CHAPTER 14

Quiver varieties

14.1. Definition and geometric properties

We fix notations and state various properties of quiver varieties in this section. The details and motivations are omitted in most of cases. See [**N-I, N-II**] and explicitly given references. The author also recommend the lecture notes by Ginzburg [**Gi-I**]. We allow the graph with edge loops here, which did not in [**N-I, N-II**]. But the proofs work unless explicitly mentioned.

14.1.1. A graph and a quiver. Let $\mathcal{G} = (I, E)$ be a finite graph, where I is the set of vertices and E the set of edges. Let $\mathbf{C} = (c_{ij})$ be the Cartan matrix of the graph, namely

$$c_{ij} = \begin{cases} 2 - 2(\text{the number of edge loops joining } i \text{ to itself}) & \text{if } i = j, \\ -(\text{the number of edges joining } i \text{ to } j) & \text{if } i \neq j. \end{cases}$$

Example 14.1. For the graph $\mathcal{G}$ with a single vertex with a single edge loop, the Cartan matrix is 0. More generally, for a single vertex with g edge loops, the Cartan matrix is $2 - 2g$. The first example will be called the *Jordan quiver* if we equip an orientation to the edge as explained below.

If the graph $\mathcal{G}$ has no edge loops, then $\mathbf{C}$ is a (symmetric) Cartan matrix. We normalize the bilinear form as $(\alpha_i, \alpha_i) = 2$ and consider the corresponding Kac-Moody Lie algebra $\mathfrak{g}$ and quantum enveloping algebra $\mathbf{U}_q(\mathfrak{g})$. We will relate the representation theory of $\mathfrak{g}$ to homology groups of quiver varieties, while we relate only the crystal structure of $\mathbf{U}_q(\mathfrak{g})$ to quiver varieties.

Remark 14.2. Even if we do not assume that the graph $\mathcal{G}$ has no edge loops, $\mathbf{C}$ is a Borcherds-Cartan matrix (or generalized Cartan matrix) in the sense of [脇本, §2.1]. The set of vertices with edge loops is identified with the set of simple imaginary roots I^{im}. We can consider the corresponding Borcherds-Kac-Moody (BKM) Lie algebra $\mathfrak{g}$, but its representations are *different* from those naturally arising quiver varieties. For example, the Jordan quiver gives the Heiseberg alebra $\langle e, f, h\rangle$ as the BKM Lie algebra, while quiver varieties for the Jordan quiver give the infinite dimensional Heisenberg algebra. In general, it seems that the BKM algebra only gives small part in the homology groups of quiver varieties (cf. [**KKS09**]). Therefore we do not treat the BKM Lie algebra so much in this book.

Let H be the set of pairs consisting of an edge together with its orientation. For $h \in H$, we denote by $\mathrm{i}(h)$ (resp. $\mathrm{o}(h)$) the incoming (resp. outgoing) vertex of h. For $h \in H$ we denote $\bar{h}$ the same edge as h with the reverse orientation. An orientation Ω of the graph $\mathcal{G}$ is a subset $\Omega \subset H$ such that $\bar{\Omega} \cup \Omega = H$, $\bar{\Omega} \cap \Omega = \emptyset$. A graph $\mathcal{G}$ together with a choice of an orientation Ω is called a *quiver*, and denoted by $\mathcal{Q} = (I, \Omega)$. The orientation defines a function $\varepsilon\colon H \to \{\pm 1\}$ given by $\varepsilon(h) = 1$ if $h \in \Omega$ and $= -1$ if $h \in \bar{\Omega}$. In fact, it turns out that a choice of the orientation is not essential for quiver varieties. It is canonically attached to the graph.

Let $\mathbf{A}_{\mathcal{Q}} = (a_{ij})$ be the adjacency matrix of the quiver $\mathcal{Q}$, i.e., $a_{ij} = \#\{h \in \Omega \mid \mathrm{o}(h) = j, \mathrm{i}(h) = i\}$. Then we have $\mathbf{C} = 2\mathbf{I} - \mathbf{A}_{\mathcal{Q}} - {}^t\mathbf{A}_{\mathcal{Q}}$, where ${}^t\mathbf{A}_{\mathcal{Q}}$ is the transpose of $\mathbf{A}_{\mathcal{Q}}$.

14.1.2. A symplectic vector space. Let $V = \bigoplus_{i\in I} V_i$ be an I-graded vector space over $\mathbb{C}$. We define its *dimension vector* by $\dim V \overset{\text{def.}}{=} (\dim V_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}$. It is usually denoted by $\mathbf{v}$.

If V^1, V^2 are I-graded vector spaces, we introduce following vector spaces

$$\mathbf{L}(V^1, V^2) \overset{\text{def.}}{=} \bigoplus_{i\in I} \operatorname{Hom}(V^1_i, V^2_i),$$
$$\mathbf{E}(V^1, V^2) \overset{\text{def.}}{=} \bigoplus_{h\in H} \operatorname{Hom}(V^1_{\mathrm{o}(h)}, V^2_{\mathrm{i}(h)}).$$

For $B = (B_h) \in \mathbf{E}(V^1, V^2)$, $C = (C_h) \in \mathbf{E}(V^2, V^3)$, we define a multiplication of B and C by

$$CB \overset{\text{def.}}{=} \left(\sum_{\mathrm{i}(h)=i} C_h B_{\bar{h}}\right)_i \in \mathbf{L}(V^1, V^3).$$

Multiplications ba, Ba of $a \in \mathbf{L}(V^1, V^2)$, $b \in \mathbf{L}(V^2, V^3)$, $B \in \mathbf{E}(V^2, V^3)$ are defined in obvious manner. If $a \in \mathbf{L}(V^1, V^1)$, its trace $\operatorname{tr}(a)$ is understood as $\sum_i \operatorname{tr}(a_i)$.

Let V, W be I-graded vector spaces. We define

$$\mathbf{M}(V, W) \overset{\text{def.}}{=} \mathbf{E}(V, V) \oplus \mathbf{L}(W, V) \oplus \mathbf{L}(V, W).$$

If the dimension vectors of V, W are only relevant in the context, we simply denote this by $\mathbf{M}(\mathbf{v}, \mathbf{w})$ where $\mathbf{v} = \dim V$, $\mathbf{w} = \dim W$. We may also use even simpler notation $\mathbf{M}$. This kind of convention will be used for other notations hereafter.

The components of an element in $\mathbf{M}$ will be denoted by B, a, b respectively. The space $\mathbf{M}$ has a holomorphic symplectic form given by

$$\omega((B, a, b), (B', a', b')) \overset{\text{def.}}{=} \operatorname{tr}(\varepsilon B B') + \operatorname{tr}(ab' - a'b),$$

where εB is an element of $\mathbf{E}(V, V)$ defined by $(\varepsilon B)_h = \varepsilon(h) B_h$.

Let $G \equiv G_V \equiv G_{\mathbf{v}}$ be the Lie group $\prod_i \mathrm{GL}(V_i)$. It acts on $\mathbf{M}$ by

$$(B, a, b) \mapsto g \cdot (B, a, b) \overset{\text{def.}}{=} (gBg^{-1}, ga, bg^{-1})$$

preserving the symplectic structure. The space $\mathbf{M}$ has a factor

$$\mathbf{M}^{\mathrm{el}} \overset{\text{def.}}{=} \bigoplus_{h:\mathrm{o}(h)=\mathrm{i}(h)} \mathbb{C}\,\mathrm{id}_{V_{\mathrm{o}(h)}} \tag{14.3}$$

on which G acts trivially. This has a 2-dimensional space for each edge loop, and hence has dimension $\sum_i (2 - c_{ii})$ in total.

14.1.3. Symplectic quotients. The moment map vanishing at the origin is given by

$$\mu(B,a,b) = \varepsilon BB + ab \in \mathbf{L}(V,V),$$

where the dual of the Lie algebra of G is identified with $\mathbf{L}(V,V)$ via the trace.

Let $\zeta_{\mathbb{C}} = (\zeta_{\mathbb{C},i}) \in \mathbb{C}^I$. We define a corresponding element in the center of $\operatorname{Lie} G$ by $\bigoplus_i \zeta_{\mathbb{C},i}\operatorname{id}_{V_i}$, where we delete the summand corresponding to i if $V_i = 0$. Let $\mu^{-1}(\zeta_{\mathbb{C}})$ be an affine algebraic variety (not necessarily irreducible) defined as the zero set of $\mu - \zeta_{\mathbb{C}}$. The group G acts on $\mu^{-1}(\zeta_{\mathbb{C}})$. We only consider the case $\zeta_{\mathbb{C}} = 0$ hereafter.

We now define stability conditions.

For $\zeta_{\mathbb{R}} = (\zeta_{\mathbb{R},i})_{i\in I} \in \mathbb{R}^I$, let $\zeta_{\mathbb{R}} \cdot \dim V \overset{\text{def.}}{=} \sum_{i\in I} \zeta_{\mathbb{R},i} \dim V_i$.

Definition 14.4. A point $(B,a,b) \in \mathbf{M}$ is *$\zeta_{\mathbb{R}}$-semistable* if the following two conditions are satisfied:

1. If an I-graded subspace S of V is contained in $\operatorname{Ker} b$ and B-invariant, then $\zeta_{\mathbb{R}} \cdot \dim S \le 0$.
2. If an I-graded subspace T of V contains $\operatorname{Im} a$ and is B-invariant, then $\zeta_{\mathbb{R}} \cdot \dim T \le \zeta_{\mathbb{R}} \cdot \dim V$.

We say (B,a,b) is *$\zeta_{\mathbb{R}}$-stable* if the strict inequalities hold in (1),(2) unless $S = 0$, $T = V$ respectively.

If $\zeta_{\mathbb{R},i} > 0$ for all i, the condition (2) is superfluous, and the condition (1) turns out to be the nonexistence of nonzero B-invariant I-graded subspaces $S = (S_i)$ contained in $\operatorname{Ker} b$ (and in this case $\zeta_{\mathbb{R}}$-stability and $\zeta_{\mathbb{R}}$-semistability are equivalent.) This is the stability condition used in [**N-I**]. We only use this $\zeta_{\mathbb{R}}$ hereafter.

Let H^{s} be the set of $\zeta_{\mathbb{R}}$-stable points in $\mu_{\mathbb{C}}^{-1}(0)$. We define

$$\mathfrak{M} \equiv \mathfrak{M}(V,W) \equiv \mathfrak{M}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} H^{\mathrm{s}}/G.$$

This can be defined as quotients in the geometric invariant theory, and hence has a natural strucure of a quasiprojective scheme.

On the other hand, we can define the *affine algebro-geometric quotient* of $\mu_{\mathbb{C}}^{-1}(0)$ by G:

$$\mathfrak{M}_0 \equiv \mathfrak{M}_0(V,W) \equiv \mathfrak{M}_0(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \mu^{-1}(0)/\!\!/G.$$

This has a structure of an affine algebraic scheme, but as a set it consists of closed G-orbits in $\mu^{-1}(0)$. By the construction, we have a map

$$\pi\colon \mathfrak{M} \to \mathfrak{M}_0,$$

by mapping a stable G-orbit to the closed G_V-orbit in its closure. By a general result of the geometric invariant theory, it is a projective morphism.

Since G acts trivially on the factor (14.3), we have the factorization

$$\mathfrak{M} = \mathbf{M}^{\mathrm{el}} \times \mathfrak{M}^{\mathrm{norm}}, \tag{14.5}$$

where $\mathfrak{M}^{\mathrm{norm}}$ is the symplectic quotient of the space of datum (B,a,b) satisfying $\operatorname{tr}(B_h) = 0$ for any h with $\mathrm{i}(h) = \mathrm{o}(h)$.

It is known that if (B,a,b) is $\zeta_{\mathbb{R}}$-stable,

- the stabilizer of (B,a,b) is trivial,
- the differential of $\mu_{\mathbb{C}}$ at (B,a,b) is surjective.

Therefore $\mathfrak{M}$ is a smooth variety of dimension

$$2\langle \mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{C}\mathbf{v}\rangle = \langle \mathbf{v}, 2\mathbf{w} - \mathbf{C}\mathbf{v}\rangle,$$

where $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{Z}^I$. In fact, as μ is a moment map, the two statements are closely related. Also we should mention that $\mathfrak{M}$ has a natural symplectic form as a symplectic quotient.

It is also known that $\mathfrak{M}$ is connected (if it is nonempty). See [**CB01**] for the proof.

14.1.4. Generators of invariants. Consider the following two types of functions (the first takes values in $\mathbb{C}$, the second in $\operatorname{End}(W)$) on $\mathfrak{M}_0$ (or $\mathfrak{M}$ via π):

(a) $\operatorname{tr}(B_{h_N} B_{h_{N-1}} \cdots B_{h_1} \colon V_{\mathrm{o}(h_1)} \to V_{\mathrm{o}(h_1)})$, where $h_1, \dots, h_N$ is an oriented cycle in our graph, i.e., $\mathrm{i}(h_1) = \mathrm{o}(h_2)$, $\mathrm{i}(h_2) = \mathrm{o}(h_3), \dots, \mathrm{i}(h_{N-1}) = \mathrm{o}(h_N)$, $\mathrm{i}(h_N) = \mathrm{o}(h_1)$.

(b) $b_{\mathrm{i}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} a_{\mathrm{o}(h_1)} \in \operatorname{Hom}(W_{\mathrm{o}(h_1)}, W_{\mathrm{i}(h_N)})$, where $h_1, \dots, h_N$ is a path in our graph, i.e., $\mathrm{i}(h_1) = \mathrm{o}(h_2)$, $\mathrm{i}(h_2) = \mathrm{o}(h_3), \dots, \mathrm{i}(h_{N-1}) = \mathrm{o}(h_N)$.

It is known that these are generators of the coordinate ring of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$. This follows from the result of LeBryun-Procesi [**LP90**] plus Crawley-Boevey's trick in [**CB01**, the end of Introduction]. It is also proved in [**Lu98a**].

14.1.5. The tangent complex. The tangent space of $\mathfrak{M}$ at a point corresponding to (B, a, b) is given by the middle cohomology of the following complex:

$$\begin{gathered} \mathscr{C}^\bullet : \mathbf{L}(V,V) \xrightarrow{\alpha} \mathbf{E}(V,V) \oplus \mathbf{L}(W,V) \oplus \mathbf{L}(V,W) \xrightarrow{\beta} \mathbf{L}(V,V), \\ \alpha(\xi) = (B\xi - \xi B) \oplus (-\xi a) \oplus (b\xi), \quad \beta(C,d,e) = \varepsilon BC + \varepsilon CB + ae + db, \end{gathered} \tag{14.6}$$

where α is the infinitesimal action of the Lie algebra of G on $\mathbf{M}$, and β is the differential $d\mu$ of the moment map μ at (B, a, b). By the discussion above, the left and right cohomology groups vanish. In the following, we will consider similar three term complexes. Since the middle term is the most important, we assign the degree 0 at the middle.

If we move the point (B, a, b) in $\mathfrak{M}$, the middle cohomology group of $\mathscr{C}^\bullet$ becomes the tangent *bundle* of $\mathfrak{M}$. In fact, we can view V as a vector bundle over $\mathfrak{M}$ by identifying it with $H^{\mathrm{s}} \times_G V$. We call it the *tautologial* bundle. The same is true for W, $\mathbf{E}(V,V)$ etc, and $\mathscr{C}^\bullet$ as a complex of vector bundles over $\mathfrak{M}$. This convention will be used hereafter.

14.1.6. Regular locus. Let us introduce an open subset of $\mathfrak{M}_0$ (possibly empty):

$$\begin{aligned} \mathfrak{M}_0^{\mathrm{reg}} \equiv \mathfrak{M}_0^{\mathrm{reg}}(V, W) &\equiv \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}, \mathbf{w}) \\ &\overset{\text{def.}}{=} \{\, [B, a, b] \in \mathfrak{M}_0 \mid (B, a, b) \text{ has the trivial stabilizer in } G \,\}. \end{aligned}$$

Here $[B, a, b]$ denotes the (closed) orbit through (B, a, b). This notation is used hereafter (also for $\mathfrak{M}$).

If $\mathfrak{M}_0^{\mathrm{reg}} \neq \emptyset$, then $[B, a, b] \in \mathfrak{M}_0^{\mathrm{reg}}$ is a $\zeta_{\mathbb{R}}$-stable orbit. And it is easy to see that there is no other $\zeta_{\mathbb{R}}$-stable orbit in the inverse image $\pi^{-1}([B, a, b])$. Therefore π is an isomorphism over $\pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}})$.

