Household Choice with Borrowing and Lending

I. Borrowing and Lending

A. Bonds and loans

1. Household i holds bonds $b_{it} \gtrless 0$

2. If $b > 0$, the household is a net lender; if $b < 0$, the household is a net borrower

3. Buying a bond is equivalent to making a loan of the same value; similarly, selling a bond is equivalent to taking a loan of the same value.

B. Bonds carry the interest rate R_t

→ for now simplify and keep the interest rate constant at R

C. For now, all borrowing and lending is in real terms: borrow and lend commodities (not money, which doesn't exist yet in our simple economy)

II. Budget Constraints

A. One period

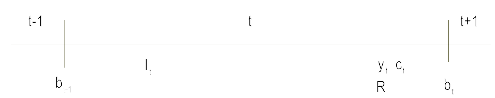

Figure 1: Discrete time

1. Total income must equal total expenditure

$$y_t + Rb_{t-1} = c_t + (b_t - b_{t-1})$$

2. Sometimes useful to rearrange terms to get the *sources and uses* version:

$$y_t + b_{t-1}(1+R) = c_t + b_t$$

B. Two periods

1. Can rearrange the budget constraint as

$$b_t = y_t + (1+R)b_{t-1} - c_t$$

2. Same relation holds for period t+1:

$$y_{t+1} + Rb_t = c_{t+1} + (b_{t+1} - b_t)$$

3. Substitute for b_t and rearrange terms to get the <u>two-period budget constraint</u>

$$c_t + \frac{c_{t+1}}{1+R} \;=\; y_t + \frac{y_{t+1}}{1+R} + b_{t-1}(1+R) - \frac{b_{t+1}}{(1+R)}$$

$$\equiv\; x_t$$

x_t is *lifetime wealth*.

a. The <u>left side</u> is the present value of consumption at the end of period t.

b. The <u>right side</u> is the present value of sources at the end of period t:

the present value of income (as of the end of period t) **plus** the value of initial bonds (as of the *end* of period t, which is why it is multiplied by 1+R) **minus** the present value of terminal bond holdings (as of the end of period t, which is why it is discounted once by the factor 1+R).

4. Alternative arrangement

$$c_{t+1} \;=\; (1+R)x_t - (1+R)c_t$$

This equation is graphed in Figure 2.

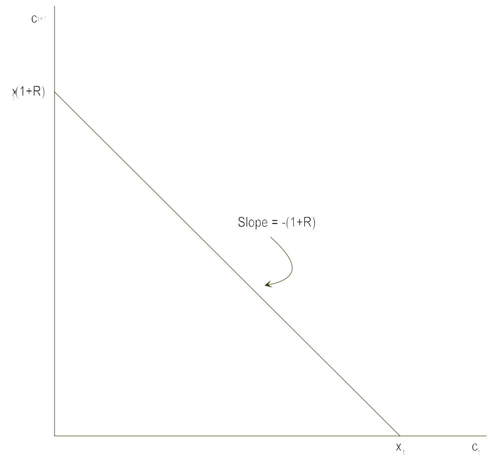

Figure 2: Two-period budget constraint

III. Intertemporal Utility

A. Discounted sum of *intratemporal* utilites:

$$U(c_t, c_{t+1}, l_t, l_{t+1}) \;=\; u(c_t, l_t) + \frac{1}{1+\rho} u(c_{t+1}, l_{t+1})$$

where ρ > 0 is the *rate of time preference*, a measure of how much a person prefers the present over the future. The rate of time preference can be thought of as the rate of impatience. Higher ρ indicates more impatience.

B. Indifference curves

1. $$dU = u_c dc_t + u_l dl_t + \frac{1}{1+\rho} u_c dc_{t+1} + \frac{1}{1+\rho} u_l dl_{t+1}$$

$$= 0 \quad \text{to be on an indifference curve}$$

2. For now, hold labor constant, so that $dl_t = dl_{t+1} = 0$.

3. Then we have

$$\frac{dc_{t+1}}{dc_t} = -\frac{u_c(c_t)}{\left[\frac{u_c(c_{t+1})}{1+\rho}\right]}$$

The right side is the ratio of marginal utilities of c_t and c_{t+1}, that is, the marginal rate of substitution between c_t and c_{t+1}.

