1. a. Find the truth table for the following and determine if the statement is a tautology, contradiction or neither.

(i) \((P \lor \neg Q) \land (\neg P \lor Q)\) \(\Rightarrow P\)

(ii) \(P \lor [P \Rightarrow (Q \land R)]\)

(iii) \((P \Rightarrow \neg Q)\) logically equivalent to \(\neg P \lor Q\)?

b. Characterize......

(i) A rational number...

(ii) A natural number...

(iii) An even integer...

(iv) An odd integer...

2. Suppose \((x_n)\) is a sequence of real numbers. Given the statement: If \((x_n)\) is monotonic and bounded then \((x_n)\) converges.

Write the converse, contrapositive, and a useful denial of the statement, each clearly labeled.

3. Critique the following "proof" of the proposition:

If \(a\) and \(b\) are integers and \(a\) divides \(b^2\), then \(a\) divides \(b\). If it is valid, indicate so.

If it is flawed, find the error(s).

Proof: Suppose \(a\) and \(b\) are integers and \(a\) divides \(b^2\). Then there is \(k \in \mathbb{Z}\) so that \(b^2 = ak\). Thus \(b = \frac{ka}{b} = \left(\frac{k}{b}\right) a \in \mathbb{Z}\). Therefore \(a\) divides \(b\).

(OVER)
4. For each of the following, determine if it is true or false. Justify your answers.
   (universe is the real numbers unless otherwise stated)
   a. \((\forall y)(\exists x)(\forall z)(xy = xz)\)
   b. \((\forall x)(\exists !y)(x \cdot y = 1)\)
   c. \((\forall x)(\forall y)(x \in \mathbb{Q} \land y \in \mathbb{Q} \Rightarrow x^y \in \mathbb{Q})\)

5. Prove each of the following:
   a. If \(xy\) is odd, then both \(x\) and \(y\) are odd. (by contraposition)
   b. If \(a, b, c \in \mathbb{N}\) and \(a\) divides \(b\) and \(a\) divides \(c\), then \(a^2\) divides \(bc\).
   c. If \(x \in \mathbb{Q}, x \neq 0\) and \(y\) is irrational then \(xy\) is irrational. (by contradiction)

6. a. Given the statement: There is a positive number \(M\) so that if \(x\) is any element of \(A\) then the absolute value of \(f(x)\) is less than or equal to \(M\).
   1. Translate the statement into symbols.
   2. Write a useful denial in symbols.
   3. Translate your denial in (2) back into English.

   b. For this problem, the universe is \(F\), the set of all fruits.
   Let \(A(x)\) be '\(x\) is an apple', \(G(x)\) be '\(x\) is green', \(T(x)\) be '\(x\) is tasty'.
   Use quantifiers (either \(\exists x \in F\) or \(\forall x \in F\)) and logical connectives to symbolize:
   1. Some green apples are tasty
   2. All fruits are tasty or no fruits are tasty
   3. Some apples are neither green nor tasty
2.25 - Test #1 (Page 1) 16 points each

1. \( \text{a) } [(P \lor \neg Q) \land (\neg P \lor \neg Q)] \Rightarrow P \)

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<th>P \lor \neg Q</th>
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\( \Rightarrow \) \text{ } \text{NEITHER} \text{ }

2. \( \text{b) rational number} \text{ } \equiv \text{ } \text{any number that can be written in the form of } \frac{a}{b} \text{ where } a \neq 0 \text{ and } b \neq 0 \)
natural number - the positive integers, counting numbers

2. an even integer - any integer "a" that can be written in the form \( a = 2k \) where \( k \) is an integer.

2. an odd integer - any integer "b" that can be written in the form \( b = 2k+1 \) where \( k \) is an integer.

\( \text{DERIA: } \) If \( (x_n) \) is monotonic and bounded, then \( (x_n) \) converges.

\( \text{CONVERSE: } \) If \( (x_n) \) converges, then \( (x_n) \) is monotonic and bounded.

\( \text{CONTRAPPOSITIVE: } \) If \( (x_n) \) does not converge, then \( (x_n) \) is not monotonic nor bounded.

\( \text{DENIAL: } \) \( (m; B) \land x \in C \Rightarrow (m; B) \land (x_n) \) is monotonic and bounded and \( (x_n) \) is not convergent.

