

Isolated Households

I. Production Function

A. The production function tells us how much output the household gets in return for its work effort (that is, for its labor input):

$$y_t = f(l_t)$$

where

y_t \equiv output per unit of time (per month, per year, etc., depending on what t measures)

l_t \equiv labor per unit of time (measured as a fraction of the period; e.g., if t is one day, then 8 hours = $1/3$)

B. Production functions have the following properties:

1. Output is non-negative

$$f(l_t) \geq 0 \text{ with } f(0) = 0, f(l) > 0 \text{ for all } l_t > 0$$

2. Marginal product of labor is positive

$$f'(l_t) > 0 \text{ for all } l_t$$

3. Marginal product of labor diminishes as labor increases

$$f''(l_t) < 0 \text{ for all } l_t$$

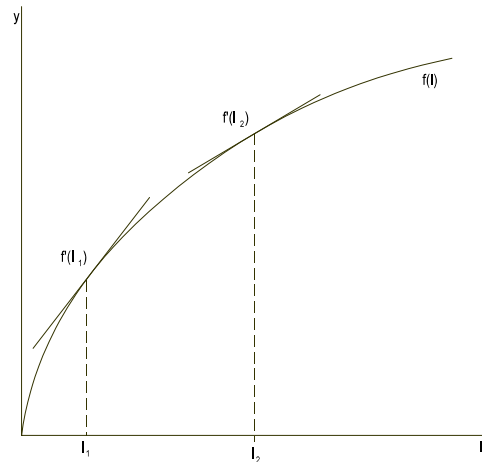


Figure 1: Production function

II. Utility Function

A. Happiness or well-being is measured by the utility function:

$$v(c_t, L_t)$$

where

c_t = consumption in period t

L_t = leisure in period t

B. The utility function has the following properties:

1. Utility is non-negative:

$$v(c, L) \geq 0$$

2. Marginal utilities are positive:

$$v_c \equiv \partial v / \partial c > 0 \text{ for all } (c_t, L_t)$$

$$v_L \equiv \partial v / \partial L > 0 \text{ for all } (c_t, L_t)$$

3. Marginal utilities diminish:

$$v_{cc} \equiv \partial^2 v / \partial c^2 < 0 \text{ for all } (c_t, L_t)$$

$$v_{LL} \equiv \partial^2 v / \partial L^2 < 0 \text{ for all } (c_t, L_t)$$

4. For simplicity, we also suppose that the cross-derivatives are zero:

$$v_{cL} \equiv \partial^2 v / \partial c \partial L = 0$$

$$v_{Lc} \equiv \partial^2 v / \partial L \partial c = 0$$

[For any utility function ever used in economics, $v_{cL} = v_{Lc}$, that is, v is of class C^2 , so that Young's theorem applies and guarantees equality of the cross derivatives.]

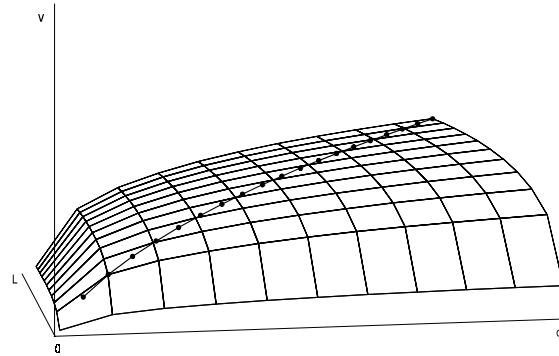


Figure 2: Utility function in 3-space

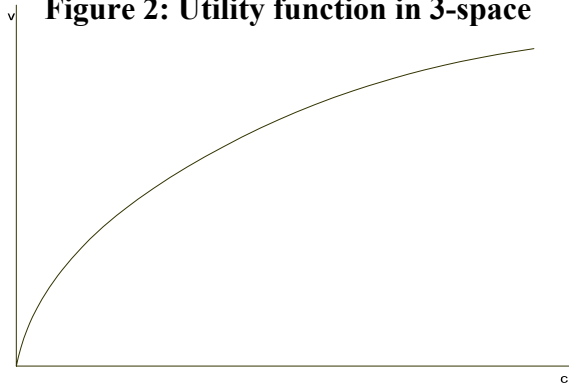


Figure 3: Utility function in v-c space

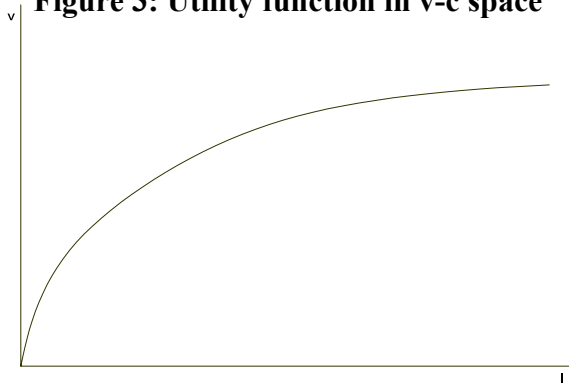


Figure 4: Utility function in v-l space

D. Utility as a function of labor.

It will be convenient to write utility as a function of labor rather than leisure. We use the restriction that total time available T must equal time spent on leisure L_t and time spent on labor l_t :

