estimator of \( \theta \), the corresponding sample bias is summarized in Table 6 for each network configuration with 50% exponential activity durations. As detailed in Section 4, the results in Table 6 and in all subsequent tables were computed from metaexperiments composed of \( m = 4096 \) basic experiments, and each basic experiment consisted of \( n = 32 \) independent simulation runs. From Table 6 we see that the estimators \( \hat{Y}(\hat{\theta}) \) and \( \hat{Y}(\hat{\theta}) \) designed for normal responses [that is, \( \hat{\theta}(1) \) and \( \hat{\theta}(2) \), respectively] possess a marked negative bias; and the bias of \( \hat{Y}(\hat{\theta}) \) appears to be an order of magnitude greater in absolute value than the bias of the conventional controlled estimator \( \hat{Y}(\hat{\theta}) \). By itself, this bias may not have much practical significance since it is less than 3% of the estimand \( \theta \) for each network configuration studied; however, this bias may significantly affect the performance of the corresponding confidence-interval estimators. This issue will be elaborated in the discussion given below on confidence-interval estimators. Finally, we note that the jackknifed estimators \( \hat{\theta}(3) \) and \( \hat{\theta}(4) \), respectively] have negligible bias.

Table 7 summarizes the sample variance of each of the four controlled point estimators of \( \theta \). The estimator \( \hat{Y}(\hat{\theta}) \) based on the known covariance structure of the controls consistently displayed more variability than the conventional controlled estimator \( \hat{Y}(\hat{\theta}) \). It is interesting to note that the jackknifed point estimators \( \hat{J}(\hat{\theta}) \) and \( \hat{J}(\hat{\theta}) \) displayed nearly the same variability as their unjackknifed counterparts.

For each controlled point estimator of \( \theta \), the sample mean square error defined by equation (20) and displayed in Table 8 provides a single figure of merit that

<table>
<thead>
<tr>
<th>Network</th>
<th>Relative dominance of ( \pi_{(i)} )</th>
<th>( \hat{\theta}(k) - \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( k = 0 )</td>
<td>( k = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>20%-40%</td>
<td>0.017</td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>-0.047</td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>-0.214</td>
</tr>
<tr>
<td>2</td>
<td>20%-40%</td>
<td>0.615</td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>0.307</td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>0.558</td>
</tr>
<tr>
<td>3</td>
<td>20%-40%</td>
<td>-0.079</td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>-0.078</td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>-0.149</td>
</tr>
</tbody>
</table>
incorporates both the bias and variance of that estimator. In all cases the conventional point estimator $\hat{Y} (\hat{\theta})$ dominated the point estimator $\overline{Y} (\hat{\theta})$ based on the known covariance structure of the controls. Moreover, the jackknifed estimator $\overline{Y} (\hat{\theta})$ was clearly superior to $\overline{Y} (\hat{\theta})$ with respect to mean square error. These conclusions are to some extent surprising since $\overline{Y} (\hat{\theta})$ and $\overline{Y} (\hat{\theta})$ were specifically designed to exploit extra analytical information about the distribution of the control vector $C$.

Finally we consider the performance of the various controlled confidence-interval estimation procedures. Table 9 summarizes the percentage reduction (relative to direct simulation) of the average half-length of nominal 90% confidence intervals generated by each procedure for each network configuration with 50% exponential activities. Inspection of this table reveals that relative dominance of the critical path was a highly significant factor. The percentage reduction in average confidence-interval half-length generally increased with increasing levels of relative dominance of $\pi_{(G)}$. This is to be expected for the same reasons discussed at the beginning of this section—at higher levels of relative dominance, the control $P_{(G)}$ for the critical path was more highly correlated with the overall completion time $Y$.

Table 7. Estimated variance of controlled point estimators (50% exponential activities).

<table>
<thead>
<tr>
<th>Network</th>
<th>Relative dominance of $\pi_{(G)}$</th>
<th>$\hat{V}(k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20%-40%</td>
<td>14.3</td>
<td>5.94</td>
<td>11.1</td>
<td>6.54</td>
<td>11.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>42.8</td>
<td>4.10</td>
<td>25.1</td>
<td>5.11</td>
<td>22.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>188.8</td>
<td>1.53</td>
<td>103.2</td>
<td>2.28</td>
<td>80.6</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20%-40%</td>
<td>3521</td>
<td>318</td>
<td>1625</td>
<td>338</td>
<td>1594</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>12410</td>
<td>130</td>
<td>4819</td>
<td>144</td>
<td>4445</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>38041</td>
<td>45.3</td>
<td>14298</td>
<td>53.1</td>
<td>12970</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20%-40%</td>
<td>10.4</td>
<td>8.54</td>
<td>9.60</td>
<td>8.69</td>
<td>9.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>11.6</td>
<td>5.50</td>
<td>8.06</td>
<td>5.71</td>
<td>7.58</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>30.9</td>
<td>0.80</td>
<td>12.6</td>
<td>0.91</td>
<td>10.3</td>
<td></td>
</tr>
</tbody>
</table>

Table 8. Estimated mean square error of controlled point estimators (50% exponential activities).

