Steady-state Simulation Analysis Using ASAP3*

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I. Introduction: The Method of Nonoverlapping Batch Means

- Given the simulation output \( \{X_i : i = 1, \ldots, n\} \), we seek a \( 100(1 - \alpha)\% \) confidence interval (CI) for the associated steady-state mean \( \mu_X \).

- Method of Nonoverlapping Batch Means (NBM)—divide the output \( \{X_i : i = 1, \ldots, n\} \) into \( k \) adjacent batches each of size \( m \) (thus \( n = km \)) and compute the \( j \)th batch mean,

\[
Y_j (m) = \frac{1}{m} \sum_{i=m(j-1)+1}^{mj} X_i \quad \text{for} \quad j = 1, \ldots, k,
\]

together with the grand mean of the batch means,

\[
\overline{Y} = \overline{Y}(m, k) = \overline{X}(n) = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{k} \sum_{j=1}^{k} Y_j(m), \quad (1)
\]
and the sample variance of the batch means,

\[ S_{m,k}^2 = \frac{1}{k-1} \sum_{j=1}^{k} \left[ Y_j(m) - \overline{Y}(m,k) \right]^2, \]  
\[ (2) \]

to yield the usual 100(1 − \alpha)\% CI for \( \mu_x \),

\[ \overline{Y}(m,k) \pm t_{1-\alpha/2,k-1} \frac{S_{m,k}}{\sqrt{k}}. \]  
\[ (3) \]

- As \( m \to \infty \) with \( k \) fixed (so that \( n \to \infty \)), the NBM CI (3) has coverage probability \( 1 - \alpha \), and thus is asymptotically valid under mild conditions on the output process \( \{X_i\} \).
II. Overview of ASAP3

ASAP3 (Steiger et al., *ACM Transactions on Modeling and Computer Simulation*, 2005) is a refinement of ASAP (Steiger and Wilson, *Management Science*, 2002) and ASAP2 (Steiger et al., *Proceedings of the 2002 Winter Simulation Conference*).

ASAP3 requires the following user-supplied inputs:

- the output process \( \{X_i : i = 1, 2, \ldots, n\} \) from which the steady-state mean \( \mu_X \) is to be estimated;

- the desired CI coverage probability \( 1 - \alpha \), where \( 0 < \alpha < 1 \); and

- an upper bound \( H^* \) on the final CI half-length, expressed
  - in absolute terms as the maximum acceptable half-length; or
  - in relative terms as the maximum acceptable fraction \( r^* \) of the magnitude of the CI midpoint \( \overline{Y} \).
ASAP3 delivers the following outputs:

- a nominal $100(1 - \alpha)\%$ CI for $\mu_X$ having the form
  \[ \bar{Y} \pm H, \quad \text{where} \quad H \leq H^*, \]
  provided no additional simulation is required; or

- a new, larger sample size $n$ to be supplied to the procedure.
ASAP3 builds a correlation-adjusted CI as follows:

- by taking an inverse Cornish-Fisher expansion for the usual $t$-ratio

$$ t = \left[ \overline{Y}(m, k') - \mu_X \right] \sqrt{\frac{S^2_{m,k'}}{k'}} $$  \hspace{1cm} (4)

using the parameter estimators obtained by fitting an AR(1) model to a truncated set of $k'$ batch means so as to compute the following:

- $\hat{\kappa}_2$ and $\hat{\kappa}_4$, estimators of the 2nd and 4th cumulants, respectively, of the $t$-ratio (4);
- $\widehat{\text{Var}}[Y(m)]$, an estimator of the variance of the batch means; and

- then computing the correlation-adjusted $100(1 - \alpha)\%$ CI for $\mu_X$,

$$ \overline{Y}(m, k') \pm \left( \frac{1}{2} + \frac{1}{2} \hat{\kappa}_2 - \frac{1}{8} \hat{\kappa}_4 + \frac{1}{24} \hat{\kappa}_4 \sqrt{z_{1-\alpha/2}^2} \right) \frac{\sqrt{\widehat{\text{Var}}[Y(m)]}}{k'} \right) z_{1-\alpha/2} = \frac{\text{Correlation adjustment}}{\text{Half-length of usual batch means CI}}.$$  \hspace{1cm} (5)
A. Steps in the Operation of ASAP3

[0] Divide the initial sample into \( k = 256 \) batches with user-defined initial batch size \( m \) (where \( m = 16 \) by default), skip the first 4 batches, and compute batch means for the remaining \( k' = k - 4 = 252 \) batches.