14.1.7. A Lagrangian subvariety. We define a $\mathbb{C}^*$-action on $\mathbf{M}$ by multiplying B_h ($h \in \Omega$) and a by $t \in \mathbb{C}^*$, while B_h ($h \in \overline{\Omega}$) and b are unchanged. Then it commutes with the G-action and preserves $\mu = 0$ and the stability condition, hence we have an induced action on $\mathfrak{M}$. Let us denote it by $t \diamond [B, a, b]$. We also have an action on $\mathfrak{M}_0$.

Let

$$\mathfrak{L} \equiv \mathfrak{L}(V, W) \equiv \mathfrak{L}(\mathbf{v}, \mathbf{w})$$

be the set of points in $\mathfrak{M}$ such that $\lim_{t\to\infty} x$ exists. If we use generators in §14.1.4, this condition is equivalent to saying following ones vanish: type (b) and type (a) associated with an oriented cycle, which contains at least one oriented edge in Ω.

If we assume Ω does not contain an oriented cycle (there exists such an orientation if the graph contains no edge loops), all generators of type (a), (b) vanish. So it means that $\mathfrak{L} = \pi^{-1}(0)$, where the origin 0 is a closed G-orbit consisting of a single point, considered as a point in $\mathfrak{M}_0$ and denote it also by 0 for brevity.

Let $\mathfrak{F}$ be the fixed point locus $\mathfrak{M}^{\mathbb{C}^*}$ and $\mathfrak{F} = \bigsqcup \mathfrak{F}_\alpha$ be the decomposition into connected components. We define

$$\mathfrak{L}_\alpha \overset{\text{def.}}{=} \text{Closure of } \left\{ [B, a, b] \in \mathfrak{L} \,\middle|\, \lim_{t\to\infty} t \diamond [B, a, b] \in \mathfrak{F}_\alpha \right\}.$$

Theorem 14.7 ([**N-I**, 5.8]). *We have*

$$\mathfrak{L} = \bigcup \mathfrak{L}_\alpha,$$

and each $\mathfrak{L}_\alpha$ *is an irreducible component of* $\mathfrak{L}$.

Hence $\pi^{-1}(0)$ is lagrangian if the graph $\mathcal{G}$ has no edge loops. In general, it is an isotropic subvariety since it is contained in $\mathfrak{L}$. In fact, in (14.5) we have $\pi^{-1}(0) \subset \mathfrak{M}^{\text{norm}}$. The author does not know whether $\pi^{-1}(0)$ is half-dimensional in $\mathfrak{M}^{\text{norm}}$ or not. For the Jordan quiver, it turns out to be true thanks to a detailed study of the topology of $\mathfrak{M}$ (see [**NY04**, S3]).

Remark 14.8. Suppose that the graph has no edge loops.

In [**Lu-II**, §12] Lusztig introduced a subvariety of $\mathbf{E}(V, V)$:

$$\Lambda_V \overset{\text{def.}}{=} \{B \in \mathbf{E}(V, V) \mid \mu(B) = 0, B \text{ is nilpotent}\},$$

and showed that it is half-dimensional in $\mathbf{E}(V, V)$ and announced that it is Lagrangian. His proof is based on an inductive argument which will be recalled in §14.3.

From this and a standard result on symplectic quotients, $\pi^{-1}(0)$ is lagrangian. But our proof is completely different.

14.1.8. A parametrization of irreducible components of $\mathfrak{L}$. Let us take a point in $\mathfrak{F}$. Take and fix a point $(B, a, b) \in H^{\mathrm{s}}$ in the orbit. Then the condition that it is fixed by the $\mathbb{C}^*$-action means that there exists $\rho(t) \in G$ such that

$$t \diamond (B, a, b) = \rho(t)^{-1} \cdot (B, a, b).$$

Since the action of G on H^{s} is free, $\rho(t)$ is uniquely determined, and $t \mapsto \rho(t)$ gives a homomorphism $\mathbb{C}^* \to G$. Its conjugacy class is independent of the choice of the point (B, a, b) in the orbit. It is unchanged if the fixed point stays in a connected component $\mathfrak{F}_\alpha$ of $\mathfrak{F}$. In fact, one can show that connected components and conjugacy classes of $\mathbb{C}^* \to G$ are bijective, if Ω does not contain an oriented cycle. (See [**Qaff**, 7.3.4]. The proof uses the compactness of $\mathfrak{F}$, and hence we need the assumption on Ω. Probably it is true always.)

Let us decompose V as

$$V = \bigoplus_{m\in\mathbb{Z}} V(m), \qquad V(m) = \bigoplus_{i\in I} V_i(m),$$

where $V(m) = \{v \in V \mid \rho(t)v = t^m v\}$. Then dimensions of $V_i(m)$, considered as an element of $\mathbb{Z}_{\geq 0}^{I\times\mathbb{Z}}$, determine the conjugacy class of ρ uniquely.

Combined with the above theorem, we see that the set of irreducible components of $\mathfrak{L}$ is a subset of $\mathbb{Z}_{\geq 0}^{I\times\mathbb{Z}}$. This observation leads to a purely combinatorial description of the crystal structure of $\mathscr{B}(\lambda)$ later (a remark after Theorem 14.27).

14.1.9. Examples. Usually $\mathfrak{M}$ and $\mathfrak{M}_0$ are just defined as above and do not have any other explicit description. The followings are few exceptions:

(1)([**N-I**, 7.3]) If the graph $\mathcal{G}$ is of type A_n ○–○–○– · · · –○, and $\dim W$ is N–0–0– · · · –0, $\mathfrak{M}$ is the cotangent bundle of an n-step flag variety parametrizing flags $0 \subset E^1 \subset E^2 \subset \cdots \subset E^n \subset \mathbb{C}^N$ with $\dim E^i = \dim V_i$. And $\mathfrak{M}_0$ is the closure of a nilpotent orbit in $\operatorname{End}(\mathbb{C}^N)$, where the orbit is determined by $\dim V$. $\pi^{-1}(0)$ is the 0-section in the cotangent bundle.

For more general $\dim W$, $\mathfrak{M}_0$ is the intersection of the Slodowy slice and the closure of a nilpotent orbit [**N-I**, §8]. The author conjectured that $\mathfrak{M}$ is the inverse image of the Slodowy slice under the natural map map $T^*(n\text{-step flag variety}) \to \operatorname{End}(\mathbb{C}^N)$. This conjecture was proved by Maffei [**Ma05**].

(2) Suppose that the graph $\mathcal{G}$ is of finite type and take W so that $\dim W$ is given by coefficients of fundamental representations in the adjoint representation. (For type A, it is $1 - 0 - \cdots - 0 - 1$. For type D or E, it is a fundamental weight corresponding to the vertex adjacent to the 0-vertex in the corresponding affine Dynkin diagram (see Table 1). Then $\mathfrak{M}_0$ is isomorphic to $\mathbb{C}^2/\Gamma$, where Γ is the finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ corresponding to the graph $\mathcal{G}$ by the McKay correspondence. And $\pi\colon \mathfrak{M} \to \mathfrak{M}_0$ is the minimal resolution of $\mathbb{C}^2/\Gamma$ ([**Kr89**]). The lagrangian $\mathfrak{L} = \pi^{-1}(0)$ is an union of $\mathbb{P}^1$ intersecting transversely. If we draw the diagram according to the rule that

- draw a vertex for each $\mathbb{P}^1$,
- connect vertices by an edge when the corresponding $\mathbb{P}^1$'s intersect,

then we get the Dynkin diagram. This result for $\pi^{-1}(0)$ is well-known in the theory of simple singularities, but one can check them in terms of representations of quivers (see [**Na96**, §5.2]).

(3) ([**Na99**, Ch. 2]) Consider the Jordan quiver. For $\dim V = n$, $\dim W = 1$, $\mathfrak{M}(n, 1)$ is Hilbert scheme of n points in $\mathbb{C}^2$. In fact, it is more natural to change the stability condition to the opposite one, i.e., $\zeta_{\mathbb{R}} < 0$. We also have $\mathfrak{M}_0(V, W) = S^n\mathbb{C}^2$, the n^{th} symmetric product of $\mathbb{C}^2$. We have $\mathfrak{M}_0^{\text{reg}}(n, 1) = \emptyset$ in this case. The morphism $\pi\colon \mathfrak{M}(n, 1) \to \mathfrak{M}_0(n, 1)$ is usually called the *Hilbert-Chow* morphism. If $\dim W = r$, $\mathfrak{M}(n, r)$ is the framed moduli space of rank r torsion free sheaves on $\mathbb{P}^2 = \mathbb{C}^2 \sqcup \ell_\infty$ with $c_2 = n$, where ℓ_∞ is the line at infinity and the framing is an isomorphism $E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$. The corresponding $\mathfrak{M}_0^{\text{reg}}(n, r)$ is the framed moduli space of *locally-free* sheaves, and $\mathfrak{M}_0(n, r)$ is the so-called Uhlenbeck

(partial) compactification, and is described as

$$\mathfrak{M}_0(n,r) = \bigsqcup_{n' \le n} \mathfrak{M}_0^{\mathrm{reg}}(n',r) \times S^{n-n'}\mathbb{C}^2$$

set-theoretically.

In this set-up, the tangent complex (14.6) computes $\operatorname{Ext}^{\bullet-1}(E, E(-\ell_\infty))$. The vanishing of left and right cohomology groups means $\operatorname{Hom}(E, E(-\ell_\infty)) = 0 = \operatorname{Ext}^2(E, E(-\ell_\infty))$.

14.1.10. $W = 0$ **version.** If $W = 0$, the $\zeta_{\mathbb{R}}$-semistability fails unless $V = 0$, since $S = V$ violates the stability inequality. Also $\mathfrak{M}_0^{\mathrm{reg}}$ is empty unless $V = 0$ as the stabilizer always contains $\mathbb{C}^*$. Therefore we need to modify the definition when $W = 0$. We replace G by $G/\mathbb{C}^*$ and consider the stability parameter $\zeta_{\mathbb{R}}$ with $\zeta_{\mathbb{R}} \cdot \dim V = 0$ instead. Then we can define $\mathfrak{M}$ and $\mathfrak{M}_0^{\mathrm{reg}}$ as above. Remark that we do not have the equivalence of $\zeta_{\mathbb{R}}$-stability and semistability in general. One can show that $\mathfrak{M}$ and $\mathfrak{M}_0^{\mathrm{reg}}$ are smooth of dimension given by

$$2 - \langle \mathbf{v}, \mathbf{C}\mathbf{v} \rangle.$$

For example, we have

Examples 14.9. (1) ([**N-I**, Proof of 6.7]) If the underlying graph $\mathcal{G}$ is of finite type, then $\mathfrak{M}_0^{\mathrm{reg}}(V,0) = \emptyset$ unless $B = 0$. In this case, $V = S_i$, i.e., V_i is the 1-dimensional vector space on the vertex i, and all other V_j's are 0.

(2) ([**KN90**, Prop. 9.2(ii)]) If the underlying graph $\mathcal{G}$ is of affine type and $\mathfrak{M}_0^{\mathrm{reg}}(V,0) \neq \emptyset$, then $V = S_i$ as above or $\dim V = \delta$. In the latter case $\mathfrak{M}_0^{\mathrm{reg}}(V,0) \cong (\mathbb{C}^2 \setminus \{0\})/\Gamma$, where Γ is the finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ corresponding to the graph $\mathcal{G}$ by the McKay correspondence.

Remark 14.10. The definition of quiver varieties has its origin in the author's joint work with Kronheimer [**KN90**]. There we considered the moduli spaces of anti-self-dual connections on the minimal resolution of $\mathbb{C}^2/\Gamma$ (in fact more generally on its deformationl) with the hyper-Kähler metric constructed in [**Kr89**]. We showed that they are given as above $\mathfrak{M}$, where the GIT quotients are replaced by hyper-Kähler quotients, and the stability parameter $\zeta_{\mathbb{R}}$ is chosen different. This is a modification of the ADHM construction describing moduli space of anti-self-dual connection on $\mathbb{R}^4$ with the Euclidean metric [**ADHM78**]. In fact, if we impose the Γ-invariance to the ADHM description, we can easily show that $\mathfrak{M}_0^{\mathrm{reg}}$ parametrizes Γ-equivariant anti-self-dual connections on $\mathbb{R}^4$, in other words, anti-self-dual connections on the orbifold $\mathbb{R}^4/\Gamma$. Thus the description in [**KN90**] says that the parameter $\zeta_{\mathbb{R}}$ allows to deform anti-self-dual connections on $\mathbb{R}^4/\Gamma$ to its minimal resolution.

Later the author observed that the above choice of $\zeta_{\mathbb{R}}$ gives the framed moduli space of Γ-equivariant torsion free sheaves on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$, where the framing is an isomorphism $E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$ in [**Na99**, Ch. 2]. This description is useful to understand the results in the following section for quiver varieties of affine type.

14.1.11. Local description. We so far mentioned two extremal cases of fibers of π: $\mathfrak{L} = \pi^{-1}(0)$ and $\pi^{-1}([B,a,b])$ for $[B,a,b] \in \mathfrak{M}_0^{\mathrm{reg}}$. We now study general cases following [**Na09**, 1(vii)]. The result is a consequence of the Kuranishi type description of the local structure of $\mathfrak{M}_0$ in [**Qaff**, §3.2] (see also [**CB03**] for more algebraic treatment).

Let $[B,a,b]\in\mathfrak{M}_0$, where (B,a,b) has a closed orbit. According to the stabilizer group, we have a decomposition

$$\begin{aligned} V &\cong V^0\oplus(V^1)^{\oplus\widehat{v}_1}\oplus\cdots\oplus(V^r)^{\oplus\widehat{v}_r},\\ (B,a,b) &\cong (B^0,a^0,b^0)\oplus(B^1)^{\oplus\widehat{v}_1}\oplus\cdots\oplus(B^r)^{\oplus\widehat{v}_r}, \end{aligned}\tag{14.11}$$

where $(B^0,a^0,b^0)\in\mu^{-1}(0)\cap\mathbf{M}(V^0,W)$ is the unique factor having $W\neq 0$, $B^k\in\mu^{-1}(0)\cap\mathbf{E}(V^k,V^k)$ $(k=1,\dots,r)$ have pairwise distinct closed orbits and $\widehat{v}_k$ is its multiplicity in (B,a,b). And the points (B^0,a^0,b^0), B^k have trivial stabilizers. (See [**N-I**, 6.5], [**N-II**, 3.27].)

The complex $\mathscr{C}^\bullet$ decomposes accordingly as $\mathscr{C}^\bullet=\bigoplus_{k,l=0}^{r}(\mathscr{C}^\bullet_{k,l})^{\oplus\widehat{v}_k\widehat{v}_l}$ with

$$\mathscr{C}^\bullet_{k,l}:\mathbf{L}(V^k,V^l)\xrightarrow{\alpha}\mathbf{E}(V^k,V^l)\oplus\mathbf{L}(W,V^l)^{\oplus\delta_{k0}}\oplus\mathbf{L}(V^k,W)^{\oplus\delta_{0l}}\xrightarrow{\beta}\mathbf{L}(V^k,V^l)$$

where $\mathbf{L}(W,V^l)$ appears in case $k=0$ and $\mathbf{L}(V^k,W)$ in case $l=0$. We also put $\widehat{v}_0=1$. Then it is easy to show that $\operatorname{Ker}\alpha=0$ unless $k=l\neq 0$ and $\operatorname{Ker}\alpha=\mathbb{C}\,\mathrm{id}$ for $k=l\neq 0$, and the similar statement for $\operatorname{Coker}\beta$.