\( b = \left( \frac{k}{b} \right) a \) where \( k \) is an integer, but \( \frac{k}{b} \) is not necessarily an integer.
4.) 

a.) \((\forall y) (\exists x)(\forall z)(xy = xz)\)

\[\text{TRUE: there does exist an } x \ (x = 0) \ \text{for all } y \text{ and all } z \text{ such that } xy = xz \ (0 \cdot y = 0 \cdot z)\]

b.) \((\forall x)(\exists ! y)(x \cdot y = 1)\)

\[\text{FALSE: if } x = 0 \text{ then there is not a unique } y \text{ such that } 0 \cdot y = 1\]

c.) \((\forall x)(\forall y)(x \in \mathbb{Q} \land y \in \mathbb{Q} \Rightarrow x^y \in \mathbb{Q})\)

\[\text{FALSE: if } x = 2 \text{ and } y = \frac{1}{2} \ (\text{both rational numbers}), \text{ then } x^y = 2^{\frac{1}{2}} = \sqrt{2}, \text{ which is not rational.}\]

5.) 

a.) \((\text{CONTRAPOSITIVE})\)

\[\text{ORIG: if } xy \text{ is odd, then } x \text{ is odd and } y \text{ is odd.}\]

\[\text{CONTRAPOSITIVE: if } x \text{ and } y \text{ are not both odd (there } x \text{ is even or } y \text{ is even, or both are even), then } xy \text{ is even.}\]

\[\text{Suppose } x \text{ is even, then } x = 2m \text{ for } m \in \mathbb{Z}. \text{ Thus } xy = (2m)y = 2(my) \text{ and my is an integer, so } xy \text{ is even.}\]

\[\text{Since the contrapositive is true ... if } xy \text{ is odd, then } x \text{ is odd and } y \text{ is odd is also true.}\]
b) if \( a, b, c \in \mathbb{N} \) and \( a/b \) and \( a/c \)
then \( a^2/\text{lcm}(b,c) \)

since \( a/b \) then \( a \cdot k = b \) (where \( k \in \mathbb{Z}^+ \))
since \( a/c \) then \( a \cdot j = c \) (where \( j \in \mathbb{Z}^+ \))

\( \text{lcm}(b,c) = ak \cdot aj = a^2(kj) \) (where \( kj \in \mathbb{Z}^+ \))
since \( \text{lcm}(b,c) = a^2(kj) \), then \( a^2/\text{lcm}(b,c) \)

\[ c. \] if \( x \in \mathbb{Q}, x \neq 0 \) and \( y \) is irrational

then \( xy \) is irrational

**Contradiction:** Suppose \( x \in \mathbb{Q}, x \neq 0 \) and \( y \in \mathbb{Q} \)
and \( xy \in \mathbb{Q} \) then \( \exists p, q, r, s \in \mathbb{Z} \)
\( (pq, rs \neq 0) \) so that \( x = \frac{p}{q} \) and
\( xy = \frac{r}{s} \)

therefore \( y = \frac{1}{x}(\frac{r}{s}) = \frac{1}{\frac{p}{q}}(\frac{r}{s}) = \frac{qs}{ps} \in \mathbb{Q} \)
(since \( qr \in \mathbb{Z} \) and \( ps \in \mathbb{Z} \) and \( ps \neq 0 \))

this is a contradiction since \( y \neq \mathbb{Q} \)

\[ x \cdot y \neq \mathbb{Q} \]

6. \( a \)
7. \( b \)
8. \( c \)
9. \( d \)
10. \( e \)
11. \( f \)
12. \( g \)
13. \( h \)
14. \( i \)
15. \( j \)
16. \( k \)
17. \( l \)
18. \( m \)
19. \( n \)
20. \( o \)
21. \( p \)
22. \( q \)
23. \( r \)
24. \( s \)
25. \( t \)
26. \( u \)
27. \( v \)
28. \( w \)
29. \( x \)
30. \( y \)
31. \( z \)

b) \( \exists x \in F [(A(x) \land G(x)) \land T(x)] \)

3. \( \forall x \in F) T(x) \lor (\forall x \in F) \neg T(x) \)

2. \( \exists x \in F)[A(x) \land \neg G(x) \lor \neg T(x)] \)