

# An optimal control solution to the liquidity constraint problem

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## Abstract

An analytical solution to the liquidity constraint problem is derived in an optimal control framework. Some properties of the solution are examined. Results in the existing literature derived there by disparate methods are obtained by a single approach, thus providing a unified theory of consumption under liquidity constraint. ©1997 Elsevier Science S.A.

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Optimal control is a standard method for solving dynamic optimization problems under certainty. To date, an optimal control solution to the liquidity constraint problem has been unavailable, forcing researchers to use less direct though often quite clever methods to analyze this problem. However, the methods used have been almost as numerous as the results themselves, making it difficult to see the relationships among the results obtained. An optimal control solution to the liquidity constraint problem is provided here. The solution includes and extends the results in the existing literature and, of course, also includes the standard unconstrained problem as a special case. It thus provides a unified theory of consumption in the presence and absence of liquidity constraint. Because the results emerge from a single general model, the relation among them is clarified. The solution should prove useful in any future work on the liquidity constraint problem.

## 1. The solution

We require that assets be non-negative at all times:

$$A_t \geq 0 \text{ for all } t \geq 0 \tag{1}$$

The household is liquidity-constrained whenever it wants its assets to be negative. The household maximizes lifetime utility

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$$\int_0^{\infty} u(c_t) e^{-\rho t} dt \quad (2)$$

subject to the usual budget lifetime budget constraint

$$\int_0^{\infty} c_t e^{-\int_0^t r_s ds} dt = A_0 + \int_0^{\infty} y_t e^{-\int_0^t r_s ds} dt \quad (3)$$

and also subject to the non-negativity constraint Eq. (1). Here,  $c$ ,  $A$ , and  $y$  are consumption, assets, and labor income,  $\rho$  is the rate of time preference, and  $r$  is the interest rate. For simplicity,  $y$  is taken to be exogenous, and uncertainty is ignored.

Kamien and Schwartz (1981, part II, section 17) present the method for solving such problems. Consider the general problem

$$\max_{w_t} \int_0^{\infty} f(t, x_t, w_t) dt$$

subject to

$$dx/dt = g(t, x, w)$$

and the general state variable constraint

$$k(t, x) \geq 0$$

plus the endpoint conditions

$$x_0 = \text{given}$$

$$x_t \rightarrow 0 \text{ as } t \rightarrow \infty$$

Here,  $x$  is the state variable, and  $w$  is the control variable.<sup>1</sup> To apply optimal control, we first re-express the state variable constraint by the equivalent constraint that  $k$  not decrease when  $k=0$ . Define

$$\phi(t, x, w) \equiv k_t(t, x) + k_x g(t, x, w)$$

We then require that  $\phi \geq 0$  whenever  $k=0$ , which is equivalent to imposing Eq. (1). The Hamiltonian is

$$H_t = f(t, x_t, w_t) + \psi_t g(t, x_t, w_t) + \epsilon_t \phi(t, x_t, w_t) \quad (4)$$

where  $\psi$  is the usual costate variable and  $\epsilon$  is the costate variable for the state variable constraint. The necessary conditions are

<sup>1</sup>The terminal condition on  $x$  is not the most general one imaginable, but it corresponds to the one required for the household, so I impose it here. Nothing that follows depends on this restriction.

$$dx_t/dt = g(t, x_t, w_t) \tag{5}$$

$$d\psi_t/dt = -\psi_t g_x(t, x_t, w_t) - \epsilon_t \phi_x(t, x_t, w_t) \tag{6}$$

$$\begin{aligned} d\epsilon_t/dt &\leq 0 \\ &= 0 \text{ if } x_t > 0 \end{aligned} \tag{7}$$

$$\begin{aligned} \epsilon_t &\geq 0 \\ &= 0 \text{ if and only if } x_{t+j} > 0 \forall j \in [0, \infty] \end{aligned} \tag{8}$$

$$x_0 = \text{given} \tag{9}$$

$$\lim_{t \rightarrow \infty} x_t = 0 \tag{10}$$

$$f_x(t, x_t, w_t) = \psi_t + \epsilon_t \tag{11}$$

Eqs. (7) and (8) give the dynamics for  $\epsilon$ , which is required never to rise, to fall only when the constraint binds, and to equal zero only if the constraint never again will bind.<sup>2</sup>