We construct a new graph $\widehat{\mathcal{G}}$ with $\widehat{I}=\{1,\dots,r\}$ with the associated Cartan matrix $\widehat{\mathbf{C}}=(\widehat{c}_{kl})$ by $\widehat{c}_{kl}\overset{\text{def.}}{=}2\delta_{kl}-\dim\operatorname{Ker}\beta/\operatorname{Im}\alpha$ for the complex $\mathscr{C}^\bullet_{kl}$. This is equal to the alternating sum of the dimensions of terms, i.e., $=\langle\mathbf{v}^k,\mathbf{C}\mathbf{v}^l\rangle$ by the above discussion. Note $\widehat{a}_{kl}=\widehat{a}_{lk}$. We also put

$$\hat{V}_k\overset{\text{def.}}{=}\mathbb{C}^{\widehat{v}_k},\qquad \hat{W}_k\overset{\text{def.}}{=}\operatorname{Ker}\beta/\operatorname{Im}\alpha\text{ for }\mathscr{C}^\bullet_{k0},$$

and consider $\mathbf{M}(\hat{V},\hat{W})$ defined for the new graph $\widehat{\mathcal{G}}$ with the $\widehat{I}$-graded vector spaces $\hat{V}$, $\hat{W}$. The stabilizer $\widehat{G}$ of (B,a,b) is naturally isomorphic to $\prod_{k\in\widehat{I}}\mathrm{GL}(\hat{V}_k)$. It acts on $\mathbf{M}(\hat{V},\hat{W})$. We have the moment map $\widehat{\mu}\colon\mathbf{M}(\hat{V},\hat{W})\to\mathbf{L}(\hat{V},\hat{V})\cong\operatorname{Lie}(\widehat{G})^*$. We consider the quotient $\widehat{\mathfrak{M}}_0(\hat{V},\hat{W})=\widehat{\mu}^{-1}(0)/\!\!/\widehat{G}$. We also consider $\widehat{\mathfrak{M}}(\hat{V},\hat{W})$, where the stability parameter $\zeta_{\mathbb{R}}$ is as above (though the graph is possibly different). There is a morphism $\widehat{\pi}\colon\widehat{\mathfrak{M}}(\hat{V},\hat{W})\to\widehat{\mathfrak{M}}_0(\hat{V},\hat{W})$.

Let

$$\begin{aligned} \hat{W}_0 &\overset{\text{def.}}{=}\operatorname{Ker}\beta/\operatorname{Im}\alpha\text{ for the complex }\mathscr{C}^\bullet_{00},\\ T &\overset{\text{def.}}{=}\widehat{\mathbf{M}}^{\mathrm{el}}(\hat{V},\hat{W})\oplus\hat{W}_0. \end{aligned}$$

The space T can be identified with the tangent space of the stratum consisting of orbits having the same decomposition type as $[B,a,b]$. Then we have the following local description around $[B,a,b]$:

$$\begin{tikzcd}
\mathfrak{M}(V,W) \arrow[r, phantom, "\supset"] & \pi^{-1}(U) \arrow[r, "\widehat{\Phi}", "\cong"'] \arrow[d, "\pi"'] & \widehat{\pi}^{-1}(\hat{U})\times U_0 \arrow[r, phantom, "\subset"] \arrow[d, "\widehat{\pi}\times\mathrm{id}"] & \widehat{\mathfrak{M}}^{\mathrm{norm}}(\hat{V},\hat{W})\times T \\
\mathfrak{M}_0(V,W) \arrow[r, phantom, "\supset"] & U \arrow[r, "\Phi", "\cong"'] & \hat{U}\times U_0 \arrow[r, phantom, "\subset"] & \widehat{\mathfrak{M}}_0^{\mathrm{norm}}(\hat{V},\hat{W})\times T \\
& {[B,a,b]} \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & 0 \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

Here Φ, $\widehat{\Phi}$ are local complex analytic isomorphisms. (In the frame work of [**CB03**] $[B,a,b]$ and 0 have neighborhoods which are isomorphic in étale topology.)

Examples 14.12. (1) Consider the quiver variety $\mathfrak{M}_0$ of finite type. By Example 14.9(1), the decomposition is $V=V^0\oplus\bigoplus_{i\in I}S_i^{\oplus\widehat{v}_i}$. The graph $\widehat{I}$ is the same as the original graph $\mathcal{G}$. Therefore any fiber $\pi^{-1}([B,a,b])$ is isomorphic to $\mathfrak{L}(\hat{V},\hat{W})$ for the same graph $\mathcal{G}$.

(2) Consider the quiver variety $\mathfrak{M}_0$ of affine type. By Example 14.9(2), any point $[B,a,b] \in \mathfrak{M}_0$ can be represented in the form

$$(B^0,a^0,b^0) \oplus \bigoplus_{i\in I} S_i^{\oplus \widehat{v}_i} \oplus \bigoplus_k (B^k)^{\oplus \lambda_k},$$

where $[B^0,a^0,b^0] \in \mathfrak{M}_0^{\mathrm{reg}}(V^0,W^0)$ and each B^k corresponds a pairwise distinct points in $(\mathbb{C}^2 \setminus \{0\})/\Gamma$. Thus the stratum containing $[B^0,a^0,b^0]$ is isomorphic to

$$\mathfrak{M}_0^{\mathrm{reg}}(V^0,W^0) \times S_\lambda^{|\lambda|}(\mathbb{C}^2 \setminus \{0\})/\Gamma,$$

where $S_\lambda^{|\lambda|}(\mathbb{C}^2 \setminus \{0\})/\Gamma$ is the stratum of the symmetric product $S^{|\lambda|}(\mathbb{C}^2 \setminus \{0\})/\Gamma$, consisting of

$$\sum_k \lambda_k x_k$$

with distinct x_k.

The corresponding graph $\widehat{\mathcal{G}}$ is the disjoint union of $\mathcal{G}$ and copies of the Jordan quiver (as many as x_k's). Therefore $\pi^{-1}([B,a,b])$ is isomorphic to the product of $\mathfrak{L}(\widehat{V},\widehat{W})$ for the same graph $\mathcal{G}$ and $\mathfrak{L}(\lambda^k, \langle \dim W, \delta\rangle)$ for the Jordan quiver.

These description can be naturally understood in the language of the framed moduli spaces of Γ-equivariant torsion free sheaves on $\mathbb{P}^2$. See [**Na02**].

14.1.12. Stratification. For a general quiver, the stratum of points having the decomposition (14.11) is of the form

$$\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}_0,\mathbf{w}) \times S_\lambda^{|\lambda|}\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,0) \times S_\mu^{|\mu|}\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^{l+1},0) \times \cdots,$$

where we collect factors having the same dimension vector, say $\mathbf{v}^1 = \mathbf{v}^2 = \cdots = \mathbf{v}^l$, and define the partition $\lambda = (\widehat{v}_1, \widehat{v}_2, \dots, \widehat{v}_l)$, and next collect factors $\mathbf{v}^{l+1} = \mathbf{v}^{l+2} = \cdots$ and so on. Therefore it is important to know the criterion for $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}_0,\mathbf{w}), \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,0) \neq \emptyset$. This was asked in [**N-II**, the end of §4]. And an answer was given by Crawley-Boevey [**CB01**]. We state his result here. In fact, he studied only the criterion for $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},0) \neq \emptyset$, but the remaining case is easily deduced from that case. (See [**Na09**, §2.4].)

THEOREM 14.13. (1) *Suppose* $W = 0$ *and consider a dimension vector* $\mathbf{v}$. *Then* $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},0) \neq \emptyset$ *if and only if the following holds*:

- $\mathbf{v}$ *is a positive root, and* $p(\mathbf{v}) > \sum_{t=1}^r p(\beta^{(t)})$ *for any decomposition* $\mathbf{v} = \sum_{t=1}^r \beta^{(t)}$ *with* $r \geq 2$ *and* $\beta^{(t)}$ *a positive root for all* t,

where $p(x) = 1 - \frac{1}{2}\langle x, \mathbf{C}x\rangle$.

(2) *Suppose* $\mathbf{w} \neq 0$, *and the graph* $\mathcal{G}$ *has no edge loops. Then* $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}) \neq \emptyset$ *if and only if the following holds*:

- $\mathbf{w} - \mathbf{C}\mathbf{v}$ *is a weight of the integrable highest weight representation* $V(\mathbf{w})$ *of the highest weight* $\mathbf{w}$, *and* $\langle \mathbf{v}, \mathbf{w} - \frac{1}{2}\mathbf{C}\mathbf{v}\rangle > \langle \mathbf{v}^0, \mathbf{w} - \frac{1}{2}\mathbf{C}\mathbf{v}^0\rangle + \sum_{t=1}^r p(\beta^{(t)})$ *for any decomposition* $\mathbf{v} = \mathbf{v}^0 + \sum_{t=1}^r \beta^{(t)}$ *with* $r \geq 1$, $\mathbf{w} - \mathbf{v}^0$ *is a weight of* $V(\mathbf{w})$, *and* $\beta^{(t)}$ *a positive root for all* t.

Remark that a positive root above is defined as an element in Q_+, which is in the Weyl group orbit of either a real simple root, or an element in the fundamental region $K = \{\alpha \in Q_+ \mid \langle \alpha, \mathbf{C}\alpha_i\rangle \leq 0$ for any $i \in I$, and $\operatorname{supp}\alpha$ is connected$\}$. It is different from the positive roots of the Borcherds-Kac-Moody Lie algebra $\mathfrak{g}$, since a multiple $m\alpha_i$ ($m \geq 2$) of a simple imaginary root is in K, but not a root of $\mathfrak{g}$. (See [脇本, §2.3].) To deduce (2) from (1), we need a fact that integrable highest weight representations of a Kac-Moody Lie algebra is irreducible (and hence unique). This is not known to be true for the Borcherds-Kac-Moody Lie algebra. (See [脇本, 命題 2.58 のあとの注].) One can make a statement including this case by using the deduction in [**Na09**, §2.4], but it is not enlightening and not given here.

14.2. Convolution on homology groups of quiver varieties

We now construct the universal enveloping algebra for the Kac-Moody Lie algebra $\mathfrak{g}$ via the convolution on homology groups of quiver varieties in this section.

14.2.1. Convolution algebra. Let us first recall the convolution product in the homology given by Ginzburg [**Gi91**]. A nice introduction is given in [**Chriss-Ginzburg**, §2.7].

We first give a (very) quick review of Borel-Moore homology groups. Besides [**Chriss-Ginzburg**], there is a treatement in [**Fu97**, §B2].

For a closed subset X of an n-dimensional oriented manifold M, we define the (rational) Borel-Moore homology of X by

$$H_i(X) \overset{\text{def.}}{=} H^{n-i}(M, M \setminus X, \mathbb{Q}), \tag{14.14}$$

where the right hand side is the relative singular cohomology group with $\mathbb{Q}$-coefficients. One can show that this is independent of the choice of the embedding $X \subset M$.

If $f\colon X \to Y$ is a proper map, there is a *push-forward* homomorphism

$$f_*\colon H_i(X) \to H_i(Y).$$

If X and Y are closed subsets of an n-dimensional oriented manifold M, we have the cup product in the relative cohomology group

$$\cup\colon H^{n-i}(M, M \setminus X) \otimes H^{n-j}(M, M \setminus Y) \to H^{2n-i-j}(M, M \setminus (X \cap Y)).$$

By (14.14), it can be transfered to the cap product in the Borel-Moore homology group:

$$\cap\colon H_i(X) \otimes H_j(Y) \to H_{i+j-n}(X \cap Y). \tag{14.15}$$

Note that this product depends on the ambient space M.

Let M^1, M^2, M^3 be oriented manifolds, and $p_{ij}\colon M^1 \times M^2 \times M^3 \to M^i \times M^j$ be the natural projection. Let $Z \subset M^1 \times M^2$ and $Z' \subset M^2 \times M^3$ be closed subsets. We have the cap product in $M^1 \times M^2 \times M^3$ by (14.15):

$$\cap\colon H_{i+d_3}(p_{12}^{-1}Z) \otimes H_{j+d_1}(p_{23}^{-1}Z') \to H_{i+j-d_2}(p_{12}^{-1}Z \cap p_{23}^{-1}Z'), \qquad d_a = \dim M^a.$$

Assume that the map

$$p_{13}\colon p_{12}^{-1}Z \cap p_{23}^{-1}Z' \to M^1 \times M^3$$

is proper. Let us denote its image by $Z \circ Z'$. We define a *convolution* by

$$*\colon H_i(Z) \otimes H_j(Z') \to H_{i+j-d_2}(Z \circ Z'); \quad c * c' = (p_{13})_*(p_{12}^*c \cap p_{23}^*c'), \tag{14.16}$$

where p_{12}^*c stands for $c \times [M^3]$, etc. This makes sense for disconnected manifolds (possibly variable dimensions) as well.

Note that the 'middle' degree part is closed under the convolution product:

$$*\colon H_{(d_1+d_2)/2}(Z) \otimes H_{(d_2+d_3)/2}(Z') \to H_{(d_1+d_3)/2}(Z \circ Z').$$

Let M be an oriented manifold and N a topological space, and $\pi\colon M \to N$ a proper continuous map. One can define Z as the fiber product

$$Z \overset{\text{def.}}{=} M \times_N M = \{(m^1, m^2) \in M \times M \mid \pi(m^1) = \pi(m^2)\},$$

and the convolution makes $H_*(Z)$ a $\mathbb{Q}$-algebra. The fundamental class of the diagonal is the unit. And $H_{\dim M}(Z)$ is a subalgebra.

For $x \in N$, consider the fiber $M_x = \pi^{-1}(x)$. We have $Z \circ M_x = M_x$, and the convolution makes $H_*(M_x)$ an $H_*(Z)$-module. And $H_j(M_x)$ is an $H_{\dim M}(Z)$-module for any j.

Remark 14.17. The convolution product can be define on any theory which has operations "pull-back" for smooth morphisms, "push forward" for proper morphisms and "intersection", e.g., the Chow rings, the equivariant K-theory, the linear space of constructible functions, etc.

14.2.2. Modified enveloping algebra and convolution on homology groups of quiver varieties.

We assume the graph $\mathcal{G}$ has no edge loops hereafter.

Convention 14.18. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra corresponding to the Cartan matrix $\mathbf{C}$. We take the root datum so that (1) $\{\alpha_i\}$ is linearly independent, and (2) there exists $\Lambda_i \in P$ with $\langle \Lambda_i, h_j\rangle = \delta_{ij}$. Associated with $\mathbf{v}$, $\mathbf{w}$, we define the corresponding elements in the weight lattice P by

$$\mathbf{v} \mapsto \sum_i v_i \alpha_i, \qquad \mathbf{w} \mapsto \sum_i w_i \Lambda_i.$$

From the above assumption, those elements determine $\mathbf{v}$, $\mathbf{w}$. We consider $\mathbf{v}$, $\mathbf{w}$ as elements of P hereafter until §14.5, where we consider the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. In that chapter, $\mathbf{w}$ is identfied with $\sum_i w_i \varpi_i$, where ϖ_i is the i^{th} level 0-fundamental weight.

Motivated by the theory in the previous section, we define an analog of the *Steinberg variety* as the fiber product $\mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$.

However, it turns out that we need to consider $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ simultaneously for various $\mathbf{v}$ (and fixed $\mathbf{w}$). We first explain how different $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ are glued. Suppose $V' \subset V$ is an I-graded subspace. We have a natural closed embedding $\mu^{-1}(0) \cap \mathbf{M}(V', W) \subset \mu^{-1}(0) \cap \mathbf{M}(V, W)$. It induces a morphism $\mathfrak{M}_0(V', W) \to \mathfrak{M}_0(V, W)$, which is a closed embedding. This is independent of the choice of the inclusion $V' \subset V$, and canonically defined. Therefore when $\mathfrak{M}_0(V^1, W)$, $\mathfrak{M}_0(V^2, W)$ are given, we consider $\mathfrak{M}_0(V^1, W)$, $\mathfrak{M}_0(V^2, W) \subset \mathfrak{M}_0(V^1 \oplus V^2, W)$ and consider the fiber

$$Z(V^1, V^2; W) \equiv Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) \equiv Z \overset{\text{def.}}{=} \mathfrak{M}(V^1, W) \times_{\mathfrak{M}_0(V^1 \oplus V^2, W)} \mathfrak{M}(V^2, W).$$

Proposition 14.19. (1) *The dimension of any irreducible component of Z is* $\le \frac{1}{2}(\dim \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) + \dim \mathfrak{M}(\mathbf{v}^2, \mathbf{w}))$.

(2) *If the graph $\mathcal{G}$ is of finite or affine type, all irreducible components of Z have dimension* $\frac{1}{2}(\dim \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) + \dim \mathfrak{M}(\mathbf{v}^2, \mathbf{w}))$.

We introduce an irreducible component of Z which will play a fundamental role later (see [**N-II**, §5]. Take a vertex i without edge loops. We take $\mathbf{v}^2 = \mathbf{v}^1 + \alpha_i$, where dimension vectors are considered as weights as above. Hence α_i, as the dimension vector, has entries 1 at i, and 0 at other vertices. We consider the variety of pairs (B, a, b) and S modulo $G_{\mathbf{v}^2}$-action such that

(a) $(B, a, b) \in \mu^{-1}(0)$ is $\zeta_{\mathbb{R}}$-stable,
(b) S is a B-invariant I-graded subspace containing the image of a with $\dim S = \mathbf{v}^1 = \mathbf{v}^2 - \alpha_i$.