[1] Begin each new iteration of ASAP3 by

- collecting additional data (if any) required by the previous iteration; and
- computing the full set of batch means \( \{Y_j(m) : j = 1, \ldots, k\} \) using the current batch size \( m \).
[2] From the current set of $k' = k - 4$ batch means $\{Y_j(m) : j = 5, \ldots, k\}$ that were truncated by skipping the first 4 batches to eliminate initialization bias, select every other group of 4 consecutive batch means to test for multivariate normality.

- If the resulting spaced batch means fail the Shapiro-Wilk multivariate normality test, then increase the batch size $m$ according to

$$m \leftarrow \left\lfloor \sqrt{2m} \right\rfloor$$

and go to step [1] to begin a new iteration.

- If the spaced batch means pass the Shapiro-Wilk multivariate normality test, then go to step [3].
[3] Fit a first-order autoregressive (AR(1)) model,

\[ \tilde{Y}_\ell = \varphi \tilde{Y}_{\ell-1} + a_\ell \quad \text{for} \quad \ell = 1, 2, \ldots, \]

(6)

to the current set of truncated batch means, where:

- the centered batch means are
  \[ \tilde{Y}_\ell \equiv Y_{\ell+4}(m) - \mu_X \quad \text{for} \quad \ell = 1, \ldots, k'; \]

- the autoregressive parameter
  \[ \varphi = \text{Corr}(\tilde{Y}_\ell, \tilde{Y}_{\ell-1}) = \text{Corr}(Y_\ell, Y_{\ell-1}); \]

and

- the residuals \( \{a_\ell\} \) are i.i.d. \( N(0, \sigma_a^2) \).
Test the null hypothesis

\[ \varphi \leq 0.8 \]

versus the alternative hypothesis

\[ \varphi > 0.8. \]

• If the null hypothesis is rejected, then
  – increase the batch size \( m \) by a factor projected to reduce \( \varphi \) to 0.80 on the next iteration; and
  – go to step [1] to begin a new iteration.

• If the null hypothesis is accepted, then go to step [4].
[4] Build a correlation-adjusted CI based on an inverse Cornish-Fisher expansion for the usual $t$-ratio

$$t = \left[ \frac{\bar{Y}(m, k') - \mu_X}{\sqrt{S^2_{m,k'}/k'}} \right]$$

using estimates $\hat{\phi}$ and $\hat{\sigma_a^2}$ for the AR(1) model (6) fitted to the current set of $k'$ truncated batch means so as to compute:

- $\hat{\kappa}_2$ and $\hat{\kappa}_4$, estimators of the 2nd and 4th cumulants, respectively, of the usual $t$-ratio; and
- $\hat{\text{Var}}[Y(m)]$, an estimator of the variance of the batch means.

From these quantities, compute the $100(1 - \alpha)\%$ CI for $\mu_X$,

$$\bar{Y}(m, k') \pm \left[ \left( \frac{1}{2} + \frac{1}{2} \hat{\kappa}_2 - \frac{1}{8} \hat{\kappa}_4 + \frac{1}{24} \hat{\kappa}_4 z^2_{1-\alpha/2} \right) \right] z_{1-\alpha/2} \sqrt{\frac{\hat{\text{Var}}[Y(m)]}{k'}}.$$
[5] If the half-length $H$ of the current CI satisfies the precision requirement

$$H \leq H^*, \quad (7)$$

then deliver that CI and stop.

If (7) is not satisfied, then do the following:

• Estimate additional batches needed to satisfy (7),

$$k'' = \max\left\{\left\lceil \left(\frac{H}{H^*}\right)^2 k' \right\rceil - k', 1\right\}.$$

• If $k' + k'' \leq 1,504$, then update the batch count

$$k' \leftarrow k' + k'',$$

leave the batch size $m$ unchanged, and go to step [1] to begin a new iteration.
• If $k' + k'' > 1,504$, then update the batch size $m$ by the multiplier $\theta$ that is the root of the equation

$$\theta \left(1 - \hat{\varphi}_+^\theta\right)^2 = \left(\frac{H}{H^*}\right)^2 \left(1 - \hat{\varphi}_+\right)^2,$$

where

$$\hat{\varphi}_+ \equiv \max\{0, \hat{\varphi}\}$$

so that we take

$$m \leftarrow \left\lceil \text{mid}\left(\sqrt{2}, \theta, 4\right)m \right\rceil;$$

leave the batch count $k$ unchanged; and go to [1] to begin a new iteration.
B. Flow Chart of ASAP3

