

1. In his textbook, Barro discusses informally the economics of Robinson Crusoe. Crusoe lives on an island, works only for himself, and hires no one to work for him. Suppose Crusoe's utility at time t is a concave function u of consumption c_t and labor l_t :

$$u_t = u(c_t, l_t)$$

$$u_c > 0, u_{cc} < 0$$

$$u_l < 0, u_{ll} < 0$$

$$u_{cc}u_{ll} - u_{cl}u_{lc} > 0$$

Crusoe produces goods y_t with his labor:

$$y_t = f(l_t)$$

$$= A_l l_t^\alpha + B_t$$

$$A, B > 0; 1 > \alpha > 0.$$

Goods are perishable and cannot be stored.

A. Assume first that Crusoe has no contact with any other people. Derive formally the solution for Crusoe's optimal time paths of consumption and labor. Describe the relation between c_t and forecastable changes in B_t . Go as far as you can without assuming anything about u_{cl} ; then see what more you can say if $u_{cl}=0$.

B. Suppose now that Crusoe discovers a friendly neighboring tribe with whom he can engage in borrowing and lending. Derive again Crusoe's optimal time paths of consumption and labor, subject to the usual condition that lifetime spending must equal lifetime production. Also, describe the relation between c_t and forecastable changes in B_t .

2. Suppose an individual has a Constant Relative Risk Aversion (CRRA) utility function with index of relative risk aversion θ :

$$U(C) = \frac{C^{1-\theta}}{1-\theta}$$

He lives two periods and faces the lifetime budget constraint

$$P_1 C_1 + P_2 C_2 = W$$

where P_i is the price of consumption in period i and W is the value of lifetime income. His rate of time preference is ρ .

(A) Find the individual's utility-maximizing values of C_1 and C_2 , given P_1 , P_2 , and W .

(B) Show that the elasticity of substitution between C_1 and C_2 is $1/\theta$. (The elasticity of substitution is the

percentage change in one ratio with respect to the percentage change in another ratio, usually a price ratio. Thus we have

$$\begin{aligned}\epsilon &= - \frac{\left[\frac{\partial(C_1/C_2)}{(C_1/C_2)} \right]}{\left[\frac{\partial(P_1/P_2)}{(P_1/P_2)} \right]} \\ &= - \frac{(P_1/P_2)}{(C_1/C_2)} \left[\frac{\partial(C_1/C_2)}{\partial(P_1/P_2)} \right] \\ &= - \frac{\partial \ln(C_1/C_2)}{\partial \ln(P_1/P_2)}\end{aligned}$$

for the problem at hand.)

3. Suppose that initial bond holdings are $b_0 = \$1000$. All commodity market receipts are zero, that is, $P y_t = 0$ for all t . The interest rate is $R = 10\%$ per year and the price level is constant over time.

(A) What is the level of permanent income? If consumption is constant, what is the amount of saving each year? What is the level of permanent income when we calculate it during period 2?

(B) Suppose now that $c_1 = 0$. What is permanent income when we calculate it during the second period? Explain this result.

4. Suppose an individual learns in period 1 that his marginal product of labor will rise in period 2 and remain at the new level until period 3, when it will return to its period 1 level and remain there forever.

A. Explain what happens to this individual's consumption and labor supply in periods 1, 2, and 3.

B. How would things be different for this individual if everyone else in the economy faced the same wage path as he does? Explain.

5. Suppose an individual has a finite time horizon of D , at which time he is certain he will die. His utility function is

$$\begin{aligned}u_t &= u(c_t, l_t) \\ u_c &> 0, u_{cc} < 0 \\ u_l &< 0, u_{ll} < 0 \\ u_{cl} &= u_{lc} = 0\end{aligned}$$

He earns income by supplying labor to an employer who pays the constant nominal wage rate of W . He can borrow and lend freely, with the only constraint being that the present value of his lifetime expenditure must equal the present value of his lifetime earnings plus initial assets. The price level P is constant, and so is the interest rate r .

Set up and solve the optimal control formulation of this individual's intertemporal optimization problem. State all necessary conditions. Draw the phase diagram and show the solution path.

6. In the simple Solow-Swan growth model with no technical progress and population growth rate n , prove rigorously that the steady state growth rates of capital K and output Y both equal n .

7. Suppose we introduce labor-augmenting technical progress into the Solow-Swan growth model by changing the production function to

$$Y_t = F(K_t, A_t L_t)$$

where

$$A_t = A_0 e^{gt}$$

and g is a positive constant (called the rate of technical progress). We suppose F is homogeneous of degree 1 in K and AL .