It can be considered as a subvariety of $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$, where the first factor is given by the restriction of (B, a, b) to S. Let us denote this by $\mathfrak{P}_i(\mathbf{v}^2, \mathbf{w})$. It was shown that $\mathfrak{P}_i(\mathbf{v}^2, \mathbf{w})$ is a closed nonsingular subvariety of dimension $(\dim \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) + \dim \mathfrak{M}(\mathbf{v}^2, \mathbf{w}))/2$. It is also clear that it is contained in $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$. Let $\omega\colon \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \to \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^1, \mathbf{w})$ be the exchange of the two factors.

We consider

$$\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})),$$

where top $= \dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) + \dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$. Since we have possibly infinitely many $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$, the sum of the diagonal may not be included in this. But this is not essential, and the convolution product makes this into an associate algebra, possibly without the unit. This algebra has a module

$$\bigoplus_{\mathbf{v}} H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v}, \mathbf{w})),$$

where top $= \dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}, \mathbf{w})$ in this case. This contains a distinguished vector $[\mathfrak{L}(0, \mathbf{w})]$, as $\mathfrak{L}(0, \mathbf{w}) = \mathfrak{M}(0, \mathbf{w})$ is a single point.

For brevity, we introducea the following notation:

$$Z(\mathbf{w}) \overset{\text{def.}}{=} \bigsqcup_{\mathbf{v}^1, \mathbf{v}^2} Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}), \qquad \mathfrak{L}(\mathbf{w}) \overset{\text{def.}}{=} \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{w}),$$

and understand that $H_{\mathrm{top}}(Z(\mathbf{w}))$, $H_{\mathrm{top}}(\mathfrak{L}(\mathbf{w}))$ are above direct sum. Similarly let $\mathfrak{M}(\mathbf{w})$ be the disjoint union of all $\mathfrak{M}(\mathbf{v}, \mathbf{w})$.

We define the modified enveloping algebra $\widetilde{\mathbf{U}}(\mathfrak{g})$ as $\widetilde{\mathbf{U}}_q$ by setting $q = 1$. Here we explicitly write the Lie algebra $\mathfrak{g}$, since we will have the loop algebra $\mathbf{L}\mathfrak{g}$ in §14.5.

THEOREM 14.20 ([**N-II**, 9.4, §10]). (1) *There exists the unique algebra homomorphism*

$$\Phi\colon \widetilde{\mathbf{U}}(\mathfrak{g}) \to H_{\mathrm{top}}(Z(\mathbf{w}))$$

such that

$$\Phi(a_\lambda) = [\Delta(\mathbf{v}, \mathbf{w})],$$

$$\Phi(e_i a_\lambda) = [\mathfrak{P}_i(\mathbf{v}, \mathbf{w})], \quad \Phi(a_\lambda f_i) = (-1)^{r(\mathbf{v}, \mathbf{w})}[\omega(\mathfrak{P}_i(\mathbf{v}, \mathbf{w}))],$$

where $r(\mathbf{v}, \mathbf{w}) = -\langle h_i, \mathbf{w} - \mathbf{v}\rangle - 1$. *Here* $\mathbf{v}$ *is chosen so that* $\lambda = \mathbf{w} - \mathbf{v}$. *If there is no such* $\mathbf{v}$, *then we put* $\Phi(a_\lambda) = \Phi(e_i a_\lambda) = \Phi(a_\lambda f_i) = 0$.

(2) *Via* Φ, $H_{\mathrm{top}}(\mathfrak{L}(\mathbf{w}))$ *considered as a representation of* $\widetilde{\mathbf{U}}(\mathfrak{g})$ *is isomorphic to* $V(\mathbf{w})$. *The highest weight vector is the fundamental class* $[\mathfrak{L}(0, \mathbf{w})]$ *of* $\mathfrak{L}(0, \mathbf{w}) =$ point.

REMARK 14.21. This result was motivated by several earlier results. One is the Ringel-Hall algebra construction of $\mathbf{U}_q^-$ from the representations of quivers, due to Ringel [**Ri90**] and its geometric reformulation by Lusztig [**Lu-I**]. Also, in the earlier paper [**N-I**] the author gave a similar construction on the space of constructible functions on $\mathfrak{L}(\mathbf{v}, \mathbf{w})$. This construction is motivated by the construciton of Lusztig [**Lu-II**, §12]. Another is Ginzburg's construction of $\widetilde{\mathbf{U}}(\mathfrak{g})$ for type A using the cotangent bundle of the n-step flag variety [**Gi91**].

14.2.3. Proof of Theorem 14.20. Let us give a sketch of the proof.

For (1) we need to check the defining relations for $\widetilde{\mathbf{U}}(\mathfrak{g})$. The relations

$$a_\lambda a_\mu = \delta_{\lambda\mu} a_\lambda$$

is clear since a_λ is the diagonal $[\Delta(\mathbf{v}, \mathbf{w})]$. The relation

$$(e_i f_j - f_j e_i) a_\lambda = 0 \quad \text{for } i \neq j$$

is checked as follows. Let $\mathbf{v} = \mathbf{v}^3$, $\mathbf{v}^2 = \mathbf{v} + \alpha_j$, $\mathbf{v}^1 = \mathbf{v} + \mathbf{v}^j - \alpha_i$ and consider the convolution for $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$ and $Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w})$. We first show that $p_{12}^{-1}(\mathfrak{P}_i(\mathbf{v}^2, \mathbf{w}))$ and $p_{23}^{-1}(\omega(\mathfrak{P}_j(\mathbf{v}^2, \mathbf{w})))$ intersect transversely, so the class

$$p_{12}^*([\mathfrak{P}_i(\mathbf{v}^2, \mathbf{w})]) \cap p_{23}^*([\omega(\mathfrak{P}_j(\mathbf{v}^2, \mathbf{w}))])$$

is represented by the set-theoretical intersection

$$Q \overset{\text{def.}}{=} p_{12}^{-1}(\mathfrak{P}_i(\mathbf{v}^2, \mathbf{w})) \cap p_{23}^{-1}(\omega(\mathfrak{P}_j(\mathbf{v}^2, \mathbf{w}))).$$

The same is true for $p_{12}^{-1}(\omega\mathfrak{P}_j(\mathbf{v}^1, \mathbf{w}))$ and $p_{23}^{-1}((\mathfrak{P}_i(\mathbf{v}^3, \mathbf{w})))$, where the convolution is considered over $Z(\mathbf{v}^1, \mathbf{v}'^2; \mathbf{w})$ and $Z(\mathbf{v}'^2, \mathbf{v}^3; \mathbf{w})$ with $\mathbf{v}'^2 = \mathbf{v}^3 - \alpha_i$. Let Q' denote the set-theoretical intersection like Q in this case.

We next show that there exists an isomorphism from Q to Q' which is compatible with the projection p_{13}. For this, note that Q parametrizes the G_{V^2}-orbits of triples $(B^2, a^2, b^2, V^1, V^3)$ where $V^1 \subset V^2$, $V^3 \subset V^2$ with $\dim V^2/V^1 = \alpha_i$, $\dim V^2/V^3 = \alpha_j$. Then

$$V'^2 = V^1 \cap V^3$$

has the correct dimension and the restriction of (B^2, a^2, b^2) to V'^2 defines a point in $\mathfrak{M}(\mathbf{v}'^2, \mathbf{w})$. The restriction can be considered by two steps, i.e., first from V^2 to V^1, and then V^1 to V'^2, and similarly V^2 to V^3, and V^3 to V'^2. Therefore we have a point in Q'. Conversely if a point in Q', and hence V^1, V'^2, V^3 are given, we define

$$V^2 = V^1 \oplus V^3 / \{v \oplus v \mid v \in V'^2\}.$$

It gives the inverse map from $Q' \to Q$. Now the assertion is clear.

The same argument shows that the restriction of this class to the complement of the diagonal in $Z(\mathbf{v}, \mathbf{v}; \mathbf{w}) \subset \mathfrak{M}(\mathbf{v}, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$ vanishes. This means $(e_i f_i - f_i e_i) a_\lambda$ is a constant multiple of a_λ. The determinantion of the constant is postponed until the next section.

Once this relation is established, the integrability of e_i, f_i automatically implies the Serre relations. The integralibility for e_i is trivial since $\dim V_i$ cannot be negative. The integrability for f_i will be explained later.

The proof of (2) is also explained later.

14.3. Semicanonical bases and their crystal structures

Note that $H_{\mathrm{top}}(Z(\mathbf{w}))$ and $H_{\mathrm{top}}(\mathfrak{L}(\mathbf{w}))$ have bases given by fundamental classes of irreducible components of various $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$, $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ of maximal dimension respectively. (There is possibly an irreducible component of smaller dimension for the former.) Following Lusztig, who considered similar base in a slightly different setting, we call these *semicanonical bases*. They are known to be different from the specialization of the canonical bases of $\widetilde{\mathbf{U}}_q$ and $V(\lambda)$ at $q = 1$ [**KS97**]. In this section, we define crystal structures on the semicanonical bases and show that they are isomorphic to the canonical base of $\widetilde{\mathbf{U}}_q$ and $V(\lambda)$ as crystals. Thus, though semicanoical and canonical bases are different, they have the same combinatorial structures.

14.3.1. Brill-Noether loci and Grassmann bundle. Let us consider a modification of the complex (14.6):

$$\begin{aligned} &\mathscr{C}_i^\bullet : V_i \xrightarrow{\sigma_i} \bigoplus_{h:\mathrm{i}(h)=i} V_{\mathrm{o}(h)} \oplus W_i \xrightarrow{\tau_i} V_i, \\ &\sigma_i = \bigoplus_{\mathrm{i}(h)=i} B_{\overline{h}} \oplus b_i, \quad \tau_i = \sum_{\mathrm{i}(h)=i} \varepsilon(h) B_h + a_i. \end{aligned} \tag{14.22}$$

Thanks to the equation $\mu = 0$, this is indeed a complex. And σ_i is injective since $\operatorname{Ker}\sigma_i = 0$ by the $\zeta_{\mathbb{R}}$-stability condition.

Note that the rank of $\mathscr{C}_i^\bullet$ is

$$\dim W_i - \sum_j c_{ij} \dim V_j = \langle h_i, \mathbf{w} - \mathbf{v} \rangle.$$

Remark 14.23. Let us explain the above complex in the description of a quiver variety as a framed moduli space of Γ-equivariant sheaves. In the McKay correspondence, each vertex $i \in I$ corresponds to an irreducible representation ρ_i of Γ. Let $\mathcal{O}_0 \otimes \rho_i$ be the skyscraper sheaf at the origin twisted by ρ_i. It is a Γ-equivariant coherent sheaf on $\mathbb{P}^2$. At a framed sheaf (E, φ) corresponding to $[B, a, b] \in \mathfrak{M}$, the above complex compute $\operatorname{Ext}^{\bullet - 1}(\mathcal{O}_0 \otimes \rho_i, E)$. This follows from the proof of [**Na99**, Chap. 2].

Following [**Lu-II**, 12.2], we introduce the following subsets of $\mathfrak{M}(\mathbf{v}, \mathbf{w})$:

$$\begin{aligned} &\mathfrak{M}_{i;r}(\mathbf{v}, \mathbf{w}) \overset{\text{def.}}{=} \left\{ [B, a, b] \in \mathfrak{M}(\mathbf{v}, \mathbf{w}) \,\middle|\, \dim \operatorname{Coker} \tau_i = r \right\}, \\ &\mathfrak{M}_{i;\le r}(\mathbf{v}, \mathbf{w}) \overset{\text{def.}}{=} \bigcup_{s \le r} \mathfrak{M}_{i;s}(\mathbf{v}, \mathbf{w}). \end{aligned} \tag{14.24}$$

Since $\mathfrak{M}_{i;\le r}(\mathbf{v}, \mathbf{w})$ is an open subset of $\mathfrak{M}(\mathbf{v}, \mathbf{w})$, $\mathfrak{M}_{i;r}(\mathbf{v}, \mathbf{w})$ is a locally closed subvariety. These are kinds of Brill-Noether loci.

Replacing V_i by $\operatorname{Im}\tau_i$, we have a morphism

$$p \colon \mathfrak{M}_{i;r}(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{i;0}(\mathbf{v} - r\alpha_i, \mathbf{w}).$$

Note that the image under π is the same as $[B, a, b]$ and $p([B, a, b])$.

The complex (14.22) is replaced by

$$\operatorname{Im}\tau_i \xrightarrow{\overline{\sigma}_i} \bigoplus_{h:\mathrm{i}(h)=i} V_{\mathrm{o}(h)} \oplus W_i \xrightarrow{\overline{\tau}_i} \operatorname{Im}\tau_i,$$

where $\overline{\sigma}_i$, $\overline{\tau}_i$ are naturally induced linear maps. Then $\operatorname{Im}\sigma_i / \operatorname{Im}\overline{\sigma}_i$ is an r-dimensional subspace in $\operatorname{Ker}\overline{\tau}_i / \operatorname{Im}\overline{\sigma}_i$. A little more analysis yields the following:

Proposition 14.25 ([**N-II**, 4.5]). *p is isomorphic to the Grassmann bundle of r-planes in the vector bundle $\operatorname{Ker}\overline{\tau}_i / \operatorname{Im}\overline{\sigma}_i$ over $\mathfrak{M}_{i;0}(\mathbf{v} - r\alpha_i, \mathbf{w})$.*

The rank of $\operatorname{Ker}\overline{\tau}_i / \operatorname{Im}\overline{\sigma}_i$ is equal to $\langle h_i, \mathbf{w} - (\mathbf{v} - r\alpha_i) \rangle$ since $\overline{\tau}_i$ is surjective.

This result is a straightforward modification of the corresponding result for Lusztig's Λ_V ([**Lu-II**, Lem. 12.5]).

14.3.2. Definition of a crystal structure. Let X be an irreducible component of $\mathfrak{L}(\mathbf{v},\mathbf{w})$. Taking a generic element $[B,a,b] \in X$, we define

$$\varepsilon_i(X) \overset{\text{def.}}{=} \dim \operatorname{Coker} \tau_i.$$

We restrict the Grassmann bundle with $r = \varepsilon_i(X)$ to $X \cap \mathfrak{M}_{i;r}(\mathbf{v},\mathbf{w})$. Then the closure of $p(X \cap \mathfrak{M}_{i;r}(\mathbf{v},\mathbf{w}))$ is an irreducible component of $\mathfrak{L}(\mathbf{v} - r\alpha_i, \mathbf{w})$. Let it denote $\widetilde{e}_i^{\max}(X)$. By its construction we have $\varepsilon_i(\widetilde{e}_i^{\max}(X)) = 0$.

Assuming $r \gneq 0$, we consider the Grassmann bundle of $(r-1)$-planes instead of r-planes, $p' \colon \mathfrak{M}_{i;r-1}(\mathbf{v} - \alpha_i, \mathbf{w}) \to \mathfrak{M}_{i;0}(\mathbf{v} - r\alpha_i, \mathbf{w})$. Then the closure of $p'^{-1}(\widetilde{e}_i^{\max}(X) \cap \mathfrak{M}_{i;0}(\mathbf{v} - r\alpha_i, \mathbf{w}))$ is an irreducible component of $\mathfrak{L}(\mathbf{v} - \alpha_i, \mathbf{w})$. We define it as $\widetilde{e}_i(X)$. If $r = \varepsilon_i(X) = 0$, we set $\widetilde{e}_i(X) = 0$.

Similarly assuming $r \lneq \operatorname{rank} \operatorname{Ker} \overline{\tau}_i / \operatorname{Im} \overline{\sigma}_i$, we consider $(r+1)$-planes in $\operatorname{Ker} \overline{\tau}_i / \operatorname{Im} \overline{\sigma}_i$ to construct an irreducible component of $\mathfrak{L}(\mathbf{v} + \alpha_i, \mathbf{w})$. We set it as $\widetilde{f}_i(X)$. If $r = \operatorname{rank} \operatorname{Ker} \overline{\tau}_i / \operatorname{Im} \overline{\sigma}_i$, we set $\widetilde{f}_i(X) = 0$.

Let $\operatorname{Irr} \mathfrak{L}(\mathbf{v},\mathbf{w})$ be the set of irreducible components of $\mathfrak{L}(\mathbf{v},\mathbf{w})$, and $\operatorname{Irr} \mathfrak{L}(\mathbf{w})$ be the disjoint union $\bigsqcup_{\mathbf{v}} \operatorname{Irr} \mathfrak{L}(\mathbf{v},\mathbf{w})$. Then the above structures, together with

$$\operatorname{wt}(X) \overset{\text{def.}}{=} \mathbf{w} - \mathbf{v}, \quad X \in \operatorname{Irr} \mathfrak{L}(\mathbf{v},\mathbf{w}), \qquad \varphi_i(X) \overset{\text{def.}}{=} \varepsilon_i(X) + \langle h_i, \operatorname{wt}(X) \rangle,$$

define a crystal structure on $\operatorname{Irr} \mathfrak{L}(\mathbf{w})$. This construction is due to Lusztig [**Lu-II**, Lem. 12.5].