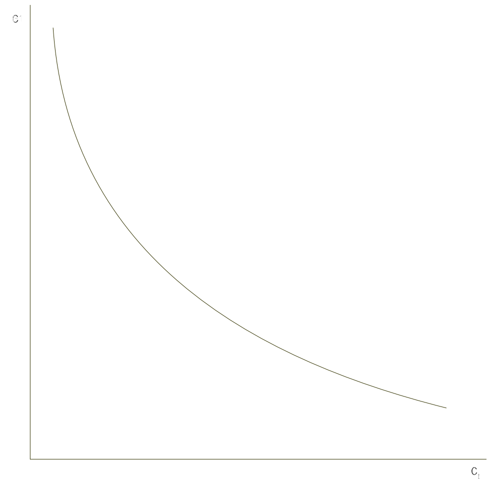

Figure 3: Intertemporal indifference curve

IV. Optimal Choice of (c_t, c_{t+1}) with Constant Labor

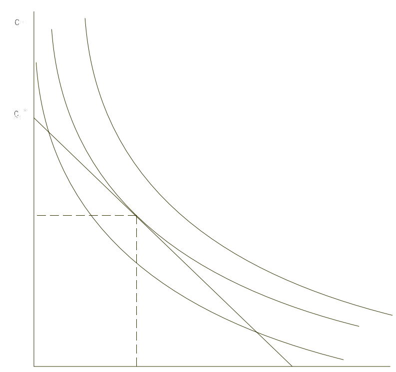

Figure 4: Optimum intertemporal choice

A. Choose (c_t , c_{t+1}) to maximize U subject to the budget constraint.

B. Feasible set consists of points on or below intertemporal budget line.

C. Choose (c_t , c_{t+1}) on the budget line that is on the highest attainable indifference curve.

D. Mathematical formulation:

1. Form Lagrangean

$$V = \left[u(c_t,\cdot) + \frac{1}{1+\rho} u(c_{t+1},\cdot)\right] + \lambda\left[x_t - \left(c_t + \frac{c_{t+1}}{1+R}\right)\right]$$

(Labor is suppressed because it is not allowed to vary in this experiment.)

2. Take derivatives with respect to c_t, c_{t+1}, and λ and set them equal to zero to get the first-order conditions

a. $$u_c(c_t) - \lambda = 0$$

b. $$\frac{u_c(c_{t+1})}{1+\rho} - \frac{\lambda}{1+R} = 0$$

c. $$x_t - \left(c_t + \frac{c_{t+1}}{1+R} \right) = 0$$

and solve for c_t, c_{t+1}, and λ.

3. Solve the first of the first-order conditions for λ, substitute into the second, and rearrange terms to get

$$\frac{u_c(c_t)}{\left[\dfrac{u_c(c_{t+1})}{1+\rho} \right]} = 1+R$$

The left side is marginal rate of substitution (slope of indifference curve), and the right side is the relative price of c_{t+1} in terms of c_t.

4. Rewrite the foregoing expression as

$$\frac{u_c(c_t)}{u_c(c_{t+1})} = \frac{1+R}{1+\rho}$$

a. $c_t = c_{t+1}$ if and only if $R = \rho$

b. $c_t < c_{t+1}$ if and only if $R > \rho$

c. $c_t > c_{t+1}$ if and only if $R < \rho$

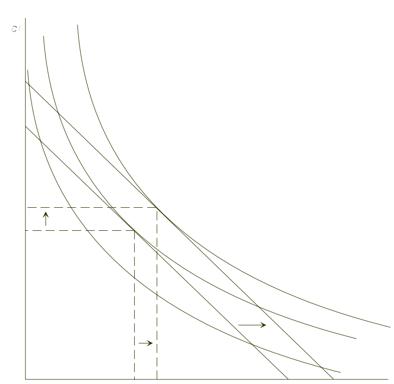

Figure 5: Pure intertemporal wealth effect

V. Response of (c_t, c_{t+1}) to Shocks

A. Pure wealth effect - change in x with R unchanged (for example, a permanent upward shift in the production function)

1. Shift the budget line

2. No change in slope of budget line

3. c_t and c_{t+1} change in same direction (both are normal goods)