(A) Solve for the steady-state of this model. Illustrate your answer by drawing the usual kind of diagram showing the intersection of the aggregate saving and augmented depreciation functions.

(B) What are the growth rates of

$$\begin{aligned} &K \\ &k \equiv K/L \end{aligned}$$

in the steady state?

8. Consider the discrete-time version of the Solow-Swan growth model. The production function exhibits constant returns to scale, has the form

$$Y_t = F(K_t, L_t)$$

and satisfies the Inada conditions:

$$\lim_{K \rightarrow 0} F_K = \infty$$

$$\lim_{K \rightarrow \infty} F_K = 0$$

$$\lim_{L \rightarrow 0} F_L = \infty$$

$$\lim_{L \rightarrow \infty} F_L = 0$$

The dynamic equations are

$$L_{t+1} = (1+n)L_t$$

$$K_{t+1} = K_t + sY_t - \delta K_t$$

(A) Find the expression for $k_{t+1} = K_{t+1}/L_{t+1}$ as a function of k_t .

(B) Draw a graph of k_{t+1} as a function of k_t . Explain whether the economy has a steady state.

9. Suppose we introduce government into the Solow-Swan growth model, with G =purchases, T =lump sum taxes, and $G=T$ always (no government debt or monetary finance). Suppose G has no investment component.

A. Modify the model to include government.

B. Explain the effects of an increase in G on the steady state values of output/labor and capital/labor.

10. Suppose the population growth rate n increases. Derive the effects this increase has on the steady state values of:

k capital/labor ratio
 c consumption/capita

in

(A) the Solow-Swan growth model

(In this model, $dk/dt = sy - (\delta + n)k$, where k is the capital/labor ratio, y is output per capita and is equal to $f(k)$ with $f = F(K/L, 1)$ and F a homogeneous degree one production function, K is the capital stock, L is labor supply, s is a constant fraction of income that is saved, δ is the depreciation rate, and n is the population growth rate.)

(B) the Blanchard-Fischer version of the continuous-time Ramsey growth model,

(In this model, the path of capital is chosen to maximize $\int u(c_t) \exp(-\rho t) dt$ subject to the constraint that $dk/dt = f(k) - c - (d+n)k$, where u is the utility function, c is consumption per capita, ρ is the rate of time preference, and all other variables are defined as in the Solow model.)

(C) and the Cass version of the continuous-time Ramsey growth model.

(Same as the Blanchard-Fischer model except that the objective function is $\int u(c_t) L_t \exp(-\rho t) dt$.)

For (A), draw a graph illustrating your answer. For (B) and (C), draw phase diagrams illustrating your answers.

11. Suppose we introduce labor-augmenting technical progress into the Cass growth model by changing the production function to

$$Y_t = F(K_t, A_t L_t)$$

where

$$A_t = A_0 e^{gt}$$

and g is a positive constant (called the rate of technical progress). We suppose F is homogeneous of degree 1 in K and AL .

State the maximization problem, state the transformed (or current value) Hamiltonian, and state all necessary conditions for a maximum.

[NOTE: It makes no sense to define utility as a function of consumption per efficiency units of labor. After all, if C and L stay fixed but technology improves, consumption per efficiency unit goes down, but obviously each household's utility should be unchanged because it still has the original level of consumption per person. Therefore, utility must be a function of $c=C/L$. However, neither c nor the capital/labor ratio $k=K/L$ has a steady state because of the technical progress term, so the analysis must consider $k^*=K/(AL)$. Keep all this in mind as you work out the answer.]

12. Suppose there is an unexpected permanent reduction in the efficiency of production, so that the production function becomes $zf(k)$ with z between zero and one. Assuming the economy started in steady state, derive and explain its optimal dynamic adjustment toward the new steady state.

13. The national budget constraint for the Cass growth model is

$$c_t + i_t \leq f(k_t)$$

where c_t is consumption and i_t is gross investment:

$$i_t = (dk/dt) + \delta k_t$$

with δ being the rate of depreciation. Our classroom discussion of the Cass model implicitly assumed that consumption always was strictly less than output; in other words, we assumed an interior solution.

Derive the general solution, allowing for corner solutions. State all necessary conditions, and draw the appropriate phase diagram.

14. Consider the following three alternative schemes for government purchases and taxation.

A. Government purchases per capita g is a constant \bar{g} and satisfies the budget constraint $\bar{g} = T$, where T is a lump-sum per capita tax.