Start

- Collect observations; compute batch means statistics

  - Stationary multivariate normality test passed?
    - Yes: Fit AR(1) model to batch means
    - No: Update test level \( \delta \)

  - Retain old batch count; compute new batch size

- Compute new batch size

- Collect observations; compute batch means statistics

- Retain old batch count; compute new batch size

  - AR(1) parameter \( \phi \leq 0.8 \)?
    - No: Compute effective degrees of freedom and inverse Cornish-Fisher expansion for \( t \)-ratio
    - Yes: Construct correlation-adjusted CI

- Deliver CI; Stop

  - CI meets precision requirements?
    - Yes: Use new batch count and retain old batch size
    - No: Compute new batch count
Algorithmic Statement of ASAP3

[0] Set iteration index $i \leftarrow 1$, $m_1 \leftarrow$ user-specified initial batch size (default = 16), initial batch count $k_1 \leftarrow 256$, initial sample size $n_1 \leftarrow k_1 m_1$ with $n_0 \leftarrow 0$, truncated initial batch count $k'_1 \leftarrow k_1 - 4$, $1 - \alpha \leftarrow$ user-specified CI coverage probability (default = 0.90), size of test for autoregressive parameter $\alpha_{\text{arp}} \leftarrow 0.01$, initial size of test for stationary multivariate normality $\delta_1 \leftarrow 0.1$ with parameter $\omega \leftarrow 0.18421$ controlling the test size in step [2] on subsequent iterations, and indicator that normality test was passed $\text{MVTestPassed} \leftarrow \text{‘no’}$;

if a relative precision requirement is given, then set $\text{RelPrec} \leftarrow \text{‘yes’}$ and $r^* \leftarrow$ the user-specified fraction of the magnitude of the CI midpoint that defines the maximum acceptable CI half-length;

if an absolute precision requirement is given, then set $\text{RelPrec} \leftarrow \text{‘no’}$ and $H^* \leftarrow$ the user-specified maximum acceptable CI half-length;

if no precision level is specified then set $\text{RelPrec} \leftarrow \text{‘no’}$, $r^* \leftarrow 0$, and $H^* \leftarrow 0$. 
Algorithmic Statement of ASAP3 (Continued)

[1] Start (or restart) the simulation to generate the data \( \{X_j : j = n_{i-1} + 1, \ldots, n_i\} \) required for the current iteration \( i \);

Compute the \( k_i \) batch means \( \{Y_{j(m_i)} : j = 1, \ldots, k_i\} \); and after skipping the initial spacer \( \{Y_1(m_i), Y_2(m_i), Y_3(m_i), Y_4(m_i)\} \), compute the truncated grand mean,

\[
\bar{Y}(m_i, k'_i) \leftarrow \frac{1}{k'_i m_i} \sum_{\ell=4m_i+1}^{n_i} X_\ell = \frac{1}{k'_i} \sum_{j=5}^{k_i} Y_{j(m_i)};
\]

if \( \text{MVTestPassed} = \text{‘yes’} \), then goto [3].
Algorithmic Statement of ASAP3 (Continued)

[2] From the truncated batch means \( \{ Y_j(m_i) : j = 5, \ldots, k_i \} \), select every other group of four successive batch means to build the \( 4 \times 1 \) vectors

\[
\{ y_\ell = [Y_{5+(\ell-1)8}(m_i), Y_{6+(\ell-1)8}(m_i), Y_{7+(\ell-1)8}(m_i), Y_{8+(\ell-1)8}(m_i)]^T : \ell = 1, \ldots, 32 \};
\]