$$T = L_t + l_t$$

to write

$$L_t = T - l_t$$

We then can write the utility function as

$$v(c_t, T - l_t)$$

We can think of this as a function of labor rather than leisure and define the new function

$$u(c_t, l_t)$$

The function u has the following properties

1. $u > 0$ because $u = v$

2. Marginal utilities

$$u_c = v_c > 0$$

$$\begin{aligned} u_l &= v_L (\partial L / \partial l) \\ &= v_L (-1) \\ &= -v_L \\ &< 0 \end{aligned}$$

3. Second derivatives

$$u_{cc} = v_{cc} < 0$$

$$\begin{aligned} u_{ll} &= \partial u_l / \partial l \\ &= \partial(-v_L) / \partial l \\ &= -v_{LL} (\partial L / \partial l) \\ &= -v_{LL} (-1) \\ &= v_{LL} \\ &< 0 \end{aligned}$$

$$u_{cl} = u_{lc} = 0$$

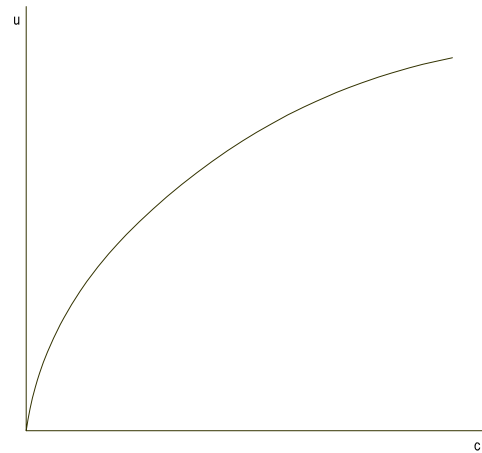


Figure 5: Utility function in u-c space

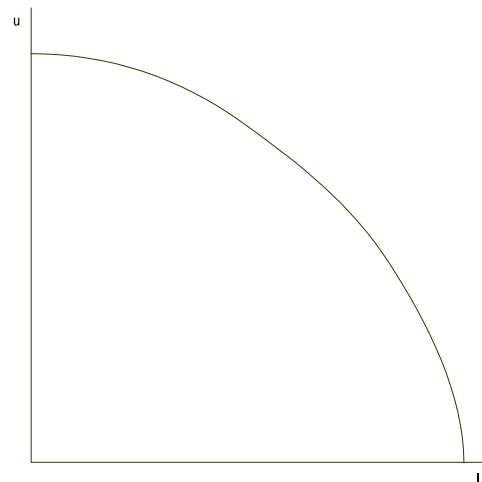


Figure 6: Utility function in u-l space

E. Indifference curves

An indifference curve is the locus of points (c, L) for which utility has a given value u^* :

$$u(c, l) = u^*$$

We want to see what the indifference curves look like.

1. To find its slope, totally differentiate:

$$\begin{aligned} u_c dc + u_l dl &= du^* \\ &= 0 \quad \text{because } u^* = \text{constant} \end{aligned}$$

Then rearrange terms to get

$$\begin{aligned} dc/dl &= -u_l / u_c \\ &> 0 \end{aligned}$$

2. Curvature:

$$\begin{aligned} \frac{d(dc/dl)}{dl} &= \frac{d(-u_l/u_c)}{dl} \\ &= -u_{ll}/u_c \quad (\text{Recall that } u_{cl} = 0 \text{ by assumption}) \\ &> 0 \end{aligned}$$

