NORTH CAROLINA STATE UNIVERSITY
MA 241 Test 2 — Summer Session II, 2004

Department of Mathematics
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1. Find the area of the region between $y = x^{2/3}$, $x = 1$, and $y = 0$.

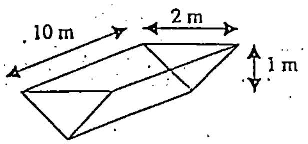
2. Rotate the region from problem (1) about the $y$-axis, and find the volume of the solid.

3. Find length of the curve parameterized by $x(t) = 3t^2$ and $y(t) = 2t^3$, over $0 \le t \le 2$. (Set up only – do not integrate nor evaluate.)

4. Find the center of mass of the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$.

5. A force of 30 N is required to stretch a spring from its' natural length of 30 *cm*, to 40 *cm*. How much work would be done in stretching the spring from 40 *cm* to 50 *cm*?

6. A water trough is 10 *m* long, with triangular ends 1 *m* high and (at the top) 2 *m* wide. The density of water is $p = 1000 kg/m^3$, and the gravitational constant is $g = 9.8 m/sec^2$. Find the hydrostatic pressure on one end of the trough, when full of water. (Set up only – do not integrate nor evaluate.)




7. a. Find the average value of the function $f(x) = -3x^2 + 4x$, on the interval [1, 3].
   b. Find a point in [1, 3] where $f(x)$ attains the mean value found in part (a).

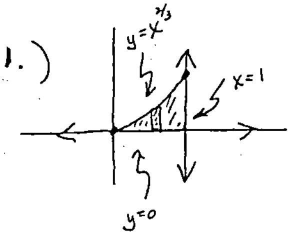
8. Given the differential equation $\frac{dy}{dx} = x^2 y$, $y(0) = 1$:
   a. Use Euler's Method with stepsize 0.5 to approximate $y(.5)$, $y(1)$, $y(1.5)$, and $y(2)$.
   b. Solve the separable differential equation for $y$, then find the exact value for $y(2)$.

9. The population of the United States in 1970 was 203 million people. It had grown to 227 million by 1980. Assuming exponential growth, predict the population in 2010 using these two data points.

10. Find the orthogonal trajectories of the family of ellipses $x^2 + 3y^2 = C$.

241 VIDEO TEST #2 SOLUTIONS 7/26/04

1.)

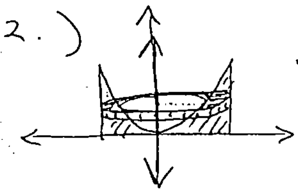



$$\int_a^b f(x)\,dx = \int_0^1 x^{2/3}\,dx$$

$$= \left.\frac{x^{5/3}}{5/3}\right]_0^1 = \left.\frac{3}{5}x^{5/3}\right]_0^1 = \frac{3}{5}\left[(1^{5/3}) - (0^{5/3})\right]$$

$$= \frac{3}{5}$$

2.)



$$V = \int_{y_1}^{y_2} \pi (R^2 - r^2)\,dy \qquad R = 1 \qquad r = x\text{-value of curve } (y = x^{2/3}) = y^{3/2}$$

$$= \pi \int_0^1 \left[(1)^2 - (y^{3/2})^2\right] dy$$

$$= \pi \int_0^1 (1 - y^3)\,dy = \pi \left.\left(1 \cdot y - \frac{y^4}{4}\right)\right]_0^1$$

$$= \pi\left[\left(1 - \frac{1^4}{4}\right) - \left(0 - \frac{0^4}{4}\right)\right] = \pi\left[\frac{3}{4}\right] = \frac{3\pi}{4} \approx 2.356$$

3.) arc length $0 \le t \le 2$

$$\frac{dx}{dt} = 6t \qquad S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\frac{dy}{dt} = 6t^2 \qquad S = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2}\,dt$$

(page2)

$$= \int_0^2 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_0^2 \sqrt{36t^2(1+t^2)}\, dt = \frac{1}{2}\int_0^2 \sqrt{1+t^2} \cdot 6t\, dt$$ bonus

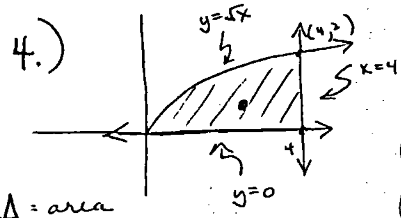
let $u = 1+t^2$

$du = 2t\,dt$

$$\Rightarrow \frac{1}{2}\cdot 6 \int u^{1/2} \cdot du = 3\left(\frac{u^{3/2}}{3/2}\right) \Rightarrow 3 \cdot \frac{2}{3}(1+t^2)^{3/2}\Big]_0^2$$

$$= 2\left[(5)^{3/2} - (1)^{3/2}\right] \approx 20.36$$

4.)