To test the hypothesis

\( \mathcal{H}_{\text{mvn}} : \{ y_\ell : \ell = 1, \ldots, 32 \} \) are i.i.d. four-dimensional normal random vectors,

evaluate \( \delta_i = \delta_1 \exp[-\omega(i - 1)^2] \), the significance level for the test, and \( W_i^* \), the multivariate Shapiro-Wilk statistic computed from the \( \{ y_\ell \} \) according to equations (10)–(12) of Steiger et al. (2004);

if \( W_i^* < w_{\delta_i}^* \), the \( \delta_i \) quantile of the distribution of \( W_i^* \) under the null hypothesis \( \mathcal{H}_{\text{mvn}} \), so that \( \mathcal{H}_{\text{mvn}} \) is rejected at significance level \( \delta_i \), then

set \( i \leftarrow i + 1, \ k_i \leftarrow 256, \ k_i' \leftarrow k_i - 4, \ m_i \leftarrow \left\lfloor \sqrt{2m_{i-1}} \right\rfloor, \) and \( n_i \leftarrow k_i m_i \);

goto [1];

else

set MVTestPassed \leftarrow ‘yes’;

goto [3].
Algorithmic Statement of ASAP3 (Continued)

[3] Fit an AR(1) model (6) to the truncated batch means \( \{ Y_j(m_i) : j = 5, \ldots, k_i \} \) so as to obtain the estimator \( \hat{\varphi} \) of the autoregressive parameter \( \varphi \);

Test the hypothesis \( H_{\text{arp}} : \varphi \leq 0.8 \) at the level of significance \( \alpha_{\text{arp}} \) by checking for the condition

\[
\hat{\varphi} \leq \sin \left( 0.927 - z_{1-\alpha_{\text{arp}}} / \sqrt{k_i'} \right);
\]

if \( H_{\text{arp}} \) is rejected at significance level \( \alpha_{\text{arp}} \), then

set \( \theta \leftarrow \text{mid} \left\{ \sqrt{2}, \ln \left[ \sin \left( 0.927 - z_{1-\alpha_{\text{arp}}} / \sqrt{k_i'} \right) \right] / \ln (\hat{\varphi}), 4 \right\} \),

\( i \leftarrow i + 1, \ k_i \leftarrow k_{i-1}, \ k_i' \leftarrow k_i - 4, \ m_i \leftarrow \lceil \theta m_{i-1} \rceil, \) and \( n_i \leftarrow k_i m_i; \) goto [1];

else

goto [4].
Algorithmic Statement of ASAP3 (Continued)

[4] Using the estimators $\hat{\varphi}$ and $\hat{\sigma}_a^2$ for the AR(1) model (6), compute $\hat{\text{Var}}[Y(m_i)]$ and $\hat{\text{Var}}[\bar{Y}(m_i, k_i')]$ from equations (15)–(16) of Steiger et al. (2004);

For the NOBM $t$-ratio (4), compute the estimated effective degrees of freedom $\hat{\nu}_{\text{eff}}$ from equation (33) of Steiger et al. (2004);

Compute $\hat{\kappa}_2$ and $\hat{\kappa}_4$, the estimators, respectively, of the second and fourth cumulants of the $t$-ratio (4), by inserting $\hat{\text{Var}}[Y(m_i)]$, $\hat{\text{Var}}[\bar{Y}(m_i, k_i')]$, and $\hat{\nu}_{\text{eff}}$ into the computing expressions for $\kappa_2$ and $\kappa_4$ given in equations (31)–(32) of Steiger et al. (2004);

Calculate the half-length of the correlation-adjusted CI,

$$H \leftarrow \left[ \left( \frac{1}{2} + \frac{1}{2} \hat{\kappa}_2 - \frac{1}{8} \hat{\kappa}_4 + \frac{1}{24} \hat{\kappa}_4^2 \right) z_{1 - \alpha/2} \right] \frac{\sqrt{\hat{\text{Var}}[Y(m_i)]}}{k'}$$

Construct the correlation-adjusted CI,

$$\bar{Y}(m_i, k_i') \pm H.$$
Algorithmic Statement of ASAP3 (Continued)

[5] if RelPrec='yes' then set $H^* \leftarrow r^* |\bar{Y}(m_i, k'_i)|$

    if $(H \leq H^*)$ or $(r^* = 0$ and $H^* = 0)$, then

        deliver $\bar{Y}(m_i, k'_i) \pm H$ and stop;

    else

        Estimate additional batches needed to satisfy the precision requirement (7),

        \[ k'' = \max \left\{ \left\lceil \left( \frac{H}{H^*} \right)^2 k'_i \right\rceil - k'_i, \ 1 \right\} ; \]