B. Pure substitution effect

1. Results from change in R

2. Changes in R also change x through the last three terms on the right:

$$x_t = y_t + \frac{y_{t+1}}{1+R} + b_{t-1}(1+R) - \frac{b_{t+1}}{(1+R)}$$

For a pure substitution effect, there must be no change in x (which would introduce a wealth effect), so suppose that the effects on the last three terms exactly cancel.

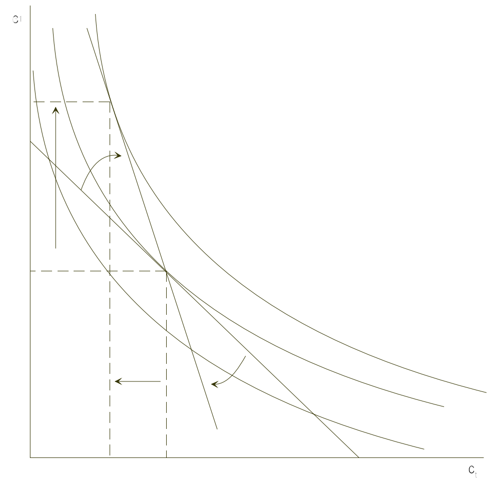

Figure 6: Pure intertemporal substitution effect

3. Budget line rotates around the original point, c_t changes in opposite direction as R, and c_{t+1} changes in the same direction as R.

VI. Optimal Choice of (l_t, l_{t+1}) with Constant Consumption

A. Now the left side of the budget constraint is fixed. We can write the budget constraint as:

$$f(l_t) + \frac{f(l_{t+1})}{1+R} = c_t + \frac{c_{t+1}}{1+R} - b_{t-1}(1+R) + \frac{b_{t+1}}{(1+R)}$$

$$\equiv z_t$$

B. Write the Lagrangean as

$$V = \left[u(\cdot, l_t) + \frac{1}{1+\rho} u(\cdot, l_{t+1})\right] + \lambda\left[f(l_t) + \frac{f(l_{t+1})}{1+R} - z_t\right]$$

(Consumption is suppressed because it is not allowed to vary in this experiment.)

C. Differentiate with respect to l_t, l_{t+1}, and λ to get the first-order conditions:

1. $$u_l(l_t) + \lambda f^{\prime}(l_t) = 0$$

2. $$\frac{u_l(l_{t+1})}{1+\rho} + \lambda \frac{f^{\prime}(l_{t+1})}{1+R} = 0$$

3. $$f(l_t) + \frac{f(l_{t+1})}{1+R} - z_t = 0$$

D. Solve the first condition for λ, substitute into the second, and rearrange to get

$$\frac{u_l(l_t)}{u_l(l_{t+1})} = \left[\frac{1+R}{1+\rho}\right]\left[\frac{f^{\prime}(l_t)}{f^{\prime}(l_{t+1})}\right]$$

1. Can work out similar results for labor as for consumption

a. Current labor l_t is positively related to R, and future labor l_{t+1} is negatively related to R. [These are the opposite relations as for consumption. That is because labor is a "bad", not a good. Leisure is the good. The equivalent results for leisure are that L_t is negatively related to R, and L_{t+1} is positively related to R, which is just like the results for consumption.]

b. Intuition is that high R makes earning income in the first period more valuable than when R is low. Income earned now can be saved and receive the interest rate R. That is more profitable the higher R is.

2. New aspect here

a. Labor depends on the marginal product of labor.

b. Have a new mechanism for intertemporal substitution

(i) Suppose the production function *temporarily* changes so that $f'(l_t)$ rises but $f'(l_{t+1})$ does not change.

(ii) Then current labor l_t rises, and future labor l_{t+1} falls.