For the liquidity constraint problem, we have the following definitions:

$$x \equiv a$$

$$w \equiv c$$

$$f(t, w_t, x_t) \equiv u(c_t)e^{-\rho t}$$

$$g(t, x_t, w_t) \equiv r_t A_t + y_t - c_t$$

$$k(t, x_t) \equiv A_t$$

$$\phi(t, x_t, w_t) \equiv r_t A_t + y_t - c_t$$

The last definition arises because, with  $k=(t,x)=A$ , we have  $k_t=0$  and  $k_x=1$ , so  $\phi$  is just  $g$ . The Hamiltonian for the liquidity constraint problem thus is

$$H_t = u(c_t)e^{-\rho t} + \psi_t(r_t A_t + y_t - c_t) + \epsilon_t(r_t A_t + y_t - c_t) \tag{12}$$

and the necessary conditions are

$$dA_t/dt = r_t A_t + y_t - c_t \tag{13}$$

$$d\psi_t/dt = -r_t \psi_t - r_t \epsilon_t \tag{14}$$

<sup>2</sup>More generally, it is required that  $\epsilon \geq \epsilon^*$  and  $\epsilon = \epsilon^*$  if and only if  $x > 0$  for all future  $t$ , where  $\epsilon^*$  is some non-negative constant. However, in the household maximization problem, it is natural for  $\epsilon^* = 0$ , in which case  $\epsilon$  simply disappears in the absence of liquidity constraint.

$$\begin{aligned} d\epsilon_t/dt &\leq 0 \\ &= 0 \text{ if } A_t > 0 \end{aligned} \quad (15)$$

$$\begin{aligned} \epsilon_t &\geq 0 \\ &= 0 \text{ if and only if } A_{t+j} > 0 \forall j \in [0, \infty] \end{aligned} \quad (16)$$

$$A_0 \text{ given} \quad (17)$$

$$\lim_{t \rightarrow \infty} A_t = 0 \quad (18)$$

$$u'(c_t)e^{-\rho t} = \psi_t + \epsilon_t \quad (19)$$

One can use the foregoing relations to study the optimal path of consumption, but as usual it is more convenient to transform the problem to current value form by dividing the Hamiltonian by the discount factor  $e^{-\rho t}$ . The current value Hamiltonian is

$$H_t = u(c_t) + \lambda_t(r_t A_t + y_t - c_t) + \eta_t(r_t A_t + y_t - c_t) \quad (20)$$

where  $\lambda_t = \psi_t e^{\rho t}$  and  $\eta_t = \epsilon_t e^{\rho t}$ . The corresponding necessary conditions are

$$dA_t/dt = r_t A_t + y_t - c_t \quad (21)$$

$$d\lambda_t/dt = (\rho - r_t)\lambda_t - r_t \eta_t \quad (22)$$

$$\begin{aligned} d\eta_t/dt &\leq \rho \eta_t \\ &= \rho \eta_t \text{ if } A_t > 0 \end{aligned} \quad (23)$$

$$\begin{aligned} \eta_t &\geq 0 \\ &= 0 \text{ if and only if } A_{t+j} > 0 \forall j \in [0, \infty] \end{aligned} \quad (24)$$

$$u'(c_t) = \lambda_t + \eta_t \quad (25)$$

and also Eqs. (17) and (18). It is convenient to define the 'total' costate variable  $\omega_t = \lambda_t + \eta_t$ . From Eqs. (22) and (23), we have

$$\begin{aligned} d\omega_t/dt &\leq (\rho - r_t)\omega_t \\ &= (\rho - r_t)\omega_t \text{ if } A_t > 0 \end{aligned} \quad (26)$$

Concavity of  $u$  (i.e.  $u'' < 0$ ) and Eq. (25) together imply a negative monotonic relation between  $c_t$  and  $\omega_{t+1}$ :

$$c_t = c(\omega_{t+1}), \quad c' < 0 \quad (27)$$

which is the solution for optimal consumption. Using Eqs. (21)–(24) together with Eqs. (17) and (18), one obtains the path for  $\omega$  and thus for  $c$ . As usual for optimal control problems, this solution generally cannot be put in closed form, but nevertheless one can obtain much information about the solution's properties.

## 2. Its properties

The properties of the solution depend on the nature of the path of  $r$ . To keep the discussion simple and stay within the space limitations, I confine attention to the case where  $r$  is constant. The solution properties in other cases are left to the reader.