Lemma 14.26. *For $X \in \operatorname{Irr} \mathfrak{L}(\mathbf{w})$ different from $\mathfrak{L}(0,\mathbf{w})$, there exists $i \in I$ such that $\varepsilon_i(X) > 0$. In particular, the crystal $\operatorname{Irr} \mathfrak{L}(\mathbf{w})$ is connected.*

This is clear from the description of §14.1.8: On a irreducible component $\mathfrak{L}_\alpha$ corresponding to a homomorphism $\rho \colon \mathbb{C}^* \to G$, we must have $a = 0$ and

$$B_h(V_{\mathrm{o}(h)}(m)) \subset \bigoplus_{n \le p} V_{\mathrm{i}(h)}(n), \qquad \text{where } p = m - 1 \text{ if } h \in \Omega \text{ and } p = m \text{ otherwise}$$

holds on an open subset. Therefore we take the maximum m with $V(m) \neq 0$ and a source vertex i of the orientation $\overline{\Omega}$, restricted to $\{i \mid V_i(m) \neq 0\}$. Then $V_i(m)$ cannot be contained in the image of τ_i.

Theorem 14.27. [**KS97, Sa02**] *The crystal $\operatorname{Irr} \mathfrak{L}(\mathbf{w})$ is isomorphic to the crystal $\mathscr{B}(\mathbf{w})$ of the integrable highest weight representation $V(\mathbf{w})$ of $\mathbf{U}_q$.*

There are several proofs of this result. One is due to [**KS97, Sa02**]. Another proof is given in [**Na01**, Th. 4.6], which will be reviewed in the next section.

Yet another proof is given in [**Na01**, 8.5], where the Kashiwara operators are described in terms of the parametrization in §14.1.8. It turns out to be the same as the parametrization of $\mathscr{B}(\mathbf{w})$ given by the so-called Kashiwara embedding [**Ka-II**].

Let $\operatorname{Irr} Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w})$ be the set of irreducible components of $Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w})$ of dimension $1/2(\dim \mathfrak{M}(\mathbf{v}^1,\mathbf{w}) + \dim(\mathbf{v}^2,\mathbf{w}))$, and $\operatorname{Irr} Z(\mathbf{w})$ be the disjoint union $\bigsqcup_{\mathbf{v}^1,\mathbf{v}^2} \operatorname{Irr} Z(\mathbf{v}^1,\mathbf{v}^2,\mathbf{w})$. We define a bi-crystal structure on $\operatorname{Irr} Z(\mathbf{w})$ in the same way as above. Namely for $\widetilde{e}_i$, $\widetilde{f}_i$ we change the first factor of $Z(\mathbf{v}^1,\mathbf{v}^2,\mathbf{w}) \subset \mathfrak{M}(\mathbf{v}^1,\mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2,\mathbf{w})$ by using the Grassmann bundle as above. For $\widetilde{e}_i^*$, $\widetilde{f}_i^*$, we change the second factor.

We have a property similar to Theorem 5.13 and (8.18), which can be proved using just the definition:

PROPOSITION 14.28 ([**N-II**, 10.1]). *Let* $X \in \operatorname{Irr} \mathfrak{L}(\mathbf{w})$. *We have*

$$f_i[X] = \pm(\varepsilon_i(X) + 1)[\widetilde{f}_i X] + \sum_{\varepsilon_i(X') > \varepsilon_i(X)+1} a_{X'}[X']$$

for some constants $a_{X'}$.

14.3.3. Proof of Theorem 14.20 – continued. We finish the proof of Theorem 14.20 in this subsection.

We first prove the integrability for f_i. Since σ_i in (14.22) must be injective by the stability condition, we have $\dim V_i \le \dim W_i + \sum_{\mathrm{i}(h)=i} \dim V_{\mathrm{o}(h)=0}$. This condition will be violated if we apply f_i too many times.

Next we compute the constant $c(i, \lambda)$ for $(e_i f_i - f_i e_i) a_\lambda = c(i, \lambda) a_\lambda$. We can restrict the class to the Brill-Noether locus of maximal possible dimension. One can show that it is $\mathfrak{M}_{i;p}(\mathbf{v}, \mathbf{w})$ with $p = \max(0, -\langle h_i, \mathbf{w} - \mathbf{v}\rangle)$ ([**N-II**, 4.6]). When we restrict the set-theoretical intersections Q and Q', appeared in §14.2.3 to the inverse image of this open set, either Q or Q' (or both of) is empty set by Proposition 14.25. Moreover the remaining Q' or Q is a fiber bundle over the diagonal, where the fiber is $\mathbb{P}^{|\langle h_i, \mathbf{w}-\mathbf{v}\rangle|-1}$. In this situation, the convolution product can be computed [**N-II**, 8.ii], and the constant is given by the Euler number of the fiber, i.e., $|\langle h_i, \mathbf{w} - \mathbf{v}\rangle|$. This shows that $c(i, \lambda) = \langle h_i, \mathbf{w} - \mathbf{v}\rangle$.

We finally prove (2). It is enough to show the highest weight property of $\bigoplus_{\mathbf{v}} H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v}, \mathbf{w}))$ since we already know it is integrable. This follows at once from Theorem 14.27. But even if we do not know Theorem 14.27, we can show it from Lemma 14.26 and Proposition 14.28 by induction on $\dim V$ and $\varepsilon_i(X)$.

14.4. Quiver varieties and tensor products

In this section we review the construction of [**Na01**]. A similar result was obtained by Malkin indepedently [**Ma03**].

Let W^2 be a nonzero proper I-graded subspace of W:

$$0 \subsetneq W^2 \subsetneq W.$$

Let $W^1 = W/W^2$. Let us denote the corresponding dimension vectors by $\mathbf{w}^1$, $\mathbf{w}^2$. We fix those data throughout in this section.

It is straightforward to generalize the construction of this case to more general case of a flag of W corresponding to tensor products of more than two representations.

14.4.1. Tensor product varieties. We introduce varieties $\mathfrak{Z}(\mathbf{v}, \mathbf{w}) \equiv \mathfrak{Z}$ and $\widetilde{\mathfrak{Z}}(\mathbf{v}, \mathbf{w}) \equiv \widetilde{\mathfrak{Z}}$ as follows. Consider the two types of functions on $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ as in §14.1.4. Then $[B, i, j] \in \mathfrak{M}(\mathbf{v}, \mathbf{w})$ is contained in $\mathfrak{Z}$ if and only if the second type (b) maps $W^2_{\mathrm{o}(h_1)}$ into $W^2_{\mathrm{i}(h_N)}$ for any path $h_1, \dots, h_N$. Similarly $[B, i, j] \in \mathfrak{M}(\mathbf{v}, \mathbf{w})$ is contained in $\widetilde{\mathfrak{Z}}$ if and only if functions of the first type (a) vanishes, and the second type (b) maps $W^2_{\mathrm{o}(h_1)}$ into 0, and $W_{\mathrm{o}(h_1)}$ to $W^2_{\mathrm{i}(h_N)}$ for any path $h_1, \dots, h_N$.

These are closed subvarieties in $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ which are π-saturated, i.e., $\pi^{-1}(\pi(\mathfrak{Z})) = \mathfrak{Z}$, $\pi^{-1}(\pi(\widetilde{\mathfrak{Z}})) = \widetilde{\mathfrak{Z}}$. We have $\widetilde{\mathfrak{Z}} \subset \mathfrak{Z}$. Also any function vanishes on $\mathfrak{L}$, so we have $\mathfrak{L} \subset \widetilde{\mathfrak{Z}}$.

Proposition 14.29 ([**Na01**, §3]). (1) *There exists a decomposition*

$$\mathfrak{Z}(\mathbf{v}, \mathbf{w}) = \bigsqcup_{\mathbf{v}^1+\mathbf{v}^2=\mathbf{v}} \mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}),$$

into locally closed subvarieties such that each piece $\mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w})$ *is a vector bundle over* $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$. *The piece* $\widetilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}) = \mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}) \cap \widetilde{\mathfrak{Z}}(\mathbf{v}, \mathbf{w})$ *is the restriction of the vector bundle to* $\mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2)$.

(2) $\widetilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2)$*, and hence* $\widetilde{\mathfrak{Z}}(\mathbf{v}, \mathbf{w})$ *also, are Lagrangian subvarieties in* $\mathfrak{M}$. *Its irreducible components are bijetive to the products of irreducible components of* $\mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1)$ *and* $\mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2)$.

Let $\widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2)$ be the disjoint union of all $\widetilde{\mathfrak{Z}}(\mathbf{v}, \mathbf{w})$. Let $\operatorname{Irr} \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2)$ be the set of its irreducible components. As a set, it is bijective to $\operatorname{Irr} \mathfrak{L}(\mathbf{w}^1) \times \operatorname{Irr} \mathfrak{L}(\mathbf{w}^2)$ thanks to the above result.

14.4.2. Tensor product crystal.

We can endow the structure of a crystal on $\operatorname{Irr} \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2)$ by the same construction as in the previous section. Then we have

Theorem 14.30 ([**Na01**, 4.6]). *The crystal* $\operatorname{Irr} \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2)$ *is isomorphic to* $\operatorname{Irr} \mathfrak{L}(\mathbf{w}^1) \otimes \operatorname{Irr} \mathfrak{L}(\mathbf{w}^2)$ *as a crystal.*

Note also that we have an inclusion $\operatorname{Irr} \mathfrak{L}(\mathbf{w}) \subset \operatorname{Irr} \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2)$, which is easy to check to be compatible with the crystal structure. Then the above theorem means that $\{\operatorname{Irr} \mathfrak{L}(\mathbf{w})\}$ forms a *compatible family* under tensor products, i.e., the component of $\operatorname{Irr} \mathfrak{L}(\mathbf{w}^1) \otimes \operatorname{Irr} \mathfrak{L}(\mathbf{w}^2)$ containing $[\mathfrak{L}(0, \mathbf{w}^1)] \otimes [\mathfrak{L}(0, \mathbf{w}^2)]$ is isomorphic to $\operatorname{Irr} \mathfrak{L}(\mathbf{w}^1 + \mathbf{w}^2)$. This property implies that $\operatorname{Irr} \mathfrak{L}(\mathbf{w})$ is isomorphic to $\mathscr{B}(\mathbf{w})$ by [**Joseph**, 6.4.21].

14.4.3. Tensor product representation.

Since $\widetilde{\mathfrak{Z}}$ is π-saturated, the convolution makes $H_*(\widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2))$ into a representation of $H_{\mathrm{top}}(Z(\mathbf{w}))$. We take its top degree part as in §14.2.2. Then its character is given by the character of the tensor product thanks to the above result. Therefore we have

Theorem 14.31 ([**Na01**, 5.2]). $H_{\mathrm{top}}(\widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2))$ *is isomorphic to the tensor produc representation* $V(\mathbf{w}^1) \otimes V(\mathbf{w}^2)$ *as a* $\mathfrak{g}$*-module via Theorem 14.20.*

Remark 14.32. This result gives only an abstract isomorphism, since we only check the equality of characters. A more natural isomorphism

$$H_{\mathrm{top}}(\mathfrak{L}(\mathbf{w}^1)) \otimes H_{\mathrm{top}}(\mathfrak{L}(\mathbf{w}^2)) \xrightarrow{\cong} H_{\mathrm{top}}(\widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2))$$

was constructed in [**Na01**, 5.9] for finite type $\mathfrak{g}$.

14.5. Convolution on equivariant K-groups of quiver varieties

In this section we replace the homology group H_{top} in Theorem 14.20 by the Grothendieck group of equivariant coherent sheaves, i.e., the equivariant K-group. More details will be explained in the subsequent chapter after the Drinfeld presentation will be given.

14.5.1. Setting. We first modify the $\mathbb{C}^*$-action in §14.1.8 so that it gives rise the convolution algebra compatible with the quantum loop algebra studied in the literature. We define a $\mathbb{C}^*$-action on $\mathbf{M}$ by multiplying all B_h $(h \in H)$, a and b by $t \in \mathbb{C}^*$. We also have a natural action of

$$G_{\mathbf{w}} \overset{\text{def.}}{=} \prod_i \mathrm{GL}(W_i)$$

by the conjugation. Let us denote this action of $G_{\mathbf{w}} \times \mathbb{C}^*$ on $\mathbf{M}$ and the induced action on $\mathfrak{M}$ by $(s,t) \star (B,a,b)$ and $(s,t) \star [B,a,b]$ for $s \in G_{\mathbf{w}}$, $t \in \mathbb{C}^*$ respectively. More concretely, they are given by

$$(s,t) \star (B,a,b) = (tB, tas^{-1}, tsb),$$

and the same formula replacing ( ) by [ ]. We also have an action on $\mathfrak{M}_0$ so that the projection π is equivariant.

Remark 14.33. This $\mathbb{C}^*$-action is different from one used in [**Qaff**, §2.7] when there are multiple edges between vertexes. Though the main result in [**Qaff**] holds for both $\mathbb{C}^*$-actions, the result in [**Qchar**] is true only for the above $\mathbb{C}^*$-action. This is the reason why we change the $\mathbb{C}^*$-action.

As in §14.2 the convolution makes

$$K^{G_{\mathbf{w}} \times \mathbb{C}^*}(Z(\mathbf{w})) = \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} K^{G_{\mathbf{w}} \times \mathbb{C}^*}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$$

into an algebra (possibly without unit), and

$$K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) = \bigoplus_{\mathbf{v}} K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{v}, \mathbf{w}))$$

into its module. We call it the *universal standard module.*

Moreover these structures are defined over

$$K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\text{point}) = R(G_{\mathbf{w}} \times \mathbb{C}^*),$$

the representation ring. Since $G_{\mathbf{w}}$ is the product of general linear groups, we have

$$R(G_{\mathbf{w}} \times \mathbb{C}^*) \cong \mathbb{Z}[q, q^{-1}] \otimes \bigotimes_{i \in I} \mathbb{Z}[z_{i,1}^{\pm 1}, \dots, z_{i,w_i}^{\pm 1}]^{S_{w_i}}$$

where q is the character of $\mathbb{C}^*$ corresponding to the identity map $\mathbb{C}^* \to \mathbb{C}^*$, w_i is the i^{th} entry of $\mathbf{w}$, and S_{w_i} is the symmetric group of w_i letters acting the Laurent polynomial ring $\mathbb{Z}[z_{i,1}^{\pm 1}, \dots, z_{i,w_i}^{\pm 1}]$ by exchanging variables. Note also that

$$R(T_{\mathbf{w}} \times \mathbb{C}^*) \cong \mathbb{Z}[q, q^{-1}] \otimes \bigotimes_{i \in I} \mathbb{Z}[z_{i,1}^{\pm 1}, \dots, z_{i,w_i}^{\pm 1}],$$

where $T_{\mathbf{w}}$ is a maximal torus of $G_{\mathbf{w}}$.

Then we have

Theorem 14.34 ([**Qaff**, 9.4.1]). *There is an* (*explicitly defined*) *algebra homomorphism*

$$\Phi \colon \widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g}) \to K^{G_{\mathbf{w}} \times \mathbb{C}^*}(Z(\mathbf{w})) \otimes_{R(\mathbb{C}^*)} \mathbb{Q}(q).$$

We do not define $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$ yet, but when $\mathfrak{g}$ is of type ADE, it is the modified quantum affine algebra $\widetilde{\mathbf{U}}_q$ associated with the weight lattice P^0_{cl} studied in Chapter 13. A general definition will be given in the next chapter. Then the construction of Φ will be sketched in §15.3.

At this moment, the reader should notice that the module structures over $R(T_{\mathbf{w}} \times \mathbb{C}^*)$ and $R(G_{\mathbf{w}} \times \mathbb{C}^*)$, which we saw in §12.2, appear already in the definition in the geometric framework.

Remark 14.35. After Theorem 14.20 (or its earlier related result in [**N-I**]) was obtained, it became natural to expect the above result, since we saw similarity with constructions of affine Hecke algebras by Kazhdan-Lusztig [**KL87**], Ginzburg [**Chriss-Ginzburg**], quantum affine algebras of type A by Ginzburg-Vasserot [**GV93**] (see also [**Va98**]), and upper half of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ by Grojnowski [**Gr94**].