        If $k_i + k'' \leq 1,504$, then

        set $i \leftarrow i + 1$,  $k_i \leftarrow k_{i-1} + k''$,  $k'_i \leftarrow k_i - 4$,  $m_i \leftarrow m_{i-1}$,

        and $n_i \leftarrow m_i k_i$;  goto [1];

    else

        Find the root $\theta$ of the equation

        \[ \theta \left( 1 - \hat{\varphi}_+ \right)^2 = \left( \frac{H}{H^*} \right)^2 \left( 1 - \hat{\varphi}_+ \right)^2 , \]

        set $\theta \leftarrow \text{mid} \left( \sqrt{2}, \theta, 4 \right)$,  $i \leftarrow i + 1$,  $k_i \leftarrow k_{i-1}$,  $k'_i \leftarrow k_i - 4$,

        $m_i \leftarrow \lceil \theta m_{i-1} \rceil$,  and $n_i \leftarrow m_i k_i$;  goto [1].
III. Experimental Performance Evaluation

A. Comparison: ASAP, ASAP3, and the Law & Carson (LC) Procedure


Overall Objective of Law & Carson (LC) Procedure

Deliver $k = 40$ batches of a batch size $m''$ sufficiently large to ensure the lag-one correlation between the batch means satisfies

$$\text{Corr} \left[ Y_j(m''), Y_{j+1}(m'') \right] \leq 0.05$$

while the associated $100(1 - \alpha)\%$ confidence interval for $\mu_X$, 

$$\bar{Y}(m'', k) \pm H_{\alpha,k} \text{ with } H_{\alpha,k} \equiv t_{1-\alpha/2,k-1} \frac{S_{m'',k}}{\sqrt{k}},$$

satisfies the user-specified relative precision requirement

$$\left| H_{\alpha,k} / \bar{Y}(m'', k) \right| \leq r^*.$$
## Algorithmic Statement of Law & Carson (LC) Procedure

**Step 0.** Set the positive integers $\ell \leftarrow 10$, $k \leftarrow 40$, $n_0 \leftarrow 600$, and $n_1 \leftarrow 800$, where $k' \equiv \ell k = 400$ and $k'' \equiv k'/2 = 200$. Set the stopping value $c \leftarrow 0.40$ and the relative precision $r^* > 0$; set $i \leftarrow 1$; and collect $n_1$ observations.

**Step 1.**

a. Divide the overall data set of $n_i$ observations into $k'$ batches of size $m \equiv n_i/k'$. Compute estimated lag-1 correlation $\tilde{\rho}_1(k', m)$ of the batch means $\{Y_j(m) : j = 1, \ldots, k'\}$. If $\tilde{\rho}_1(k', m) \geq c$, then go to **Step 2**. If $\tilde{\rho}_1(k', m) \leq 0$, then go to **Step 1c**. Otherwise, go to **Step 1b**.

b. Divide the overall data set of size $n_i$ into $k'' \equiv k'/2$ batches of size $m' \equiv 2m$. Compute the estimated lag-1 correlation $\tilde{\rho}_1(k'', m')$ of the batch means $\{Y_j(m') : j = 1, \ldots, k''\}$. If $\tilde{\rho}_1(k'', m') < \tilde{\rho}_1(k', m)$, then go to **Step 1c**. Otherwise, go to **Step 2**.

c. Divide the overall data set of $n_i$ observations into $k$ batches of size $m'' \equiv \ell m$. Compute $\bar{Y}(m'', k)$ and $S^2_{m'', k}$ as in (1) and (2). Compute the half-length $H_{\alpha,k} = t_{1-\alpha/2,k-1}S_{m'', k}/\sqrt{k}$ of the CI (3). If $\left| H_{\alpha,k}/\bar{Y}(m'', k) \right| \leq r^*$, then deliver (3) and stop. Otherwise, go to **Step 2**.

**Step 2.** Update the iteration counter $i \leftarrow i + 1$ and the total sample size $n_i \leftarrow 2n_i - 2$. Collect the additional $n_i - n_{i-1}$ observations and go to **Step 1a**.
B. Test Processes

- Response Times in Central Server Model 3 of Law and Carson (1979)
- Waiting Times in $M/M/1$ Queue with Utilization 0.8
- Waiting Times in $M/H_2/1$ Queue with Utilization 0.8
- Waiting Times in $M/M/1$ Queue with Utilization 0.8
- Waiting Time in $M/M/1/M/1$ Queue with Utilization 0.8 at Each Station
- Reward in Two-State Markov Chain with $P_{00} = P_{11} = 0.99$
C. Performance Measures