VII. Infinite Horizon

A. Budget constraint for n+1 periods:

$$y_t + \frac{y_{t+1}}{1+R} + \ldots + \frac{y_{t+n}}{(1+R)^n} + b_{t-1}(1+R) \;=\; c_t + \frac{c_{t+1}}{1+R} + \ldots + \frac{c_{t+n}}{(1+R)^n} + \frac{b_{t+n}}{(1+R)^n}$$

B. Budget constraint for infinitely many periods:

$$y_t + \frac{y_{t+1}}{1+R} + \frac{y_{t+2}}{(1+R)^2} + \ldots + b_{t-1}(1+R) \;=\; c_t + \frac{c_{t+1}}{1+R} + \frac{c_{t+2}}{(1+R)^2} + \ldots$$

or

$$b_{t-1}(1+R) + \sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+R)^i} \;=\; \sum_{i=0}^{\infty} \frac{c_{t+i}}{(1+R)^i}$$

C. Utility over an infinite horizon:

$$U(c_t, c_{t+1}, \ldots; l_t, l_{t+1}, \ldots) \;=\; u(c_t, l_t) + \frac{1}{1+\rho} u(c_{t+1}, l_{t+1}) + \frac{1}{(1+\rho)^2} u(c_{t+2}, l_{t+2}) + \ldots$$

$$= \sum_{i=0}^{\infty} \frac{u(c_{t+i}, l_{t+i})}{(1+\rho)^i}$$

D. Optimal choice

1. Cannot draw a graph because there are too many dimensions

2. Intuition same as for 2-period case

3. Lagrangean is

$$V \;=\; \sum_{i=0}^{\infty} u(c_{t+i}, l_{t+i}) \left(\frac{1}{1+\rho}\right)^i + \lambda \left[b_{t-1}(1+R) + \sum_{i=0}^{\infty} f(l_{t+i}) \left(\frac{1}{1+R}\right)^i - \sum_{i=0}^{\infty} c_{t+i} \left(\frac{1}{1+R}\right)^i \right]$$

4. Using the Lagranean and the first-order conditions that can be obtained from it, one can show that

a. Consumption will be rising over time, constant, or falling over time as R is greater than, equal to, or less than ρ.

b. Labor will be falling over time, constant, or rising over time as R is greater than, equal to, or less than ρ (assuming the production function is the same for all time periods).