CHAPTER 15

Quanum loop algebras of Kac-Moody Lie algebras

As we mentioned in Chapter 9, an (untwisted) affine Lie algebra has two presentations, one as a Kac-Moody Lie algebra, another as a central extension of the loop Lie algebra of a finite dimensional simple complex Lie algebra. Similarly the quantum affine algebra has two presentations. The first one was is what we have studied already. The second one, which is usually called the *Drinfeld presentation* [**Dr88**], was known to be useful to study finite dimensional representations of quantum affine algebras.

The Drinfeld presentation is used to define the homomorphism Φ in Theorem 14.34 and more useful than the usual one for us. In this chapter, we review it quickly and then explain how structures studied in Part 2 can be seen from the geometry of quiver varieties.

15.1. Drinfeld presentation

The Drinfeld presentation can be used to define the quantum affinization $\mathbf{U}_q(\widehat{\mathfrak{g}})$ associated with any Kac-Moody Lie algebra $\mathfrak{g}$. This was used to define, for example, quantum toroidal algebras [**GKV95, VV96, VV98, Sa98, STU98**], where $\mathfrak{g}$ is an affine Lie algebra. The quiver varieties make sense for any (symmetric) Kac-Moody Lie algebras and there are no essential differences between finite types and other types. Therefore the author introduced the definition and various concepts on representations for $\mathbf{U}_q(\widehat{\mathfrak{g}})$ for a Kac-Moody Lie algebra $\mathfrak{g}$ in a uniform way in [**Qaff**].

In this section, we give a quick review for the definition of the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ of the Kac-Moody algebra $\mathfrak{g}$. The quantum affinization $\mathbf{U}_q(\widehat{\mathfrak{g}})$ can be also defined in the same way as its one-dimensional central extension, but will not be used in the book, so its definition will be omitted. Since the algebras defined via quiver varieties are automatically symmetric, we treat only the symmetric case, though non-symmetric case can be treated in the same way.

The *quantum loop algebra* $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ of $\mathfrak{g}$ is an associative algebra over $\mathbb{Q}(q)$ generated by $e_{i,r}, f_{i,r}$ ($i \in I$, $r \in \mathbb{Z}$), q^h ($h \in P^*$), $h_{i,m}$ ($i \in I$, $m \in \mathbb{Z} \setminus \{0\}$) with the following defining relations:

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \quad [q^h, h_{i,m}] = 0, \tag{15.1}$$

$$\psi_i^\pm(z)\psi_j^\pm(w) = \psi_j^\pm(w)\psi_i^\pm(z), \tag{15.2}$$

$$\psi_i^-(z)\psi_j^+(w) = \psi_j^+(w)\psi_i^-(z), \tag{15.3}$$

$$q^h e_{i,r} q^{-h} = q^{\langle h,\alpha_i\rangle} e_{i,r}, \quad q^h f_{i,r} q^{-h} = q^{-\langle h,\alpha_i\rangle} f_{i,r}, \tag{15.4}$$

$$(z - q^2 w)\psi_i^s(z) x_i^\pm(w) = (q^2 z - w) x_i^\pm(w) \psi_i^s(z), \tag{15.5}$$

$$(z - q^{-1}w)^{-\langle \alpha_i, h_j\rangle} \psi_j^s(z) x_i^\pm(w) = (q^{-1}z - w)^{-\langle \alpha_i, h_j\rangle} x_i^\pm(w) \psi_j^s(z), \quad \text{if } i \neq j \tag{15.6}$$

$$[x_i^+(z), x_j^-(w)] = \frac{\delta_{ij}}{q-q^{-1}}\left\{\delta\left(\frac{w}{z}\right)\psi_i^+(w) - \delta\left(\frac{z}{w}\right)\psi_i^-(z)\right\}, \tag{15.7}$$

$$(z - q^{\pm 2}w)x_i^\pm(z)x_i^\pm(w) = (q^{\pm 2}z - w)x_i^\pm(w)x_i^\pm(z), \tag{15.8}$$

$$(z - q^{-1}w)^{b'} x_i^\pm(z)x_j^\pm(w) = (q^{-1}z - w)^{b'} x_j^\pm(w)x_i^\pm(z), \quad \text{if } i \neq j, \tag{15.9}$$

$$\begin{aligned}\sum_{\sigma\in S_b}\sum_{p=0}^{b}(-1)^p \begin{bmatrix} b \\ p \end{bmatrix}_q & x_i^\pm(z_{\sigma(1)})\cdots x_i^\pm(z_{\sigma(p)})x_j^\pm(w) \\ & \times x_i^\pm(z_{\sigma(p+1)})\cdots x_i^\pm(z_{\sigma(b)}) = 0, \quad \text{if } i \neq j,\end{aligned} \tag{15.10}$$

where $s = \pm$, $b = 1 - \langle h_i, \alpha_j\rangle$, $b' = -\langle h_i, \alpha_j\rangle$, and S_b is the symmetric group of b letters. Here $\delta(z)$, $x_i^+(z)$, $x_i^-(z)$, $\psi_i^\pm(z)$ are generating functions defined by

$$\delta(z) \overset{\text{def.}}{=} \sum_{r=-\infty}^{\infty} z^r, \qquad x_i^+(z) \overset{\text{def.}}{=} \sum_{r=-\infty}^{\infty} e_{i,r}z^{-r}, \qquad x_i^-(z) \overset{\text{def.}}{=} \sum_{r=-\infty}^{\infty} f_{i,r}z^{-r},$$

$$\psi_i^\pm(z) \overset{\text{def.}}{=} q^{\pm h_i}\exp\left(\pm(q-q^{-1})\sum_{m=1}^{\infty} h_{i,\pm m}z^{\mp m}\right).$$

We also need the following generating function later:

$$p_i^\pm(z) \overset{\text{def.}}{=} \exp\left(-\sum_{m=1}^{\infty}\frac{h_{i,\pm m}}{[m]_q}z^{\mp m}\right).$$

We have $\psi_i^\pm(z) = q^{\pm h_i}p_i^\pm(qz)/p_i^\pm(q^{-1}z)$.

Remark 15.11. We slightly change the defining relation from [**Qaff**]. When $\mathfrak{g}$ is of finite type, $b' = 0$ or 1, and the relation is the same as one in [**Qaff**]. This is because we change the $\mathbb{C}^*$-action from [**Qaff**] when there are multiple edges between vertexes as mentioned in Remark 14.33.

Let $\mathbf{U}_q(\mathbf{L}\mathfrak{g})^+$ (resp. $\mathbf{U}_q(\mathbf{L}\mathfrak{g})^-$) be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ generated by elements $e_{i,r}$'s (resp. $f_{i,r}$'s). Let $\mathbf{U}_q(\mathbf{L}\mathfrak{g})^0$ be the $\mathbb{Q}(q)$-subalgebra generated by elements q^h, $h_{i,m}$.

The modified quantum loop algebra $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$ is defined as in the case of $\widetilde{\mathbf{U}}_q$ by replacing q^h by a_λ ($\lambda \in P$). We have

$$a_\lambda a_\mu = \delta_{\lambda\mu}a_\lambda, \quad e_{i,r}a_\lambda = a_{\lambda+\alpha_i}e_{i,r}, \quad f_{i,r}a_\lambda = a_{\lambda-\alpha_i}f_{i,r}, \quad h_{i,m}a_\lambda = a_\lambda h_{i,m}.$$

There is a homomorphism $\mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathbf{L}\mathfrak{g})$ defined by

$$q^h \mapsto q^h, \quad e_i \mapsto e_{i,0}, \quad f_i \mapsto f_{i,0}.$$

Let $e_{i,r}^{(n)} \overset{\text{def.}}{=} e_{i,r}^n/[n]_q!$, $f_{i,r}^{(n)} \overset{\text{def.}}{=} f_{i,r}^n/[n]_q!$. Let $_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by $e_{i,r}^{(n)}$, $f_{i,r}^{(n)}$, q^h and the coefficients of $p_i^\pm(z)$ for $i \in I$, $r \in \mathbb{Z}$, $n \in \mathbb{Z}_{>0}$, $h \in P^*$. (It should be true that $_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is free over $\mathbb{Z}[q,q^{-1}]$ and that the natural map $_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to \mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is an isomorphism. But the author does not know how to prove them.) This subalgebra was introduced by Chari-Pressley [**CP97**]. Let $_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})^+$ (resp. $_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})^-$) be $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by $e_{i,r}^{(n)}$ (resp. $f_{i,r}^{(n)}$) for $k \in I$, $r \in \mathbb{Z}$, $n \in Z_{>0}$. We have $_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})^\pm \subset {_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})}$. Let $_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})^0$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by q^h, the coefficients of $p_i^{\pm}(z)$ and

$$\begin{bmatrix} q^{h_i}; n \\ r \end{bmatrix} \overset{\text{def.}}{=} \prod_{s=1}^{r} \frac{q^{h_i} q^{n-s+1} - q^{-h_i} q^{-n+s-1}}{q^s - q^{-s}}$$

for all $h \in P$, $i \in I$, $n \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$. One can easily shown that ${}_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})^0 \subset {}_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ (see e.g., [**Lusztig**, 3.1.9]).

For $\varepsilon \in \mathbb{C}^*$, let $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ be the *specialized quantum loop algebra* defined by ${}_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ via the algebra homomorphism $\mathbb{Z}[q, q^{-1}] \to \mathbb{C}$ that takes q to ε. We have an algebra homomorphism $\mathbf{U}_\varepsilon(\mathfrak{g}) \to \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$.

Let $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^{\pm}$, $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$ be the specialization of ${}_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})^{\pm}$, ${}_{\mathbf{A}}\mathbf{U}_q(\mathbf{L}\mathfrak{g})^0$ respectively. We have a weak form of the triangular decomposition

$$\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) = \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- \cdot \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0 \cdot \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^+, \tag{15.12}$$

which follows from the definition (cf. [**CP97**, 6.1]).

15.2. Drinfeld generators in terms of PBW bases

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. We denote the index set of its simple roots by I as in the previous section in this chapter. We denote the index set of the corresponding (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}$ by $\widehat{I} = I \sqcup \{0\}$ though these notation contradicts with what we have used in Part 2. We also write $P = \operatorname{Hom}(\bigoplus_{i\in I} \mathbb{Z}h_i, \mathbb{Z})$ the weight lattice of $\mathfrak{g}$, $\widehat{P} = \operatorname{Hom}(\bigoplus_{i\in \widehat{I}} \mathbb{Z}h_i \oplus \mathbb{Z}d, \mathbb{Z})$ the weight lattice of $\widehat{\mathfrak{g}}$.

Drinfeld [**Dr88**] defined a homomorphism from the Drinfeld presentation in $e_{i,r}$, $f_{i,r}$ $(i \in I,\ r \in \mathbb{Z})$, q^h $(h \in P)$, $h_{i,m}$ $(i \in I,\ m \in \mathbb{Z} \setminus \{0\})$ to the ordinary one in terms of Chevalley generators e_i, f_i $(i \in \widehat{I})$, q^h $(h \in \widehat{P})$ and stated that it is an isomorphism. But the proof was not given. Here we given an isomorphism in terms of the PBW base following Beck [**Be94**]. Note that Jing gave a proof following Drinfeld's definition [**Ji98**].

Choose a map $o\colon I \to \{\pm 1\}$ such that $o(i) = -o(j)$ if $a_{ij} < 0$. Then

———- To be written. ———-

15.3. Convolution product on equivariant K-groups – continued

In this section we explain how the homomorphism in Theorem 14.34 is defined. It is similar to the construction in Theorem 14.20. We define the images of generators of $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$ and check the defining relations.

15.3.1. Image of generators. As in Theorem 14.20 the image of a_λ is the diagonal $[\Delta(\mathbf{v}, \mathbf{w})]$ with $\lambda = \mathbf{w} - \mathbf{v}$, considered as an equivariant K-homology class this time.

To describe the image of $p_i^{\pm}(z)a_\lambda$ as a class, we prepare some notation. Let $\mathscr{C}_i^\bullet$ be the complex in (14.22). Recall that we considered it as a complex of vector bundle over $\mathfrak{M}(\mathbf{v}, \mathbf{w})$. Note that $\langle h_i, \mathbf{w} - \mathbf{v}\rangle$ is equal to $\operatorname{rank} \mathscr{C}_i^\bullet$. The image of $p_i^{\pm}(z)a_\lambda$ is given by more refined information of $\mathscr{C}_i^\bullet$.

We consider the generating function of its exterior powers

$$\textstyle\bigwedge_{-1/z}[\mathscr{C}_i^\bullet] \in [\mathcal{O}_{\mathfrak{M}(\mathbf{v},\mathbf{w})}] + K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w})) \otimes z^{-1}\mathbb{Z}[[z^{-1}]].$$

Note that this generating function $\bigwedge_{-1/z}$ is defined for the K-cohomology class E by extending the definition $\bigwedge_{-1/z} E = \sum_{i=0}^{\operatorname{rank} E} (-z)^i \bigwedge^i E$ for a vector bundle

E, thanks to the property $\bigwedge_{-1/z} F = \bigwedge_{-1/z} E \otimes \bigwedge_{-1/z} G$ for an exact sequence $0 \to E \to F \to G \to 0$. We can also consider

$$\bigwedge_{-z}[\mathscr{C}_i^{\bullet\vee}] \in [\mathcal{O}_{\mathfrak{M}(\mathbf{v},\mathbf{w})}] + K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w})) \otimes z\mathbb{Z}[[z]],$$

which is formally equal to $(-z)^{\operatorname{rank}\mathscr{C}_i^\bullet}(\det\mathscr{C}_i^\bullet)^\vee \bigwedge_{-1/z}\mathscr{C}_i^\bullet$. Here $\vee$ denotes the dual. Now we define the image of $p_i^\pm(z)a_\lambda$ by

$$p_i^+(z)a_\lambda \mapsto \Delta_* \bigwedge_{-1/z}[\mathscr{C}_i^\bullet], \qquad p_i^-(z)a_\lambda \mapsto \Delta_* \bigwedge_{-z}[\mathscr{C}_i^{\bullet\vee}],$$

where Δ is the inclusion $\mathfrak{M}(\mathbf{v},\mathbf{w}) \to Z(\mathbf{v},\mathbf{v};\mathbf{w})$.

Similarly the images of $x_i^\pm(z)a_\lambda$ are given by refinements of $\mathfrak{P}_i(\mathbf{v},\mathbf{w})$, $\omega\mathfrak{P}_i(\mathbf{v},\mathbf{w})$ given as the iamge of $e_i a_\lambda$, $f_i a_\lambda$ in Theorem 14.20.

Recall that $\mathfrak{P}_i(\mathbf{v},\mathbf{w})$ parametrizes pairs (B,a,b) and a B-invariant subspace S with $\dim S = \mathbf{v} - \alpha_i$. Considering $\mathfrak{P}_i(\mathbf{v},\mathbf{w}) \subset \mathfrak{M}(\mathbf{v}-\alpha_i,\mathbf{w}) \times \mathfrak{M}(\mathbf{v},\mathbf{w})$, we have two tautological vector bundles, denoted by V^1 and V^2, from the first and second components respectively. By the definition of $\mathfrak{P}_i(\mathbf{v},\mathbf{w})$, V^1 is a subbundle of V^2 and the quotient V^2/V^1 defines a line bundle over $\mathfrak{P}_i(\mathbf{v},\mathbf{w})$. Up to a slight correction factors, $e_{i,r}a_\lambda$ is given by the r^{th} tensor power of this line bundle, extended to $Z(\mathbf{v}-\alpha_i,\mathbf{v};\mathbf{w})$ by the push-forward homomorphism of the inclusion.

15.3.2. Proof of the defining relations. The proof of the defining relations of the quantum loop algebra is the technical heart of the paper [**Qaff**]. It is long and requires a detailed calculation at the last step, so we only review it very briefly.

The first three relations (15.1,15.2,15.3) (with replacement of q^h by a_λ) are obvious since all classes are defined over the diagonal.

The relations (15.4,15.5,15.6) can be checked by studying how complex $\mathscr{C}_j^\bullet$ differs in the first and second components of $\mathfrak{P}_i(\mathbf{v},\mathbf{w}) \subset \mathfrak{M}(\mathbf{v}-\alpha_i,\mathbf{w}) \times \mathfrak{M}(\mathbf{v},\mathbf{w})$. If V^1, V^2 are tautological bundles for the first and second components as in the previous subsection, V^1 is a subbundle of V^2 and the quotient V^2/V^1 is the line bundle whose powers correspond to $x_{i,r}$'s. From this description, it is easy to check these relations.

The relation (15.7) for $i \neq j$ is similar to the corresponding relation for homology case.