• Confidence Interval (CI) Properties
  ▶ Empirical Coverage Probability
  ▶ Average Relative Precision
  ▶ Average Half-Length
  ▶ Variance of Half-Length

• Average Required Sample Size

D. Analysis of Results

Complete results are available via

<ftp.ncsu.edu/pub/eos/pub/jwilson/asap3pe.pdf>
Table 1: Performance of NBM Procedures for the $M/M/1/LIFO$ Queue Waiting Time Process with Utilization $\tau = 0.8$ Based on Independent Replications of Nominal 90% CIs

<table>
<thead>
<tr>
<th>Precision Requirement</th>
<th>Nominal 90% CIs</th>
<th>ASAP</th>
<th>ASAP3</th>
<th>LC</th>
</tr>
</thead>
<tbody>
<tr>
<td>NO PRECISION</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># replications</td>
<td>100</td>
<td>400</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>avg. sample size</td>
<td>5,025</td>
<td>53,958</td>
<td>3,120</td>
<td></td>
</tr>
<tr>
<td>coverage</td>
<td>72.0%</td>
<td>87.0%</td>
<td>64.0%</td>
<td></td>
</tr>
<tr>
<td>avg. rel. precision</td>
<td>0.210</td>
<td>0.082</td>
<td>0.236</td>
<td></td>
</tr>
<tr>
<td>avg. CI half length</td>
<td>0.652</td>
<td>0.261</td>
<td></td>
<td></td>
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<tr>
<td>var. CI half length</td>
<td>0.074</td>
<td>0.106</td>
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<tr>
<td>±15% PRECISION</td>
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<tr>
<td># replications</td>
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<td>400</td>
<td>100</td>
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</tr>
<tr>
<td>avg. sample size</td>
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<td>54,017</td>
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<tr>
<td>coverage</td>
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<td>86.8%</td>
<td>76.0%</td>
<td></td>
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<tr>
<td>avg. rel. precision</td>
<td>0.119</td>
<td>0.081</td>
<td>0.131</td>
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<tr>
<td>avg. CI half length</td>
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<tr>
<td>var. CI half length</td>
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<tr>
<td>±7.5% PRECISION</td>
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<tr>
<td># replications</td>
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<tr>
<td>avg. sample size</td>
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<tr>
<td>coverage</td>
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<td>87.5%</td>
<td>84.0%</td>
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<tr>
<td>avg. rel. precision</td>
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<tr>
<td>avg. CI half length</td>
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<tr>
<td>var. CI half length</td>
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<td>5.1E–4</td>
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Table 2: Performance of NBM Procedures for the $M/H_2/1$ Queue Waiting Time Process with Utilization $\tau = 0.8$ Based on Independent Replications of Nominal 90% CIs

<table>
<thead>
<tr>
<th>Precision Requirement</th>
<th>Nominal 90% CIs</th>
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<tbody>
<tr>
<td></td>
<td>ASAP</td>
<td>ASAP3</td>
<td>LC</td>
</tr>
<tr>
<td>NO PRECISION</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># replications</td>
<td>100</td>
<td>400</td>
<td>100</td>
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<tr>
<td>avg. sample size</td>
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<td>86,144</td>
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<tr>
<td>coverage</td>
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<td>87.8%</td>
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<tr>
<td>avg. rel. precision</td>
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<tr>
<td>avg. CI half-length</td>
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</tr>
<tr>
<td>var. CI half-length</td>
<td>43.730</td>
<td>0.5962</td>
<td></td>
</tr>
<tr>
<td>±15% PRECISION</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># replications</td>
<td>100</td>
<td>400</td>
<td>100</td>
</tr>
<tr>
<td>avg. sample size</td>
<td>148,820</td>
<td>76,214</td>
<td>86,144</td>
</tr>
<tr>
<td>coverage</td>
<td>88.0%</td>
<td>88.0%</td>
<td>88.0%</td>
</tr>
<tr>
<td>avg. rel. precision</td>
<td>0.102</td>
<td>0.1308</td>
<td>0.106</td>
</tr>
<tr>
<td>avg. CI half-length</td>
<td>0.802</td>
<td>1.0329</td>
<td></td>
</tr>
<tr>
<td>var. CI half-length</td>
<td>0.055</td>
<td>0.0273</td>
<td></td>
</tr>
<tr>
<td>±7.5% PRECISION</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># replications</td>
<td>100</td>
<td>400</td>
<td>100</td>
</tr>
<tr>
<td>avg. sample size</td>
<td>405,854</td>
<td>228,482</td>
<td>229,632</td>
</tr>
<tr>
<td>coverage</td>
<td>93.0%</td>
<td>90.0%</td>
<td>90.0%</td>
</tr>
<tr>
<td>avg. rel. precision</td>
<td>0.053</td>
<td>0.07054</td>
<td>0.067</td>
</tr>
<tr>
<td>avg. CI half-length</td>
<td>0.421</td>
<td>0.5623</td>
<td></td>
</tr>
<tr>
<td>var. CI half-length</td>
<td>0.021</td>
<td>2.0E–3</td>
<td></td>
</tr>
</tbody>
</table>
Table 3: Performance of NBM Procedures for the $M/M/1/M/1$ Queue Waiting Time Process with Utilization $\tau = 0.8$ at Each Station Based on Independent Replications of Nominal 90% CIs