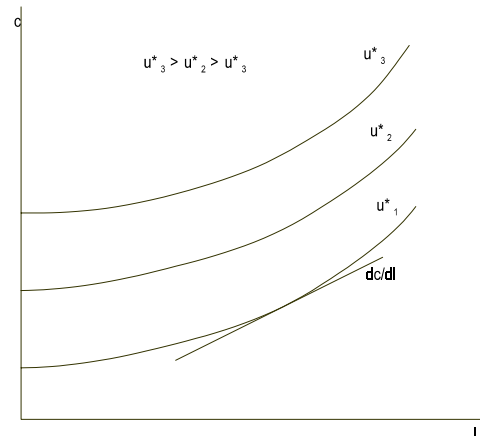


Figure 7: Indifference curves

III. Optimal Choice of c and l

A. Goal: maximize utility

⇒ get to highest indifference curve possible

B. Utility maximization is subject to two *constraints*:

1. The amounts of leisure and labor, L and l, must be non-negative and must sum to the total amount of time available:

$$\begin{aligned} L &\geq 0 \\ l &\geq 0 \\ L + l &= T \end{aligned}$$

2. The amount of consumption available must be less than or equal to the amount of output (= income) produced:

$$c \leq y = f(l)$$

Formally, anything on or below the production function is *feasible*. However, it should be obvious that anything below the production function is sub-optimal.

C. Choose (c^*, l^*) where the production function is just touching an indifference curve (which will be the highest feasible indifference curve)

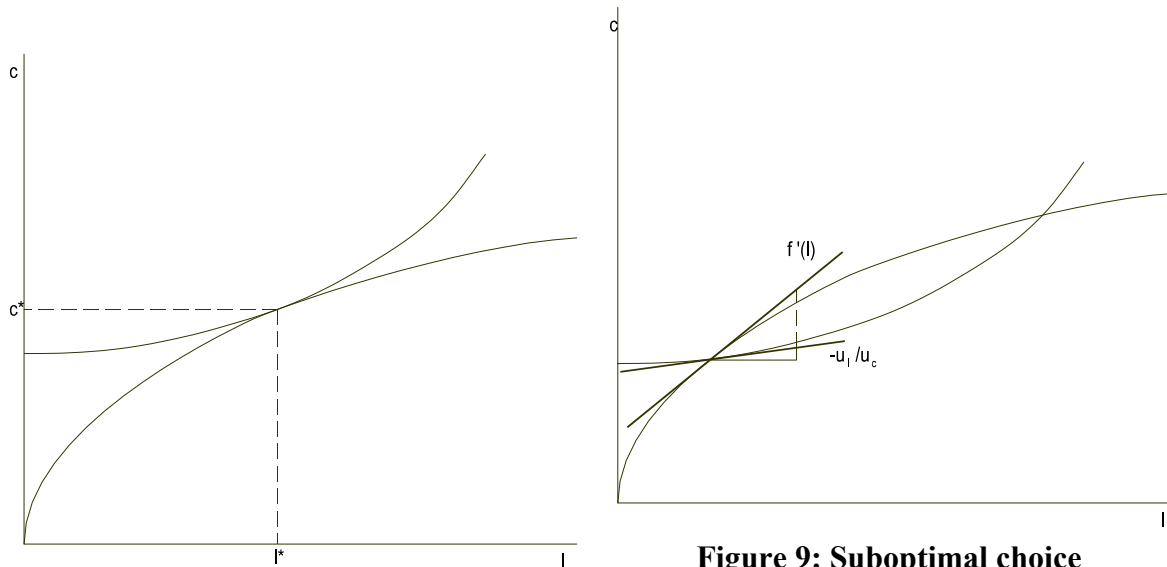


Figure 8: Optimal choice

Figure 9: Suboptimal choice

D. Mathematical formulation

1. Note that $c = f(l)$ and substitute into utility function to get $u[f(l), l]$
2. Take derivative with respect to l , set equal to 0, and solve for l :

$$u_c f'(l) + u_l = 0$$

Note that this can be rearranged as

$$f'(l) = -u_l / u_c$$

which is consistent with result in Figure 8.

3. Substitute solution for l into production function to get c :

$$c = f(l)$$

4. Alternatively, form Lagrangean:

$$V = u(c, l) + \lambda[f(l) - c]$$

and take derivatives with respect to c , l , and λ to get the *first-order conditions*

$$u_c - \lambda = 0$$

$$u_l + \lambda f'(l) = 0$$

$$f(l) - c = 0$$

Note that the first two first-order conditions together imply once again

$$f'(l) = -u_l / u_c$$

E. Alternative point of view

1. Any other point is suboptimal.
2. See Figure 9
 - a. Slope of indifference curve: indicates exchange toward which one is indifferent. This slope is called the *marginal rate of substitution*.
 - b. Slope of production function: indicates exchange one is able to make. This slope is the relative price of leisure in terms of consumption. It shows how much consumption the household could get if it gave up one more unit of leisure (that is, if it supplied one more unit of labor).
 - c. In Figure 9, slope of production function $>$ slope of indifference curve
 - \Rightarrow household is able to supply (infinitesimally) more labor and receive in return more consumption than it requires to be willing to supply that much extra labor
 - \Rightarrow household definitely will supply more labor
 - \Rightarrow original point was not optimal