The exchange relation (15.9) is proved by showing the relevant intersections are transverse, and studying the set-theoretical intersections. The set theoretical intersections for the left hand side and the right hand are slightly different, but both have natural vector bundles with sections, whose zero loci are the same. From this we get (15.9) and the correction factors $(z-q^{-1}w)^{b'}$, $(q^{-1}z-w)^{b'}$ come from those bundles.

The Drinfeld-Serre relation (15.10) can be reduced to the ordinary quantum Serre relation for $e_{i,0}$, $e_{j,0}$ by an application of the projection formula in the K-theory. The ordinary quantum Serre relation follows from the integrability, once (15.7) (only for $e_{i,0}$, $f_{j,0}$) is estabilished as in the homology case.

Thus only those relations (15.7) with $i = j$ and (15.8). This is the hardest one to check. We notice that these are defining relations for $\mathbf{U}_q(\mathbf{L}\,\mathfrak{sl}_2)$ in side $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. Thus it is natural to hope that a reduction to the $\mathfrak{sl}_2$ case might be possible.

When the graph is of type A_1, consists of a single vertex with no edges, the quiver varieties are cotangent bundles of Grassmann manifolds of various dimensions. In this special case, the equivariant K-theory can be expressed by polynomials and the required computation was already done by Vasserot [**Va98**]. The quiver variety $\mathfrak{M}$ for a general graph and $T^* \operatorname{Gr}$ for A_1 are related in the following way:

- We take the quotient only by $GL(V_i)$, instead of G_V, there is a principal $\prod_{j \neq i} \operatorname{GL}(V_j)$-bundle $\widehat{\mathfrak{M}}$ over $\mathfrak{M}$.
- We only impose the equation $\mu = 0$ only on $\operatorname{Hom}(V_i, V_i)$, not on the whole $\mathbf{L}(V, V)$, we define a nonsingular variety $\widetilde{\mathfrak{M}}^\circ$, where the stability condition is still imposed. Then $\widehat{\mathfrak{M}}$ is a nonsingular closed subvariety of $\widetilde{\mathfrak{M}}^\circ$.
- We only impose the injectivity of σ_i in (14.22) instead of the stability condition to define $\widetilde{\mathfrak{M}}$. Then $\widetilde{\mathfrak{M}}^\circ$ is an open subvariety of $\widetilde{\mathfrak{M}}$. Moreover, $\widetilde{\mathfrak{M}}$ is the product of the cotangent bundle of the Grassmann manifold and an Euclidean space.

Now we relate the convolution product on $\widetilde{\mathfrak{M}}$ to one on $\mathfrak{M}$ step by step. The first step $\widetilde{\mathfrak{M}}$ to $\widetilde{\mathfrak{M}}^\circ$ is just restriction to the open subvariety, and is trivial. Other steps are given by functors of the form $F \times \operatorname{id}$, where F is the pull-back homomorphism with respect to the inclusion or the projection of the $\prod_{j \neq i} \operatorname{GL}(V_j)$-bundle. The detail can be found in [**Qaff**, §8].

15.4. Structures of universal standard modules

In the remainder of this chapter we assume $\mathfrak{g}$ is of type ADE.

Recall that the universal standard module $K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$ is a module of $K^{G_{\mathbf{w}} \times \mathbb{C}^*}(Z(\mathbf{w}))$. Via Φ (and taking the tensor product $\otimes_{R(\mathbb{C}^*)} \mathbb{Q}(q)$), we consider it as a reprsentation of $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$. We compare it with the extremal weight module $V(\lambda)$ associated with the weight $\lambda = \mathbf{w}$.

15.4.1. Universal standard modules and extremal weight modules. Recall (Chapter 12) that $V(\lambda)$ is a subrepresentation of the tensor product $\widetilde{V}(\lambda)$ of level 0 fundamental representations, and we have an intermediate subrepresentation $\check{V}(\lambda)$ so that $V(\lambda) \subset \check{V}(\lambda) \subset \widetilde{V}(\lambda)$. The latter two $\check{V}(\lambda)$, $\widetilde{V}(\lambda)$ have $R(T_{\mathbf{w}}) = \bigotimes_{i \in I} \mathbb{Z}[z_{i,1}^{\pm 1}, \dots, z_{i,w_i}^{\pm 1}]$-module structures, and $V(\lambda)$ has an induced $R(G_{\mathbf{w}}) = \bigotimes_{i \in I} \mathbb{Z}[z_{i,1}^{\pm 1}, \dots, z_{i,w_i}^{\pm 1}]^{S_{\lambda_i}}$-module structure.

THEOREM 15.13 ([**Na04**, Th. 2]). *Suppose $\mathfrak{g}$ is of type ADE. There is a unique ${}_{\mathbf{A}}\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$-isomorphism from ${}_{\mathbf{A}}V(\lambda)$ to $K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$, sending v_λ to the class $[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$ of the structure sheaf of $\mathfrak{L}(0, \mathbf{w}) = \mathrm{point}$, and respecting the $R(G_{\mathbf{w}})$-module structures.*

Here ${}_{\mathbf{A}}\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$ is the $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$-integral form associated with P^0_{cl} (denoted by ${}_{\mathbf{A}}\widetilde{\mathbf{U}}_q$ before). And ${}_{\mathbf{A}}V(\lambda)$ is the integral form of $V(\lambda)$.

When λ is a level 0 fundamental weight, this result was obtained by [**VV-III**] independently.

PROOF. First we show that the assertion follows, once the same is checked for $V(\lambda)$ and $K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) \otimes_{R(\mathbb{C}^*)} \mathbb{Q}(q)$. We know that ${}_{\mathbf{A}}V(\lambda) = {}_{\mathbf{A}}\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g}) v_\lambda$. On the other hand, it is known (see [**Qaff**, 12.3.2]) that

$$K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) = {}_{\mathbf{A}}\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g}) R(G_{\mathbf{w}} \times \mathbb{C}^*) [\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]. \tag{15.14}$$

Moreover, we can remove $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ since $R(G_{\mathbf{w}} \times \mathbb{C}^*)[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$ are spanned by elements $S_{\mathbf{c}_0}[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}] \in {}_{\mathbf{A}}\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$ by results in Chapter 12. This shows the above assertion.

Let us start the proof for the assertion after $\otimes\mathbb{Q}(q)$. Since $\mathfrak{L}(\mathbf{v},\mathbf{w})$ is Lagrangian, we have $H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v},\mathbf{w})) = 0$ if and only if $\mathfrak{L}(\mathbf{v},\mathbf{w}) = \emptyset$. Together with Theorem 14.20(2), $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{v},\mathbf{w})) = 0$ unless $\mathbf{w}-\mathbf{v}$ is contained in the convex hull of $W_0(\mathbf{w})$, where W_0 is the *finite* Weyl group. This implies that $[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$ is an extremal vector. Therefore we have a $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$-homomorphism Ψ sending v_λ to $[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$.

Next we consider the special case $\mathbf{w} = \varpi_i$. We have $R(G_{\mathbf{w}}) = \mathbb{Z}[z^{\pm}]$ in this case. From the definition of the $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$-module structure on $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$, explained in §15.3, we see that $P_{i,1}[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}] = z[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$. Since we have $P_{i,1}v_{\varpi_i} = zv_{\varpi_i}$ for the extremal weight module, the homomorphism Ψ commutes with z. Therefore we have an induced homomorphism $W(\varpi_i) = V(\varpi_i)/z \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))/z$. Since the left hand side is irreducible by Theorem 11.7(2), it must be injective. But $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$ is generated by $[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$ as a $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$-module, as we mentioned in (15.14) (with $\otimes_{R(\mathbb{C}^*)}\mathbb{Q}(q)$). Therefore Ψ is surjective also. This completes the proof for the special case $\lambda = \varpi_i$.

Now consider a general $\lambda = \mathbf{w}$. We consider the tensor product variety $\widetilde{\mathfrak{Z}}(\mathbf{w}^1;\cdots;\mathbf{w}^r)$ associated with the decomposition $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2 + \cdots + \mathbf{w}^r$, where each $\mathbf{w}^a$ is a fundamental weight. If we replace H_{top} by the equivariant K-group $K^{T_{\mathbf{w}}\times\mathbb{C}^*}$, we have ([**Na01**, 6.12])

$$K^{T_{\mathbf{w}}\times\mathbb{C}^*}(\widetilde{\mathfrak{Z}}(\mathbf{w}^1;\cdots;\mathbf{w}^r)) \cong \bigotimes_{R(T_{\mathbf{w}}\times\mathbb{C}^*)} K^{T_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}^a)),$$

where the order of tensor product is $a = 1$ to $a = r$ from the left. Since we know the result holds for fundamental weights, the right hand side is isomorphic to $\widetilde{V}(\lambda)$ so that $R(T_{\mathbf{w}})$-structures are compatible.

We consider the composite of homomorphisms

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) \to K^{T_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) \to K^{T_{\mathbf{w}}\times\mathbb{C}^*}(\widetilde{\mathfrak{Z}}(\mathbf{w}^1;\cdots;\mathbf{w}^r)),$$

where the first one is the restriction of the action from $G_{\mathbf{w}}$ to $T_{\mathbf{w}}$ and the second one is the push-forward homomorphism for the inclusion $\mathfrak{L}(\mathbf{w}) \subset \widetilde{\mathfrak{Z}}(\mathbf{w}^1;\cdots;\mathbf{w}^r)$. Both are compatible with the convolution product, and hence they are $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$-homomorphisms. Also both are compatible with $R(G_{\mathbf{w}})$ and $R(T_{\mathbf{w}})$-structures, where $R(G_{\mathbf{w}}) \to R(T_{\mathbf{w}})$, given by the restriction, is nothing but forgetting the $\prod_i S_{\lambda_i}$-invariance. It is known that both homomorphisms are injective. (See [**Qaff**, §7] for the first, [**Na01**, 3.10] for the second.)

Now $V(\lambda)$ is characterized as $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})v_\lambda$ in $\widetilde{V}(\lambda)$ and the same is true for $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) \otimes \mathbb{Q}(q)$, as we mentioned after (15.14). Therefore we get the assertion. □

The proof of (15.14) in [**Qaff**, 12.3.2] shows that the same is true for any closed subgroup of $G_{\mathbf{w}}$, in particular, for $T_{\mathbf{w}}$. It shows that $K^{T_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$ appeared during the proof is isomorphic to the $\mathbf{A}$-integral form of $\check{V}(\lambda)$.

From this result, we see that the canonical base element $S_{\mathbf{c}_0}v_\lambda$ corresponds to $S_{\mathbf{c}_0} \otimes \mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}$, where $S_{\mathbf{c}_0}$ in the latter is regarded as a representation of $G_{\mathbf{w}}$. More generally, $T_w(S_{\mathbf{c}_0}v_\lambda)$ for w in the affine Weyl group is represented by a similar class, where $\mathscr{O}_{\mathfrak{L}(0,\mathbf{w})}$ is replaced by $\mathscr{O}_{\mathfrak{L}(\mathbf{v},\mathbf{w})}$ with $\mathbf{w}-\mathbf{v}=w\mathbf{w}$. (Note that $\mathfrak{L}(\mathbf{v},\mathbf{w})$ in this case consists of a single point by [**Na03**].) Thus all extremal vectors in the canonical base $\mathscr{B}(\lambda)$ are very simple geometric objects, and easy to understand. For more results on canonical bases, see §15.4.3.

15.4.2. Geometric interpretation of cells. Recall that we have introduced two-sided ideals $\widetilde{\mathbf{U}}_q[{\geq}\lambda]$, $\widetilde{\mathbf{U}}_q[{>}\lambda]$ of the modified enveloping algebra in Definition 13.1. ($\widetilde{\mathbf{U}}_q$ there is $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$ above.) Let us introduce a slightly different version: $\widetilde{\mathbf{U}}_q[{\nleq}\lambda]$ consists of elements acting on $V(\lambda')$ by 0 for any $\lambda'\leq\lambda$. We have $\widetilde{\mathbf{U}}_q[{>}\lambda]\subset\widetilde{\mathbf{U}}_q[{\nleq}\lambda]$. Similarly we define $\widetilde{\mathbf{U}}_q[{\nless}\lambda]$. We have $\widetilde{\mathbf{U}}_q[\lambda]=\widetilde{\mathbf{U}}_q[{\geq}\lambda]/\widetilde{\mathbf{U}}_q[{>}\lambda]\cong\widetilde{\mathbf{U}}_q[{\nless}\lambda]/\widetilde{\mathbf{U}}_q[{\nleq}\lambda]$.

Take $\lambda=\mathbf{w}$ as in the previous subsection.

Theorem 15.15. (1) *The homomorphism Φ in Theorem 14.34 factors through $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})/\widetilde{\mathbf{U}}_q[{\nleq}\lambda]$. The induced homomorphism*

$$\Phi\colon\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})/\widetilde{\mathbf{U}}_q[{\nleq}\lambda]\to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))\otimes_{R(\mathbb{C}^*)}\mathbb{Q}(q)$$

is injective.

(2) *We have a commutative diagram*

$$\begin{tikzcd}
\widetilde{\mathbf{U}}_q[\lambda]=\widetilde{\mathbf{U}}_q[{\nless}\lambda]/\widetilde{\mathbf{U}}_q[{\nleq}\lambda] \arrow[r, "\cong"] \arrow[d] & K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})\times\mathfrak{L}(\mathbf{w}))\otimes_{R(\mathbb{C}^*)}\mathbb{Q}(q) \arrow[d] \\
\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})/\widetilde{\mathbf{U}}_q[{\nleq}\lambda] \arrow[r, "\Phi"] & K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))\otimes_{R(\mathbb{C}^*)}\mathbb{Q}(q),
\end{tikzcd}$$

where the right down arrow is the push-forward homomorphism associated with the inclusion $\mathfrak{L}(\mathbf{w})\times\mathfrak{L}(\mathbf{w})\subset Z(\mathbf{w})$.

Note that the right down arrow in (2) is a homomorphism with respect to convolution products. Moreover, the Künneth theorem holds for $\mathfrak{L}(\mathbf{w})$ by [**Na01**, 3.4], hence we have

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})\times\mathfrak{L}(\mathbf{w}))\cong K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))\otimes_{R(G_{\mathbf{w}}\times\mathbb{C}^*)}K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})).$$

The right hand side clearly contains element $\beta=b_1sb_2^{\#}$ in $\mathscr{B}[\lambda]$ in Theorem 13.3(3), as we have $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))\cong{}_{\mathbf{A}}V(\lambda)$ by Theorem 15.13. Thus (2) is clear if (1) is proved.

Proof. For $\mu\nleq\lambda$, $\Phi(a_\mu)$ is defined to be 0 since $\mathfrak{M}(\mathbf{v},\mathbf{w})=\emptyset$ unless $\mu=\mathbf{w}-\mathbf{v}\leq\mathbf{w}=\lambda$. Therefore $b_1sb_2^{\#}$ for $\beta=(b_1,s,b_2)\in\mathscr{B}[\mu]$ is also sent to 0. Since such elements for various $\mu\nleq\lambda$ form a base of $\widetilde{\mathbf{U}}_q[{\nleq}\lambda]$ by Theorem 13.3(3), the homomorphism Φ factors.

If $x\in\widetilde{\mathbf{U}}_q[\lambda]$ is mapped to 0 in $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$, it acts on $V(\lambda)=K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))\otimes\mathbb{Q}(q)$ by 0. More generally it acts on $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\pi^{-1}(x))$ for any $x\in\mathfrak{M}_0^{\mathrm{reg}}(\mathring{\mathbf{v}},\mathbf{w})$, where $\pi^{-1}(x)$ is a shorthanded notation of the union of $\pi^{-1}(x)\subset\mathfrak{M}(\mathbf{v},\mathbf{w})$ for various $\mathbf{v}$. By §14.1.11, this $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\pi^{-1}(x))$ is isomorphic to $V(\mathbf{w}-\mathring{\mathbf{v}})$. Now we have $\mathfrak{M}_0^{\mathrm{reg}}(\mathring{\mathbf{v}},\mathbf{w})\neq\emptyset$ if and only if $\mathbf{w}-\mathring{\mathbf{v}}\leq\mathbf{w}$ and it is dominant by Theorem 14.13. (It was proved earlier for finite type in [**N-II**, 10.5].) Therefore $x\in\widetilde{\mathbf{U}}_q[{\nleq}\lambda]$. $\square$

Remarks 15.16. (1) It is likely that Φ in (1) is an isomorphism.