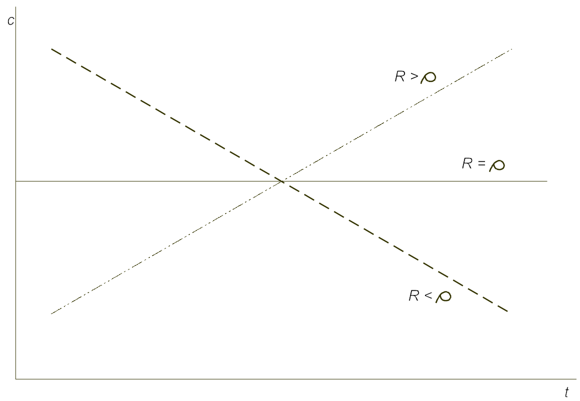

Figure 7: Consumption over time

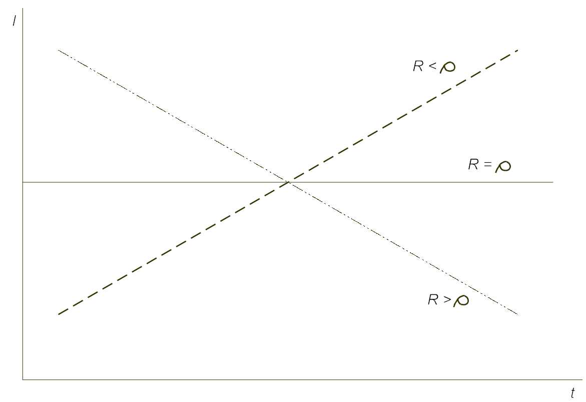

Figure 8: Labor over time

E. Response to shocks

1. Permanent increase in R

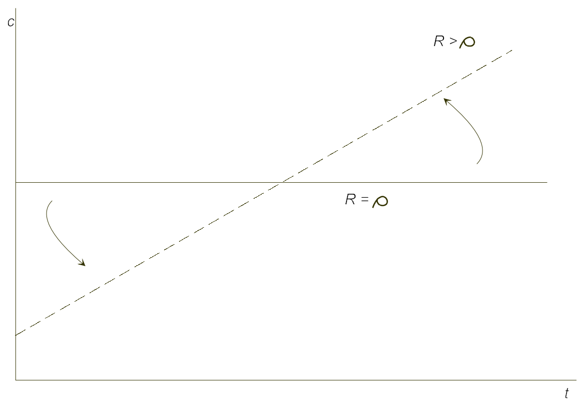

Figure 9: Permanent increase in R - Consumption response

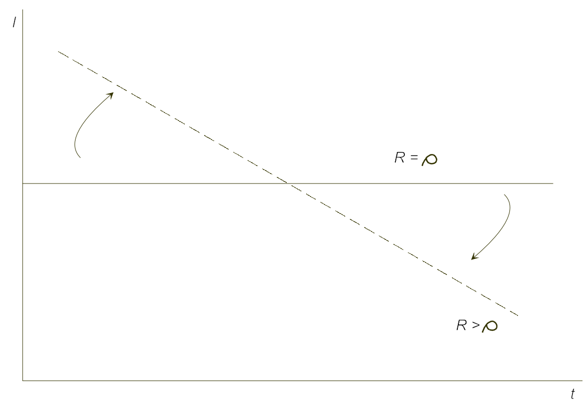

Figure 10: Permanent increase in R - Labor response

2. Temporary increase in R

a. Suppose R increases at time t and is expected to return to its original value at time t+j

b. After t+j, shape of consumption and labor paths must be the same as before the change in R