A = area

$$A = \int_0^4 \sqrt{x} \cdot dx$$

$$A = \int_0^4 x^{1/2} dx$$

$$A = \frac{2}{3}x^{3/2}\Big]_0^4 = \frac{2}{3}[8-0]$$

$$A = \frac{16}{3}$$

$$\bar{x} = \frac{1}{A}\int x \cdot f(x)\, dx$$

$$\bar{x} = \frac{1}{\frac{16}{3}}\int_0^4 x \cdot \sqrt{x}\, dx$$

$$\bar{x} = \frac{3}{16}\int_0^4 x^{3/2}\, dx$$

$$\bar{x} = \frac{3}{16}\left[\frac{x^{5/2}}{5/2}\right]_0^4 = \frac{3}{16} \cdot \frac{2}{5}\left[4^{5/2} - 0^{5/2}\right]$$

$$\bar{x} = \frac{3}{40}[32] = \frac{3 \cdot 32}{40} = \frac{12}{5} = \bar{x}$$

$$\bar{y} = \frac{1}{A}\int \frac{1}{2}[f(x)]^2 dx$$

$$\bar{y} = \frac{1}{\frac{16}{3}} \cdot \frac{1}{2}\int_0^4 (\sqrt{x})^2 dx = \frac{3}{16} \cdot \frac{1}{2}\int_0^4 x\, dx$$

$$\bar{y} = \frac{3}{32}\left[\frac{x^2}{2}\right]_0^4 = \frac{3}{32} \cdot \frac{1}{2}\left[4^2 - 0^2\right]$$

$$= \frac{3}{64}[16] = \frac{3 \cdot 16}{64} = \frac{3}{4} = \bar{y}$$

$$(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4}\right)$$

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5.) $f(x) = K \cdot x$ $\quad \left(N = \frac{K \cdot m}{s^2}\right)$

$30\,N = K \cdot (.10)\,m$

$K = \frac{30}{.1} = 300$

30 cm = .3 m
40 cm = .4 m
50 cm = .5 m

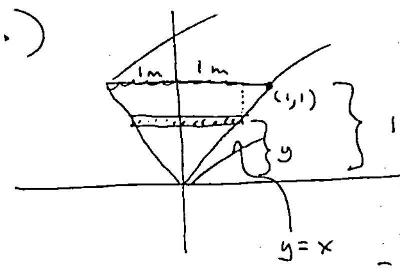
$$\text{work} = \int_{.1}^{.2} K \cdot x\, dx$$

$$\text{work} = \int_{.1}^{.2} 300x \cdot dx = 300\frac{x^2}{2}\Big]_{.1}^{.2}$$

$$= 150\left((.2)^2 - (.1)^2\right) = 150(.04 - .01)$$

$$= 150(.03) = 4.5\ J$$

6.)




$\int$ density depth area

$$\int_0^1 \left[1000 \frac{kg}{m^3}\right]\left[(1-y)\,m\right]\left[(2x\ m) \cdot (\Delta y\right.$$

$$= \int_0^1 \left(9.8 \frac{m}{sec^2}\right)\left(1000 \frac{kg}{m^3}\right)(1-y) \times (2x) \times dy \quad y = x$$

$$= (9.8)(1000)(2)\int_0^1 y(1-y)\, dy \quad \frac{kg \cdot m}{s^2}$$

$$= (9.8)(1000)(2)\int_0^1 (y - y^2)\, dy$$

bonus

$$= (9.8)(1000)(2)\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= (9.8)(2000)\left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{0}{2} - \frac{0}{3}\right)\right]$$

$$= (9.8)(2000)\left(\frac{1}{6}\right) \approx 3266.67\ N$$

bonus

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7.) a.) ave value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{3-1}\int_1^3 (-3x^2+4x)\,dx$

$= \frac{1}{2}\left[-\frac{\cancel{3}x^3}{\cancel{3}} + \frac{\cancel{4}^{2}x^2}{\cancel{2}_{1}}\right]_1^3$

$= \frac{1}{2}\left[-x^3+2x^2\right]_1^3 = \frac{1}{2}\left[\left(-(3)^3+2(3)^2\right)-\left(-1^3+2(1)\right)\right]$

$= \frac{1}{2}\left[-27+18+1-2\right] = \frac{1}{2}\left[-29+19\right] = -5$

b.) $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ (c in [1, 3])