<table>
<thead>
<tr>
<th>Precision Requirement</th>
<th>Nominal 90% CIs</th>
<th>ASAP</th>
<th>ASAP3</th>
<th>LC</th>
</tr>
</thead>
<tbody>
<tr>
<td>NO PRECISION</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># replications</td>
<td>100</td>
<td>400</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>avg. sample size</td>
<td>3,152</td>
<td>19,133</td>
<td>3,120</td>
<td></td>
</tr>
<tr>
<td>coverage</td>
<td>85.0%</td>
<td>91.0%</td>
<td>64.0%</td>
<td></td>
</tr>
<tr>
<td>avg. rel. precision</td>
<td>0.454</td>
<td>0.154</td>
<td>0.236</td>
<td></td>
</tr>
<tr>
<td>avg. CI half-length</td>
<td>3.250</td>
<td>0.983</td>
<td></td>
<td></td>
</tr>
<tr>
<td>var. CI half-length</td>
<td>14.06</td>
<td>0.165</td>
<td></td>
<td></td>
</tr>
<tr>
<td>±15% PRECISION</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># replications</td>
<td>100</td>
<td>400</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>avg. sample size</td>
<td>46,610</td>
<td>25,522</td>
<td>13,944</td>
<td></td>
</tr>
<tr>
<td>coverage</td>
<td>93.0%</td>
<td>91.5%</td>
<td>76.0%</td>
<td></td>
</tr>
<tr>
<td>avg. rel. precision</td>
<td>0.103</td>
<td>0.119</td>
<td>0.131</td>
<td></td>
</tr>
<tr>
<td>avg. CI half-length</td>
<td>0.649</td>
<td>0.755</td>
<td></td>
<td></td>
</tr>
<tr>
<td>var. CI half-length</td>
<td>0.030</td>
<td>0.037</td>
<td></td>
<td></td>
</tr>
<tr>
<td>±7.5% PRECISION</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># replications</td>
<td>100</td>
<td>400</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>avg. sample size</td>
<td>117,339</td>
<td>58,844</td>
<td>49,920</td>
<td></td>
</tr>
<tr>
<td>coverage</td>
<td>90.0%</td>
<td>91.3%</td>
<td>87.0%</td>
<td></td>
</tr>
<tr>
<td>avg. rel. precision</td>
<td>0.050</td>
<td>0.069</td>
<td>0.065</td>
<td></td>
</tr>
<tr>
<td>avg. CI half-length</td>
<td>0.318</td>
<td>0.441</td>
<td></td>
<td></td>
</tr>
<tr>
<td>var. CI half-length</td>
<td>0.008</td>
<td>1.7E–3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Efficiency Analysis

In the “ideal” case that \( \{ X_i : i = 1, 2, \ldots \} \) is stationary and Gaussian with known steady-state variance parameter (SSVP)

\[
\gamma_X \equiv \lim_{n \to \infty} n \text{Var} \left[ \frac{\bar{X}(n)}{n} \right] = \sum_{\ell = -\infty}^{\infty} \text{Cov}(X_i, X_{i+\ell}),
\]

the nominal \( 100(1 - \alpha)\% \) CI, \( \bar{X}(n) \pm z_{1-\alpha/2} \sqrt{\gamma_X/n} \), is asymptotically valid:

\[
\lim_{n \to \infty} \Pr \left\{ \mu_X \in \bar{X}(n) \pm z_{1-\alpha/2} \sqrt{\gamma_X/n} \right\} = 1 - \alpha.
\]