<table>
<thead>
<tr>
<th>Network</th>
<th>Relative dominance of $\pi_{(G)}$</th>
<th>$\hat{MSE}(k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20%-40%</td>
<td>14.3</td>
<td>6.24</td>
<td>15.5</td>
<td>6.54</td>
<td>11.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>42.8</td>
<td>4.43</td>
<td>39.1</td>
<td>5.11</td>
<td>22.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>189.</td>
<td>1.70</td>
<td>153</td>
<td>2.30</td>
<td>80.7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20%-40%</td>
<td>3521</td>
<td>326.</td>
<td>1898</td>
<td>339</td>
<td>1594</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>12407</td>
<td>142</td>
<td>5737</td>
<td>150</td>
<td>4451</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>38032</td>
<td>66.8</td>
<td>16987</td>
<td>71.1</td>
<td>12989</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20%-40%</td>
<td>10.4</td>
<td>8.61</td>
<td>10.4</td>
<td>8.70</td>
<td>9.26</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50%-70%</td>
<td>11.6</td>
<td>5.56</td>
<td>9.56</td>
<td>5.71</td>
<td>7.58</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80%-100%</td>
<td>30.9</td>
<td>0.83</td>
<td>17.5</td>
<td>0.93</td>
<td>10.3</td>
<td></td>
</tr>
</tbody>
</table>
Table 9. Percentage reduction in average confidence-interval half-length for nominal 90% confidence intervals (50% exponential activities).

<table>
<thead>
<tr>
<th>Network</th>
<th>Relative dominance of ( \pi_{(k)} )</th>
<th>( \hat{H}(0) )</th>
<th>( 100(\hat{H}(0) - \hat{H}(k))/\hat{H}(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>( k = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>20%–40%</td>
<td>6.30</td>
<td>36.6</td>
</tr>
<tr>
<td></td>
<td>50%–70%</td>
<td>10.8</td>
<td>70.4</td>
</tr>
<tr>
<td></td>
<td>80%–100%</td>
<td>22.6</td>
<td>92.3</td>
</tr>
<tr>
<td>2</td>
<td>20%–40%</td>
<td>98.6</td>
<td>70.1</td>
</tr>
<tr>
<td></td>
<td>50%–70%</td>
<td>186.0</td>
<td>90.1</td>
</tr>
<tr>
<td></td>
<td>80%–100%</td>
<td>326.0</td>
<td>96.9</td>
</tr>
<tr>
<td>3</td>
<td>20%–40%</td>
<td>5.38</td>
<td>9.9</td>
</tr>
<tr>
<td></td>
<td>50%–70%</td>
<td>5.64</td>
<td>33.5</td>
</tr>
<tr>
<td></td>
<td>80%–100%</td>
<td>9.27</td>
<td>89.8</td>
</tr>
</tbody>
</table>

Table 10 summarizes the percentage of confidence intervals that actually covered the estimand \( \theta \) for each controlled estimation procedure and for each network configuration with 50% exponential activities. Since a total of \( m = 4096 \) confidence intervals were generated by each estimation procedure for each network configuration, the standard error of the estimated coverage probability \( \hat{I}(k) \) in Table 10 is

\[
SE[\hat{I}(k)] = \left( \frac{E[\hat{I}_{w}(k)][1 - E[\hat{I}_{w}(k)]]}{m} \right)^{1/2} \leq \left[ \frac{(0.5)^2}{4096} \right]^{1/2} = 0.0078. \quad (26)
\]

The experiment count \( m \) within each metaexperiment was specifically set to achieve the level of stability in the performance measure \( \hat{I}(k) \) prescribed by (26) so that \( 2 \cdot SE[\hat{I}(k)] \leq 0.015 \) for \( k = 1, 2, 3, 4 \). Moreover, if the nominal confidence level is actually achieved for the \( k \)th confidence-interval estimator so that \( E[\hat{I}(k)] = 0.90 \), then we have \( 2 \cdot SE[\hat{I}(k)] < 0.01 \).