c. Between t and t+j, consumption must be rising and labor must be falling

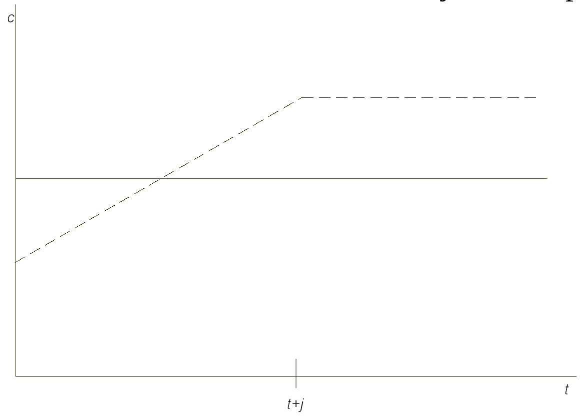

Figure 11: Temporary increase in R - Consumption response

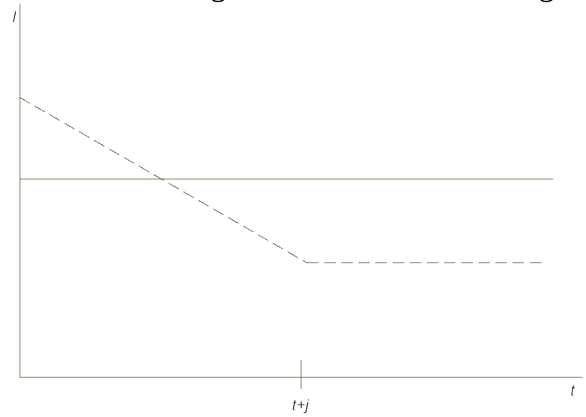

Figure 12: Temporary increase in R - Labor response

3. Permanent increase in the marginal product of labor f ' with R = ρ at all times

a. Consumption is higher for all t

b. Labor is higher for all t

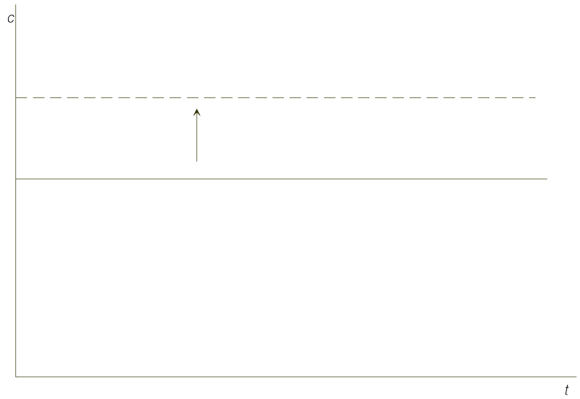

Figure 13: Permanent increase in f' - Consumption response

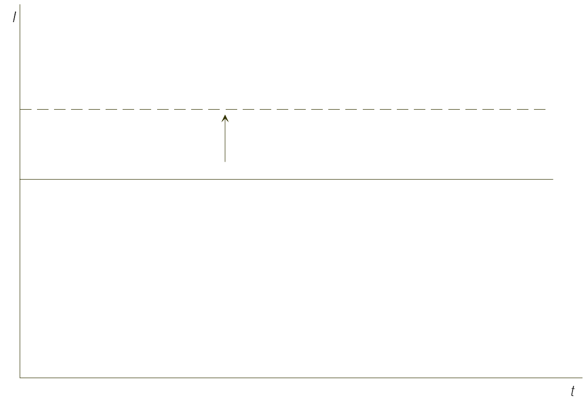

Figure 14: Permanent increase in f' - Labor response

4. Temporary increase in the marginal product of labor

a. Labor is higher while f ' is higher, than lower than before the shock, in response to the intertemporal substitution effect that makes earlier labor more productive than later labor

b. The higher f ', creates more lifetime income, which is used to buy more consumption. Consumption is higher at all times. Because R = ρ at all times, consumption is constant over time.

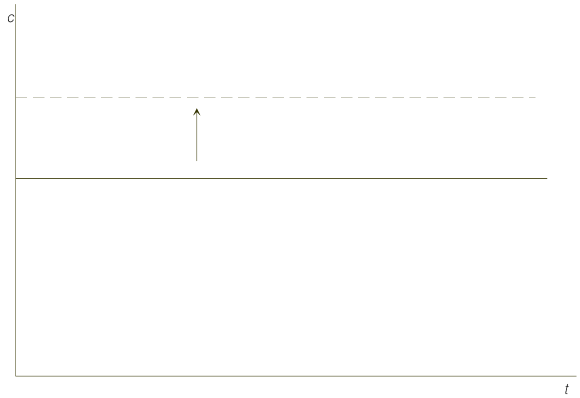

Figure 15: Temporary increase in f' - Consumption response

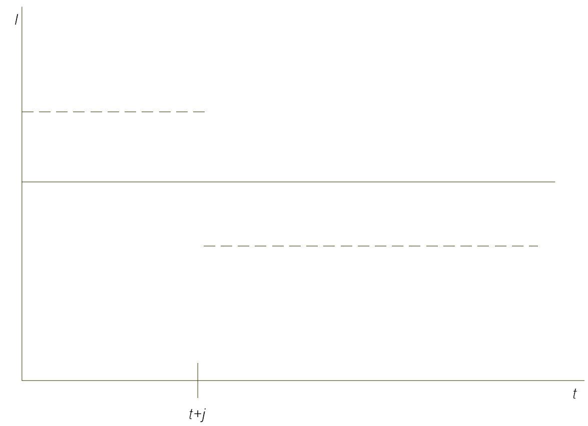

Figure 16: Temporary increase in f' - Labor response

F. Comparison of effects of temporary and permanent shocks

1. Suppose the household starts in a situation where R = ρ and therefore consumption is constant over time.

a. The budget constraint is

$$b_{t-1}(1+R) + \sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+R)^i} = \sum_{i=0}^{\infty} \frac{c_{t+i}}{(1+R)^i}$$

$$= \sum_{i=0}^{\infty} \frac{c^*}{(1+R)^i}$$

$$= c^* \sum_{i=0}^{\infty} \frac{1}{(1+R)^i}$$

$$= c^* \frac{1+R}{R}$$

The last line arises from the following result for an infinite series:

$$\sum_{i=0}^{\infty} a^{i} = \frac{1}{1-a} \quad \text{if } |a| < 1$$

In the budget constraint, we have

$$a = \frac{1}{1+R} < 1$$

b. Then we can conclude that

$$c^{*} = \frac{R}{1+R}\left[b_{t-1}(1+R) + \sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+R)^{i}}\right]$$

$$= R\,b_{t-1} + \frac{R}{1+R}\left[\sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+R)^{i}}\right]$$