Thus in the ideal case, an efficient procedure yielding an asymptotically valid \( 100(1 - \alpha)\% \) CI for \( \mu_X \) with relative precision \( r^* \) will require

\[
n^* = \frac{z^2_{1-\alpha/2} \gamma_X}{(r^* \mu_X)^2}
\]
observations.
Table 4: Comparison of ASAP3’s Average Sample Size $\bar{n}$ with the Efficient Sample Size $n^*$ Required by an Asymptotically Valid 90% CI for $\mu_X$ with Relative Precision $r^*$

<table>
<thead>
<tr>
<th>Output Process</th>
<th>$r^*$</th>
<th>$n^*$</th>
<th>$\bar{n}$</th>
<th>$\bar{n}/n^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M/M/1$ Queue Waiting Times</td>
<td>15%</td>
<td>53,306</td>
<td>103,742</td>
<td>1.946</td>
</tr>
<tr>
<td>$\tau = 0.9, \mu_X = 9, \gamma_X = 35,901$</td>
<td>7.5%</td>
<td>213,222</td>
<td>287,568</td>
<td>1.349</td>
</tr>
<tr>
<td>(Steiger et al., 2005)</td>
<td>3.75%</td>
<td>852,886</td>
<td>969,011</td>
<td>1.136</td>
</tr>
<tr>
<td>$M/M/1$ Queue Waiting Times</td>
<td>15%</td>
<td>14,853</td>
<td>43,796</td>
<td>2.949</td>
</tr>
<tr>
<td>$\tau = 0.8, \mu_X = 3.2, \gamma_X = 1,264.64$</td>
<td>7.5%</td>
<td>59,412</td>
<td>72,060</td>
<td>1.213</td>
</tr>
<tr>
<td></td>
<td>3.75%</td>
<td>237,650</td>
<td>256,186</td>
<td>1.078</td>
</tr>
<tr>
<td>$M/H_2/1$ Queue Waiting Times</td>
<td>15%</td>
<td>45,486</td>
<td>76,214</td>
<td>1.676</td>
</tr>
<tr>
<td>$\tau = 0.8, \mu_X = 8, \gamma_X = 24,204.8$</td>
<td>7.5%</td>
<td>181,942</td>
<td>228,482</td>
<td>1.256</td>
</tr>
<tr>
<td></td>
<td>3.75%</td>
<td>727,765</td>
<td>798,234</td>
<td>1.097</td>
</tr>
<tr>
<td>Cost Function of Highly Correlated 2-state DTMC</td>
<td>15%</td>
<td>1,323</td>
<td>9,553</td>
<td>7.221</td>
</tr>
<tr>
<td>$\mu_X = 7.5, \gamma_X = 618.75$</td>
<td>7.50%</td>
<td>5,292</td>
<td>12,283</td>
<td>2.321</td>
</tr>
<tr>
<td></td>
<td>3.75%</td>
<td>21,168</td>
<td>46,469</td>
<td>2.195</td>
</tr>
<tr>
<td></td>
<td>1.875%</td>
<td>84,669</td>
<td>105,190</td>
<td>1.242</td>
</tr>
</tbody>
</table>
IV. Conclusions and Recommendations

A. Main Findings

• ASAP3 was designed as an improved batch means procedure that:
  – generally avoids undercoverage problems often observed with ASAP and LC as well as other popular procedures such as ABATCH and LBATCH;
  – generally avoids overcoverage problems sometimes observed with ASAP; and
  – completely eliminates excessive variability of CI half-length and final sample size sometimes observed with ASAP, especially with no precision requirement.
• ASAP3 has been tested on an extensive suite of test problems.
• ASAP3 outperforms ABATCH, ASAP, LBATCH, and LC in all test problems used so far.
• ASAP3 delivers well-behaved CIs for highly correlated or highly nonnormal processes, provided it is used with a precision requirement.
B. Recommendations for Future Work

- Examine the theoretical asymptotic validity and efficiency of the CIs delivered by ASAP3 as the precision specification $H^*$ or $r^*$ tends to zero.

- Test ASAP3 on queueing network models of production and telecommunications systems having realistic levels of complexity, congestion, and workstation utilization.