If there were no asset constraint,  $\eta$  would be zero always,  $d\omega/dt \geq 0$  as  $r \leq \rho$ , so  $dc/dt \leq 0$  as  $r \leq \rho$  from Eq. (27). This is the standard unconstrained solution. The same general tendencies emerge when the asset constraint is present, but  $c$  tends to increase faster (or decrease more slowly) than in the unconstrained case. This result is easiest to see if  $r = \rho$ . Then  $c$  is constant without the constraint. With the constraint,  $\omega$  can remain constant or can fall, by Eq. (26), so Eq. (27) implies that consumption can rise or stay constant. Thus  $c$  can rise, even if  $r = \rho$ , with the constraint present. Similar results hold when  $r \neq \rho$ . The intuition is straightforward. The household cannot have negative assets, so if assets are zero, consumption cannot be above current income. If current income subsequently rises,  $c$  can rise with it. Thus there are periods when  $c$  is lower than it would be without the constraint. Subsequently,  $c$  must be higher than it would have been without the constraint in order to satisfy the lifetime budget constraint. Thus there must be a transition period when  $c$  rises faster or at least falls more slowly than it would have done in the absence of the constraint. These conclusions extend those of Heller and Starr (1979) and show that the results of Deaton (1991) on falling consumption in the presence of liquidity constraint arise because Deaton assumes  $r < \rho$ , not because of the liquidity constraint. The presence of liquidity constraint in his model actually causes  $c$  to fall less rapidly over time than it would without the constraint.

Henceforth, I confine attention to the case where  $r = \rho$ , again to simplify discussion. Consider the path of income shown in Fig. 1. The household starts at time  $t_0$ , works until retirement at  $t_R$ , and dies at time  $t_D$ . Income initially falls, but then rises much higher before falling to zero at retirement. Permanent income at  $t_0$  is  $y^*$ . With no constraint,  $c$  would equal  $y^*$ . With constraint, the household initially is liquidity-constrained because current income initially is below permanent income and the

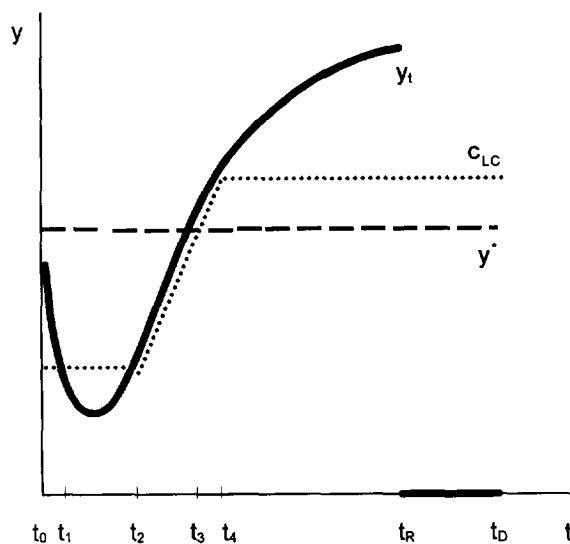


Fig. 1. Income and consumption paths under liquidity constraint: a simple case.

household would like to borrow but cannot. Consumption initially cannot track current income because income initially falls, which consumption cannot do. In fact,  $c$  must stay constant for a time. Suppose  $c$  were set equal to income at  $t_0$ . The next instant it would exceed income (because  $c$  must either rise or stay constant, whereas  $y$  falls) and assets would become negative, which is prohibited. Consequently,  $c$  must be set below current income at  $t_0$ . Then assets immediately become positive,  $\omega$  also is positive by Eq. (26), and so  $c$  is constant by Eq. (27). At time  $t_2$ , assets are exhausted and from then on  $c$  equals current income until  $t_4$ , after which it is constant at the value of permanent income at  $t_4$ ;  $\omega$  falls from  $t_2$  to  $t_4$ , after which it is constant.

Consumption follows the dotted path in Fig. 2. Over the interval  $t_0-t_1$ , the household saves in order to level its consumption over the interval  $t_0-t_2$ . Indeed, in this particular case the household saves precisely *because* it cannot borrow. This is the result of Mariger (1987) and Zeldes (1989) that liquidity-constrained households need not have zero assets. Later, the household dissaves, even though it still is liquidity-constrained (it wants to borrow but cannot). The liquidity-constrained household therefore may hold positive assets and even dissave.