B. Government purchases per capita g fluctuates as necessary to satisfy the budget constraint $g = \tau f(k)$, where the tax rate τ is constant.

C. Government purchases per capita g is a constant $\bar{g} = \tau f(k)$ with the tax rate τ fluctuating as necessary to satisfy this constraint.

Derive and compare the steady states in the Ramsey infinite-horizon model for these three cases.

15. Suppose, as emphasized by Barro and others, that government purchases partly substitute for privately purchased consumption at the rate of α and also have a marginal product of β , where $0 < \alpha, \beta, \alpha + \beta < 1$. Government purchases are financed by lump-sum taxes distributed evenly over the population. Individuals have identical preferences and derive utility from consumption and leisure. Production requires capital and labor. The dynamic equation for capital is

$$dK/dt = F(K, L) - C - \delta K$$

where δ is the depreciation rate.

A. Set up the individual's maximization problem (equivalently, the central planner's problem), stating the function to be maximized, the constraints under which it is maximized, the Hamiltonian, the dynamic equations for all state and costate variables, the first order conditions, and the endpoint conditions.

B. Show the effects on the individual's choices of a change in government purchases in which $dG = dG^* > 0$, where G and G^* are current and permanent purchases.

C. Show the effects on the individual's choices of a change in government purchases in which $dG^* > 0 = dG$.

D. Show the effects on the individual's choices of a change in government purchases in which $dG_1 = -dG_2 > 0$.

E. Show the effects on the individual's choices of a change in government purchases in which $dG > dG^* = 0$.

16. Assume

(i) a Cobb-Douglas production function with labor share α ,

(ii) two-period lives, with utility provided only by consumption according to the logarithmic utility function

$$U = \ln c_{1t} + (1+\theta)^{-1} \ln c_{2t+1},$$

with θ = rate of time preference,

(iii) one unit of labor is supplied inelastically in the first period of life and no labor is supplied in the second period of life,

(iv) the population growth rate is n , and

(v) the physical depreciation of capital is 100%.

Using the overlapping generations model, solve for the steady state capital stock.

17. Suppose two economies ($i=1,2$) have the same production function $Y_{it} = K_{it}^\theta$ and the same capital dynamics $dK_i/dt = s_i Y_{it}$. Suppose $\theta > 1$, so we have increasing returns to scale. Suppose also that the two economies have the same initial capital stock but that $s_1 > s_2$. Show that Y_1/Y_2 is continuously rising.

18. Consider a firm that owns its own capital K_t and that buys its investment goods elsewhere (i.e., does not make them itself). Suppose this firm is a price-taker in all markets. The production function is

$$Y_t = F(K_t) \text{ with } F' > 0, F'' < 0$$

For simplicity, we ignore all factors of production other than capital. There is no cost of installing new capital. Also, the firm is prohibited from borrowing, capital depreciates at rate δ , the market interest rate is r (which the firm uses only for discounting), the price of output is 1, and the price of investment is P . All prices are constant for simplicity. The firm's objective is to maximize the present value of profits.

Using a continuous-time model, state the firm's intertemporal optimization problem and the necessary conditions for solving it. Describe the nature of the firm's investment path, and give a brief economic explanation of it.

19. Assume that utility exhibits constant relative risk aversion and that the conditional distribution of C_{t+1} is lognormal. Use the first-order condition

$$[(1+r)/(1+\rho)]E[U'(C_{t+1})|t] = U'(C_t)$$

to derive the relation between expected consumption next period and consumption this period. Show how that relation depends on the riskless interest rate, the subjective discount rate, and the variance of next period's consumption.

HINTS:

(1) CRRA utility is HARA.

(2) For X normally distributed,

$$E(e^X) = e^{E(X) + \frac{\sigma^2 X}{2}}$$

20. Consider a one-period bond A. The nominal repayment at maturity is

$$A(1+R) = A(1+r^e)(1+\pi^e)$$

where R=nominal interest rate, r^e =expected real interest rate, and π^e =expected inflation rate.

(A) Suppose the period is divided into n sub-periods with interest compounded over these sub-periods (i.e., interest is paid at the end of each sub-period and subsequent sub-periods pay interest on the principal plus accumulated interest). Show that the nominal repayment at maturity is

$$A[1+(r/n)^e]^n [1+(\pi/n)^e]^n$$

Note in particular that when n=1, the usual formula for the nominal interest rate,

$$R = r + \pi$$

is only approximately true, not exact.

(B) Derive the repayment at maturity when compounding is continuous. What about the formula $R=r+\pi$ in this case?