IV. Optimal Response to Shocks

A. Pure wealth (or income) effect

1. A shock creates a pure wealth effect if
 - a. The household is able to have more of one good without reducing the quantity of any other good
 - b. There are no changes in relative prices
2. We restrict attention to the case where consumption and leisure are both *normal* goods: both are positively related to household wealth or income. (Individual consumption goods may not be normal, but total consumption is always normal. All our analysis will concern total consumption.)

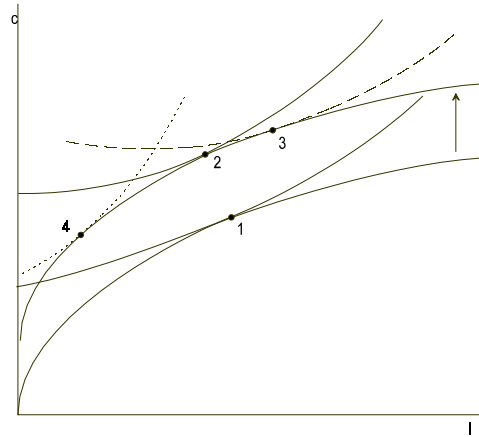


Figure 10: Pure wealth effect

3. Example: upward parallel shift of production function. (A parallel shift of a curve means that the slopes of the tangent lines on the two curves are the same for any given value of the independent variable (which, for the production function, is labor.)

B. Pure substitution effect

1. A shock creates a pure substitution effect if
 - a. The household receives exactly the same amount of any one good if it keeps all other goods constant
 - b. The shock changes relative prices
2. Example: a change in technology that twists the production function so that it passes through the original point but has a higher slope there.

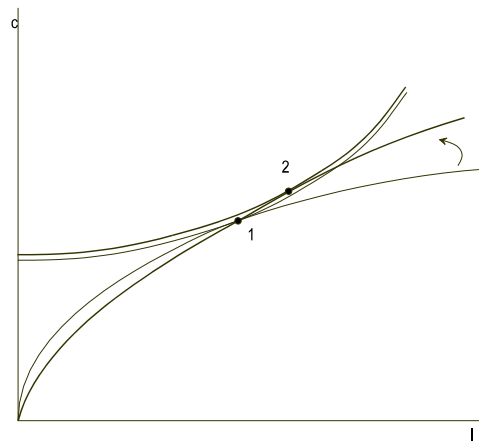


Figure 11: Pure substitution effect

- a. The higher slope means that the relative price of leisure in terms of consumption is higher.
- b. The household substitutes the relatively less expensive good (consumption) for

the relatively more expensive good (leisure).

C. Combination effect

1. Example: upward tilt of the production function around zero

2. Both wealth and substitution effects

a. Both effects cause consumption to rise, so the change in consumption is unambiguous.

b. The two effects have opposite effects on labor, so the change in labor is ambiguous.

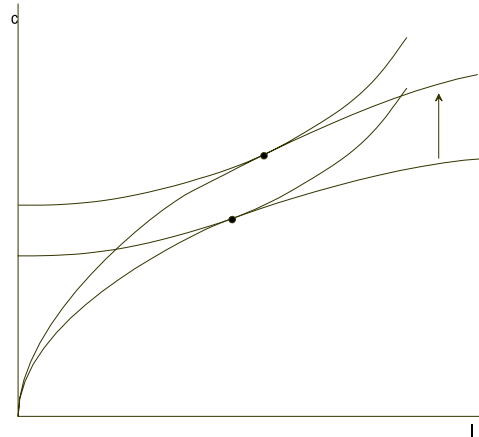


Figure 12: Combined effect

V. Demand and Supply

We have determined the household's optimal consumption c^* , labor l^* , and output $y^* = f(l^*)$. We can think of these as quantities demanded and supplied:

1. Consumption demand

$$\begin{aligned} c_t^* &= \text{quantity of consumption demanded at time } t \\ &\equiv c_t^D \end{aligned}$$

2. Labor supply

$$\begin{aligned} l_t^* &= \text{quantity of labor supplied at time } t \\ &\equiv l_t^S \end{aligned}$$

3. Output supply

$$\begin{aligned} f(l_t^S) &= \text{quantity of output supplied at time } t \\ &\equiv y_t^S \end{aligned}$$