Fig. 2 shows a general path for income. We can use it to describe other properties of the optimal consumption path. In general, there may be several periods over which consumption is constant and several others over which it equals current income. Consumption therefore follows a stair-step pattern. Because consumption never falls, periods over which it equals current income must be periods when income is rising.<sup>3</sup> Consequently, periods when consumption is constant must start and stop at times when current income is rising (*except*, of course, that they may start or stop at the initial and terminal times  $t_0$  and  $t_D$ ). Thus, in Fig. 2, a period of constant consumption cannot end at a point like 3, where current income is falling. Also, consumption is a continuous function of time because  $c$  is a

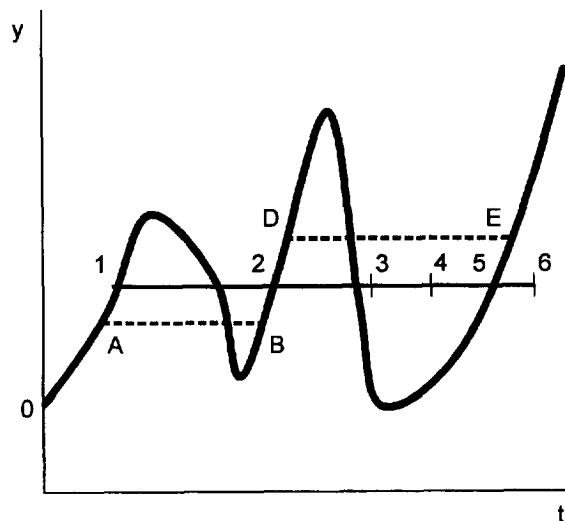


Fig. 2. Income and consumption paths under liquidity constraint: general case.

<sup>3</sup>Or at least not falling. Throughout the following discussion, I assume the income path has no flat spots except during retirement, when it is zero. Introducing flat spots changes none of the conclusions but requires attention to tedious but unenlightening details.

continuous function of  $\omega$ , by Eq. (27), and  $\omega$  is a continuous function of time, by Eq. (26). Then, by continuity, segments of the consumption path that are constant must start and end where the constant segment intersects the path of current income; otherwise, there would be a jump in consumption. Endpoints like 4 and 6 in Fig. 2 thus are ruled out. The same kind of reasoning rules out similar points at the start of the constant segment, so that both the initial point and terminal point of the constant segment must lie on rising sections of the current income path. Finally, any period of constant consumption will be the longest one among the set of consumption paths that satisfy all other optimality properties. The formal proof is tedious, but the intuition is easy enough. Consider the two alternative consumption paths shown in Fig. 2, the solid path 0–1–5–E and the dotted path 0–A–B–D–E. Suppose both paths are feasible, so that assets never are negative and are exhausted exactly at the ends of the paths. The dotted path postpones early consumption (along section A–B–2) to have greater consumption later (along section 2–D–E), making consumption less smooth than along the solid path and hence less desirable. Thus the longest feasible path of constant consumption is the optimal one.

Finally, compare the Euler equations for constrained and unconstrained households. In making this comparison, we no longer require that  $r = \rho$ . The general Euler equation is

$$\frac{u'(c_t)}{u'(c_{t+i})} = \frac{\lambda_t + \eta_t}{\lambda_{t+i} + \eta_{t+i}}$$

Using Eq. (22), we can rewrite this equation as

$$\begin{aligned} \frac{u'(c_t)}{u'(c_{t+i})} &= \frac{e^{(r-\rho)i}\lambda_{t+i} + e^{(\rho-r)t} \int_t^{t+i} e^{(r-\rho)s} r\eta \, ds + \eta_t + [e^{(r-\rho)i}\eta_{t+i} - e^{(r-\rho)i}\eta_{t+i}]}{\lambda_{t+i} + \eta_{t+i}} \\ &= e^{(r-\rho)i} + \frac{e^{(\rho-r)t} \int_t^{t+i} e^{(r-\rho)s} r\eta \, ds + \eta_t - e^{(r-\rho)i}\eta_{t+i}}{\lambda_{t+i} + \eta_{t+i}} \\ &= e^{(r-\rho)i} + Z_t \end{aligned} \tag{28}$$

This is the general continuous-time Euler equation, applying to all households, and corresponds to that discussed by Zeldes (1989) in discrete time. For households with positive assets, Eq. (23) holds as an equality, in which case  $\eta_{t+i} = e^{\rho i} \eta_t$ ,  $z = 0$ , and Eq. (28) simplifies to the usual

$$\frac{u'(c_t)}{u'(c_{t+i})} = e^{(r-\rho)i} \tag{29}$$

Note that Eq. (29) applies to any household with positive assets, whether liquidity-constrained or not. This proves the assertion of Zeldes (1989) that even liquidity-constrained households may satisfy the standard Euler equation.

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