Examination of Table 10 reveals that the level of relative dominance of the critical path had a more complex effect on coverage probability than on con-

Table 10. Actual coverage percentages for nominal 90% confidence intervals (50% exponential activities).

<table>
<thead>
<tr>
<th>Network</th>
<th>Relative dominance of ( \pi_{(k)} )</th>
<th>( k = 0 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20%–40%</td>
<td>89.5</td>
<td>85.4</td>
<td>74.7</td>
<td>89.2</td>
<td>91.8</td>
</tr>
<tr>
<td></td>
<td>50%–70%</td>
<td>88.5</td>
<td>81.8</td>
<td>68.3</td>
<td>87.4</td>
<td>94.0</td>
</tr>
<tr>
<td></td>
<td>80%–100%</td>
<td>88.0</td>
<td>67.7</td>
<td>67.5</td>
<td>70.4</td>
<td>93.7</td>
</tr>
<tr>
<td>2</td>
<td>20%–40%</td>
<td>88.8</td>
<td>86.3</td>
<td>81.6</td>
<td>87.9</td>
<td>94.4</td>
</tr>
<tr>
<td></td>
<td>50%–70%</td>
<td>88.8</td>
<td>79.1</td>
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<td>80%–100%</td>
<td>88.8</td>
<td>63.4</td>
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<td>3</td>
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<td>88.4</td>
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<td>80%–100%</td>
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ence-interval half-length. The conventional control-variate estimators (6) and (12) [that is, \( \hat{\Theta}(1) \) and \( \hat{\Theta}(3) \), respectively] performed poorly at high levels of relative dominance of \( \pi_{(i)} \); note particularly the performance in network 3 with relative dominance of \( \pi_{(i)} \) in the range 80%-100%. In this latter case, the critical-path control \( P_{(i)} \) virtually always coincided with the overall completion time \( Y \) so that the resulting confidence intervals of the form (6) and (12) were either very small or degenerate; and when this phenomenon was coupled with the bias induced by the marked degree of nonnormality in the response and the controls, the net effect was a substantial loss of confidence-interval coverage. With respect to the confidence-interval estimation procedures based on the known covariance matrix of the controls, estimator (10) [that is, \( \hat{\Theta}(2) \)] also suffered from this effect while its jackknifed version (14) [that is, \( \hat{\Theta}(4) \)] did not. Bauer [2] observed similar performance of the conventional controlled confidence-interval estimator (6) in a variety of queueing network simulations.

Several conclusions emerged from our comparison of the performance of the four controlled confidence-interval procedures. The conventional estimator (6) and its jackknifed version (12) yielded consistently larger half-length reductions than the corresponding estimators (10) and (14) based on the known covariance structure of the controls. Half-length reductions of up to 97% were observed with (6) and (12). However, such reductions were not realized without cost. The actual coverage probability achieved by estimators (6), (10), and (12) fell more than two standard errors below the nominal level 0.90 in every case reported in Table 10 except for the jackknifed estimator (12) in network 1 with relative dominance of \( \pi_{(i)} \) in the range 20%-40%. On the other hand, the jackknifed estimator (14) achieved at least the nominal coverage level 0.90 for every network configuration on which it was tested. Of course, the reductions in confidence-interval half-length achieved by (14) were much more modest than the reductions achieved by the other controlled estimation procedures.

6. CONCLUSIONS

In this article we have examined four control-variate estimation procedures to be used in lieu of the conventional direct-simulation analysis of stochastic activity networks. All of these procedures are designed to improve upon the performance of direct simulation with respect to the accuracy of both point and confidence-interval estimators of mean completion time. As a fundamental principle for evaluating the performance improvements yielded by each of these procedures, we believe that confidence-interval coverage must be maintained at its nominal level while optimizing the accuracy of the corresponding point and confidence-interval estimators; thus it is unacceptable to achieve a large improvement in point-estimator accuracy at the expense of a significant loss of confidence-interval coverage. The second basic consideration in evaluating the performance of these control-variate estimation procedures is the additional computational overhead that they incur. In the simulation study reported here, the computational cost of these controlled estimation procedures was negligible compared to the cost of simulating the stochastic activity networks; moreover, the cost of the controlled estimation procedures was very insensitive to the size of the selected networks. Thus considerable efficiency gains can be realized by
the use of path control variates in large-scale applications. To estimate the mean completion time of a stochastic activity network, we recommend the use of (13) and (14) with these path controls because the resulting estimators can yield moderate to substantial variance reductions without significant degradation of coverage. Research is currently in progress to develop refined point and confidence-interval estimators that make more effective use of the known covariance structure of the path controls and that are robust against departures from joint normality in the response and the path controls.

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REFERENCES