$f(c) = -5$

$-3c^2+4c = -5$

$0 = 3c^2-4c-5$ $(3c\quad)(c\quad)$

$c = \frac{4 \pm \sqrt{16-4(3)(-5)}}{2(3)} = \frac{4\pm\sqrt{16+60}}{6}$

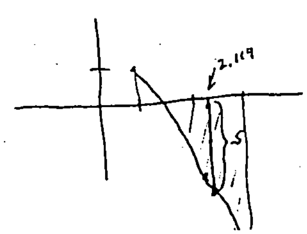
$c = \frac{4\pm\sqrt{76}}{6} = \frac{4+\sqrt{76}}{6}$ or $\cancel{\frac{4-\sqrt{76}}{6}}$ not in [1, 3]

$2.119$ $\quad -.786$

$c \approx 2.119$

| x | y |
|---|---|
| 1 | 1 |
| 2 | -4 |
| 3 | -15 |

$y = -3x^2+4x$




8.) $\frac{dy}{dx} = x^2y \qquad y(0)=1$

a.) $\Delta x = .5 \qquad x_0 = 0 \quad y_0 = 1$

$x_1 = x_0 + \Delta x = 0 + .5 = .5$

$y_1 = y_0 + \frac{dy}{dx}\cdot\Delta x = 1 + [(0)^2(1)]\cdot(.5) = 1$

$(x_1, y_1) = (.5, 1)$

$x_2 = x_1 + \Delta x = .5 + .5 = 1.0$

$y_2 = y_1 + \frac{dy}{dx}\Delta x = 1 + [(.5)^2(1)](.5) = 1 + .125 = 1.125$

$(x_2, y_2) = (1.0, 1.125)$

$x_3 = x_2 + \Delta x = 1.0 + .5 = 1.5$

$y_3 = y_2 + \frac{dy}{dx}\cdot\Delta x = 1.125 + [(1.125)(1.0)^2]\cdot(.5) = 1.125 + .5625$
$= 1.6875$

$(x_3, y_3) = (1.5, 1.6875)$

$x_4 = x_3 + \Delta x = 1.5 + .5 = 2.0$

$y_4 = y_3 + \frac{dy}{dx}\cdot\Delta x = 1.6875 + [(1.5)^2(1.6875)]\cdot(.5)$
$= 3.586$

$(x_4, y_4) = (2.0, 3.586)$

b.) $\frac{1}{y}\cancel{dx}\frac{dy}{\cancel{dx}} = x^2\cancel{y}\cdot dx\cdot\frac{1}{\cancel{y}}$

$\int \frac{1}{y}\,dy = \int x^2\,dx$

$\ln y = \frac{x^3}{3} + C$

$e^{\ln y} = e^{\frac{x^3}{3}+C}$

$y = e^{x^3/3}\cdot e^{C}$

$y = K\cdot e^{x^3/3}$

$y(0) = 1$

$1 = K\cdot e^{0/3}$

$1 = K\cdot 1$

$K = 1$

$y = e^{x^3/3}$

$y(2) = ?$

$y = e^{8/3}$

$y \approx 14.4$

9.) $y = y_0 \cdot e^{ct}$

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1970 ($t=0$) $P = 203$ million

1980 ($t=10$) $P = 227$ million

2010 ($t=40$) $P = ??$

1970: $t=0$ $P=203$

$y = 203 \cdot e^{ct}$

1980: $t=10$ $P=227$

$227 = 203\, e^{c(10)}$

$\frac{227}{203} = \frac{\cancel{203}\, e^{10c}}{\cancel{203}}$

$\frac{227}{203} = e^{10c}$

$\ln\left(\frac{227}{203}\right) = 10c \Rightarrow \frac{\ln\left(\frac{227}{203}\right)}{10} = c \approx .011174$

$y = 203 \cdot e^{.011174t}$

2010: ($t=40$) $P = ?$

$y = 203\, e^{.011174(40)} \approx 317.4$ million

10.) $x^2 + 3y^2 = C$

$2x + 6y \cdot \frac{dy}{dx} = 0$

$\frac{6y \cdot \frac{dy}{dx}}{6y} = \frac{-2x}{6y}$

$\frac{dy}{dx} = \frac{-x}{3y}$

ORTHOG. TRAJ.: $m_\perp = \frac{3y}{x} = \frac{dy}{dx}$

$\frac{1}{\cancel{y}} \cdot \cancel{dx} \frac{3\cancel{y}}{x} = \frac{dy}{\cancel{dx}} \cdot \frac{1}{y} \cdot \cancel{dx}$

$\frac{3}{x} dx = \frac{1}{y} dy \Rightarrow 3\int \frac{1}{x} dx = \int \frac{1}{y} dy$

$3 \cdot \ln x = \ln y + K$

$\ln x^3 = \ln y + K$

$e^{\ln x^3} = e^{\ln y + K}$

$x^3 = e^{\ln y} \cdot e^K$

$x^3 = y \cdot A$ or $y = B \cdot x^3$