2. Now suppose the household wins the national lottery, which pays z income per period.

a. Following the same steps, we can conclude that

$$c_{\text{new}}^{*} = R\,b_{t-1} + \frac{R}{1+R}\left[\sum_{i=0}^{\infty} \frac{y_{t+i}+z}{(1+R)^{i}}\right]$$

$$= R\,b_{t-1} + \frac{R}{1+R}\left[\sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+R)^{i}} + \sum_{i=0}^{\infty} \frac{z}{(1+R)^{i}}\right]$$

$$= R\,b_{t-1} + \frac{R}{1+R}\left[\sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+R)^{i}}\right] + \frac{R}{1+R}\left[\sum_{i=0}^{\infty} \frac{z}{(1+R)^{i}}\right]$$

$$= c_{\text{old}}^{*} + z\,\frac{R}{1+R}\left[\sum_{i=0}^{\infty} \frac{1}{(1+R)^{i}}\right]$$

$$= c_{\text{old}}^{*} + z$$

3. Suppose instead that the household receives a gift of z *in the first period only*. Then

$$c^*_{\text{new}} = b_{t-1} + \frac{R}{1+R}\left[\frac{y_t + z}{(1+R)^0} + \sum_{i=1}^{\infty}\frac{y_{t+i}}{(1+R)^i}\right]$$

$$= b_{t-1} + \frac{R}{1+R}\left[z + \sum_{i=0}^{\infty}\frac{y_{t+i}}{(1+R)^i}\right]$$

$$= b_{t-1} + \frac{R}{1+R}\left[\sum_{i=0}^{\infty}\frac{y_{t+i}}{(1+R)^i}\right] + z\frac{R}{1+R}$$

$$= c^*_{\text{old}} + z\frac{R}{1+R}$$

$$< c^*_{\text{old}} + z$$

4. In both cases, income in the first period is y_t + z, but consumption responds differently, depending on households' expectations of future income.

5. In the second case, the increase in income is temporary. Households still want constant consumption (because R = ρ), so they want to save most of the extra income z in order to increase their consumption in the future.

VIII. Demand and Supply

We have determined the household's optimal time paths for consumption, labor, and output. Once again, we can think of these as demands and supplies. In particular, the demands and supplies in period t depend on R_t, x_t, and other variables (e.g., future interest rates R_{t+i} or gifts that raise lifetime wealth x) :

1. $$c_t^D \equiv c_t^*$$

$$= c^D(R_t, \ldots) \qquad \text{where} \quad \partial c^D / \partial R_t < 0$$

2. $$l_t^S \equiv l_t^*$$

$$= l^S(R_t, \ldots) \qquad \text{where} \quad \partial c^D / \partial R_t > 0$$

3. $$y_t^S \equiv f(l_t^S)$$

$$= y^S(R_t, ...) \qquad \text{where} \quad \partial y^S/\partial R_t = [df(l^S)/dl^S][dl^S/dR] > 0$$

IX. Time-Shift

It will be convenient for later work to convert everything to present values *at the beginning* of period t.

A. Budget constraint

We must discount everything by one more period than we have done so far. Therefore, divide both sides of the budget constraint by 1+R to obtain

$$b_{t-1} + \sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+R)^{i+1}} = \sum_{i=0}^{\infty} \frac{c_{t+i}}{(1+R)^{i+1}}$$

B. Utility function

Similarly, we must discount utility by one more period, so divide by 1+ρ to obtain

$$U(c_t, c_{t+1}, ...; l_t, l_{t+1}, ...) = \sum_{i=0}^{\infty} \frac{u(c_{t+i}, l_{t+i})}{(1+\rho)^{i+1}}$$

Note that this change of dating merely rescales all quantities and has no effect on the previous analysis.