(2) The above $K^{G_{\mathbf w}\times\mathbb C^*}(\mathfrak L(\mathbf w)\times\mathfrak L(\mathbf w))$ can be generalized as follows. Take $\mathfrak M_0^{\mathrm{reg}}(\overset{\circ}{\mathbf v},\mathbf w)$ as in the proof, and consider a closed subvariety of $Z(\mathbf w)$ given by

$$Z_{\overset{\circ}{\mathbf v}}(\mathbf w)=\{(x^1,x^2)\in\mathfrak M(\mathbf w)\times\mathfrak M(\mathbf w)\mid\pi(x^1)=\pi(x^2)\in\overline{\mathfrak M_0^{\mathrm{reg}}(\overset{\circ}{\mathbf v},\mathbf w)}\},$$

where $\overline{}$ is the closure. If $\overset{\circ}{\mathbf v}=0$, we get $\mathfrak L(\mathbf w)\times\mathfrak L(\mathbf w)$. Then it is likely that

$$\widetilde{\mathbf U}_q[{}^{\not<(\mathbf w-\overset{\circ}{\mathbf v})}]/\widetilde{\mathbf U}_q[{}^{\nleq}\lambda]\to K^{G_{\mathbf w}\times\mathbb C^*}(Z_{\overset{\circ}{\mathbf v}}(\mathbf w))\otimes_{R(\mathbb C^*)}\mathbb Q(q)$$

is an isomorphism. See [**TX06**] for the corresponding result for the affine Hecke algebra of type A.

15.4.3. Geometric characterization of the canonical base elements. By Theorem 12.12 we have a characterization of the canonical base elements up to sign. In this subsection, we review the geometric definition of the bilinear form and the bar involution due to Varagnolo-Vasserot [**VV-III**].

The existence of those structures and the characterization of the canonical base were conjectured by Lusztig [**Lu00**]. (More precisely, he conjectured them for the equivariant K-group for $T_{\mathbf w}$.) Varagnolo-Vasserot proved the conjecture for level 0 fundamental representations. But once results in Chapter 12 are established, the results are true for any level 0 weight.

——————- To be written. ——————-

The following was proved in [**Qaff**, 7.3.5] earlier than the above results, and holds for arbitrary $\mathfrak g$.

Theorem 15.17. *$K^{G_{\mathbf w}\times\mathbb C^*}(\mathfrak L(\mathbf v,\mathbf w))$ is free of finite rank over $R(G_{\mathbf w}\times\mathbb C^*)$.*

Thus it is natural to hope that there exists a geometrically defined base on $K^{G_{\mathbf w}\times\mathbb C^*}(\mathfrak L(\mathbf v,\mathbf w))$ for any $\mathfrak g$.

Remark 15.18. Lusztig's conjecture [**Lu00**] is motivated by his similar conjecture for the existence of the canonical base of the equivariant K-groups of Springer fibers [**Lu98b**]. This conjecture is proved recently by Bezrukavnikov-Mirkovic [**BM10**]. It is natural to hope that their technique works also for quiver varieties. This is communicated also by Bezrukavnikov to the author.

CHAPTER 16

Character formula

In this chapter we explain the character formula of irreducible representations of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ in terms of the intersection cohomology groups of an abelian group fixed point set of quiver varieties, called graded or cyclic quiver varieties.

16.1. ℓ-integrable highest weight representations

Irreducible finite dimensional representations of quantum affine algebras are classified by the so-called Drinfeld polynomials. This result is due to Drinfeld [**Dr88**], Chari-Pressley [**CP94, CP95**], and [**CP97**] for the case of roots of unity.

In this section we generalize it to the case of the quantum loop algebra for a Kac-Moody Lie algebra. This is the concept of the title of this section. It turns out that there is no essential difference, as is expected from the theory of quiver varieties as explained in the beginning of §15.1.

16.1.1. Definition and classification of irreducible ℓ-integrable highest weight representations. We first introduce a concept of *ℓ-weight*, where ('ℓ' stands for the loop). The point is that $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ contains a larger commutative subalgebra than the usual $\mathbf{U}_q$, since $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$ contains not only the Cartan part q^h ($h \in P$) but also all $h_{i,m}$'s.

Let M be an $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module with the weight space decomposition $M = \bigoplus_{\lambda\in P} M_\lambda$ as a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module. Since the commutative subalgebra $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$ preserves each M_λ, we can further decompose M into a sum of generalized simultaneous eigenspaces for $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$:

$$M = \bigoplus M_\Psi, \tag{16.1}$$

where Ψ stands a pair $(\Lambda, (P_i^\pm(z))_i)$ and M_Ψ consists of vectors $m \in M$ satisfying the following conditions:

$$\begin{gathered} q^h * m = \varepsilon^{\langle h,\Lambda\rangle} m \quad \text{for } h \in P^*, \\ \begin{bmatrix} q^{h_i}; 0 \\ r \end{bmatrix} * m = \begin{bmatrix} \langle h_i, \Lambda\rangle \\ r \end{bmatrix}_\varepsilon \quad \text{for } i \in I,\ r \in \mathbb{Z}_{>0}, \\ (p_i^\pm(z) - P_i^\pm(z)\operatorname{Id})^N * m = 0 \quad \text{for } i \in I \text{ and sufficiently large } N. \end{gathered} \tag{16.2}$$

If $M_\Psi \neq 0$, we call M_Ψ an *ℓ-weight space*, and the corresponding $\Psi = (\Lambda, (P_i^\pm(z))_i)$ an *ℓ-weight*. This is a refinement of the weight space decomposition.

This concept is a natural generalization of usual weight spaces, except that we consider *generalized* simultaneous eigenspaces, not the actural eigenspaces. It turns out that having simultaneous eigenspaces give us a *strong* constraint on representations, which are called *tame* representations [**NT98**]. Most of irreducible representations are *not* tame, so the above definition is natural.

We say a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-representation M is an *ℓ-highest weight representation* with *ℓ-highest weight* $\Psi = (\Lambda, (P_i^\pm(z))_i)$ if there exists a vector $m_0 \in M$ satisfying

$$e_{i,r} * m_0 = 0, \qquad \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- * m_0 = M,$$

together with (16.2) with $N = 1$. (Since the usual highest weight space is 1-dimensional, m_0 must be the actual eigenvector.) This m_0 is called the *ℓ-highest weight vector*.

By using (15.12) and a standard argument, one can show that there is a simple ℓ-highest weight representation M of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ with ℓ-highest weight vector m_0 satisfying the above for any Ψ. Moreover, such M is unique up to isomorphism.

A representation M of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ is said to be *ℓ-integrable* if

(a) M has a weight space decomposition $M = \bigoplus_{\lambda \in P} M_\lambda$ as a representation of $\mathbf{U}_\varepsilon(\mathfrak{g})$ such that $\dim M_\lambda < \infty$,
(b) for any $m \in M$, there exists $n_0 \geq 1$ such that $e_{k,r_1} \cdots e_{k,r_n} * m = f_{k,r_1} \cdots f_{k,r_n} * m = 0$ for all $r_1, \dots, r_n \in \mathbb{Z}$, $k \in I$ and $n \geq n_0$.

For example, if $\mathfrak{g}$ is finite dimensional, and M is a finite dimensional representation, then M satisfies the above conditions after twisting with a certain automorphism of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ ([**CP94**, 12.2.3]).

Proposition 16.3. *An irreducible ℓ-highest weight module V with an ℓ-highest weight $\Psi = (\Lambda, P_i^\pm(z))$ is ℓ-integrable if and only if Λ is dominant and there exists an I–tuple of polynomials $P = (P_i(u))_{i \in I}$ with $P_i(0) = 1$ such that*

$$\begin{gathered} \langle h_i, \Lambda \rangle = \deg P_i, \\ P_i^+(z) = P_i(1/z), \quad P_i^-(z) = c_{P_i}^{-1} z^{\deg P_i} P_i(1/z), \end{gathered} \tag{16.4}$$

where c_{P_i} is the top term of P_i, i.e., the coefficient of $u^{\deg P_i}$ in P_i.

This result was stated without proof by Drinfeld for the Yangian [**Dr88**]. The proof of the 'only if' part when $\mathfrak{g}$ is finite dimensional was given by Chari-Pressley [**CP94**, 12.2.6]. The 'if' part was proved by them later in [**CP95**] when $\mathfrak{g}$ is finite dimensional. The proof can be generalized to the quantum loop algebra in a straightforward way, so I leave it as an exercise to the reader. The result is also a simple consequence of a geometric construction, and need not to know the proof.

Definition 16.5. The polynomials P_i are called *Drinfeld polynomials.*

Motivated by Proposition 16.3, we introduce the following notion:

Definition 16.6. An ℓ-weight $\Psi = (\Lambda, P_i^\pm(z))$ is said to be *ℓ-dominant* if Λ is dominant and there exist a polynomial $P(u) = (P_i(u))_i \in \mathbb{C}[u]^I$ for with $P_i(0) = 1$ such that (16.4) holds.

When the Drinfeld polynomials are given by

$$P_i(u) = \begin{cases} 1 - su & \text{if } i \neq i_0, \\ 1 & \text{otherwise,} \end{cases}$$

for some $i_0 \in I$, $s \in \mathbb{C}^*$, the corresponding irreducible ℓ-highest representation is called an *ℓ-fundamental representation.* When $\mathfrak{g}$ is finite dimensional, it corresponds to an level 0 fundamental representation $W(\varpi_{i_0})$ studied in §11.2 if $s = \pm 1$. For general s, it is just equal to $V(\varpi_i)/(z - \pm s)$.

16.1.2. Drinfeld polynomials of the universal standard module. Let us consider the universal standard module $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$. As is noticed in Theorem 15.13, it has a distinguished vector $[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$. Note that the class $[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$ is killed by all $x_i^+(z)$ $(i \in I)$ as $\mathfrak{L}(-\alpha_i, \mathbf{w}) = \emptyset$. Thus it satisfies the first condition for the ℓ-highest weight vector. The second condition is not satisfied as $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(0,\mathbf{w})) = R(G_{\mathbf{w}} \times \mathbb{C}^*)$ is not 1-dimensional as a complex vector space. But it is of rank 1 as $R(G_{\mathbf{w}} \times \mathbb{C}^*)$, and we have the property (15.14). Thus $[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$ is the ℓ-highest weight vector if we consider $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ as 'coefficients'. Note also that the complex $\mathscr{C}_i^\bullet$ (14.22) is just W_i at $\mathfrak{L}(0,\mathbf{w})$. Therefore we have

$$p_i^+(z)[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}] = \textstyle\bigwedge_{-1/z} W_i \otimes [\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}], \quad p_i^-(z)[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}] = \textstyle\bigwedge_{-z} W_i^\vee \otimes [\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}].$$

Therefore the Drinfeld polynomials are

$$P_i(u) = \textstyle\bigwedge_u W_i,$$

where the polynomials take value in $R(G_{\mathbf{w}} \times \mathbb{C}^*)$.

16.2. Character formulas in terms of interesection cohomology groups

We now use the machinery to analyze the convolution algebra due to Ginzburg [**Chriss-Ginzburg**].

We study irreducible representations of the specialized quantum loop algebra $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ introduced at the end of §15.1, which comes from representations of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ via Φ in Theorem 14.34.

Note that Φ is not injective nor surjective in general. Therefore there is *a priori* no reason why irreducible representations of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ remain irreducible as $\widetilde{\mathbf{U}}_q(\mathbf{L}\mathfrak{g})$-representations. Fortunately, the highest weight theory helps to remedy this failure. For example, an ℓ-highest weight representation M is irreducible if and only if it does not contain a vector m, except the ℓ-highest weight vector, with the property $e_{i,r} * m = 0$ for all i, r. Thus we only need to check this property in the convolution algebra side.

16.2.1. Semisimple elements in $G_{\mathbf{w}} \times \mathbb{C}^*$ and Drinfeld polynomials. The central subalgebra $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ acts on an irreducible representation by scalars. Therefore we have homomorphism $R(G_{\mathbf{w}} \times \mathbb{C}^*) \to \mathbb{C}$, which is given by an evaluation at a semisimple element $\tilde{s} = (s, \varepsilon)$ be a semisimple element in $G_{\mathbf{w}} \times \mathbb{C}^*$. Let $\chi_{\tilde{s}} \colon R(G_{\mathbf{w}} \times \mathbb{C}^*) \to \mathbb{C}$ denote the corresponding homomorphism. Considering $\mathbb{C}$ as an $R(G_{\mathbf{w}} \times \mathbb{C}^*)$-module by this evaluation homomorphism, we denote it by $\mathbb{C}_{\tilde{s}}$. We are lead to analyze the specialized convolution algebra

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w})) \otimes_{R(G_{\mathbf{w}}\times\mathbb{C}^*)} \mathbb{C}_{\tilde{s}},$$

and its module

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) \otimes_{R(G_{\mathbf{w}}\times\mathbb{C}^*)} \mathbb{C}_{\tilde{s}}.$$

This specialization of the universal standard module is called the *standard module*.

By the discussion in §16.1.2, the class $[\mathcal{O}_{\mathfrak{L}(0,\mathbf{w})}]$ in the standard module is the ℓ-highest weight vector and its Drinfeld polynomials are

$$P_i(u) = \det(1_{W_i} - u s_i),$$

where s_i is the $\mathrm{GL}(W_i)$-component of s. Note that the conjugacy class of the semisimple element s is parametrized by $P_i(u)$. Therefore the Drinfeld polynomial corresponds bijectively to the homomorphism $\chi_{\tilde{s}} \colon R(G_{\mathbf{w}} \times \mathbb{C}^*) \to \mathbb{C}$.

16.2.2. Graded/cyclic quiver varieties. Let A be the Zariski closure of $\{\tilde{s}^n \mid n \in \mathbb{Z}\}$. Let $\mathfrak{M}(\mathbf{w})^A$, $\mathfrak{M}_0(\mathbf{w})^A$ be the A-fixed point subvarieties of $\mathfrak{M}(\mathbf{w})$, $\mathfrak{M}_0(\mathbf{w})$ respectively. This is caled a *graded* or *cyclic* quiver variety, according to either ε *is* or *is not* a root of unity. We also have $Z(\mathbf{w})^A = \mathfrak{M}(\mathbf{w})^A \times_{\mathfrak{M}_0(\mathbf{w})^A} \mathfrak{M}(\mathbf{w})^A$, the A-fixed point subvariety in the analog of the Steinberg variety.

By the concentration theorem in the localized equivariant K-theory, and the bi-invariant theory, suitably corrected so that they commute with convolution products, we have a chain of algebra homomorphisms

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w})) \otimes_{R(G_{\mathbf{w}}\times\mathbb{C}^*)} \mathbb{C}_{\tilde{s}} \to K(Z(\mathbf{w})^A) \otimes_{\mathbb{Z}} \mathbb{C} \to H_*(Z(\mathbf{w})^A, \mathbb{C}).$$

See [**Chriss-Ginzburg**, §5 ?] for the detail.

As in §14.1.8, to a fixed point $[B, a, b] \in \mathfrak{M}(\mathbf{v}, \mathbf{w})^A$ and its representative (B, a, b), we can assign a homomorphism $\rho\colon A \to G_{\mathbf{v}}$ such that

$$\tilde{s} \star (B, a, b) = \rho(\tilde{s})^{-1} \cdot (B, a, b). \tag{16.7}$$

We denote by $\mathfrak{M}^\bullet(\rho)$ the set consisting of fixed points $[B, a, b]$ such that (16.7) holds for some representative (B, a, b). It depends only on the conjugacy class of ρ, and conversely the conjugacy class of ρ is determined by the fixed point $[B, a, b]$. We thus have $\mathfrak{M}(\mathbf{w})^A = \bigsqcup_{\langle\rho\rangle} \mathfrak{M}^\bullet(\rho)$, where $\langle\rho\rangle$ denote the conjugacy class.

Let us give more concrete description of $\mathfrak{M}^\bullet(\rho)$. Since $\tilde{s}$ acts on V via ρ and we consider the eigenspace with eigenvalue $\lambda \in \mathbb{C}^*$:

$$V(\lambda) \overset{\text{def.}}{=} \{v \in V \mid \rho(\tilde{s}) \cdot v = \lambda v\},$$

which decomposes as $V(\lambda) = \bigoplus V_i(\lambda)$. We have the eigenspace decomposition $V = \bigoplus V(\lambda)$. We also have the decomposition $W = \bigoplus W(\lambda)$ as the eigenspace decomposition with respect to s. Then (16.7) means

$$B_h(V_{\mathrm{o}(h)}(\lambda)) \subset V(\varepsilon^{-1}\lambda), \quad a_i(W_i(\lambda)) \subset V_i(\varepsilon^{-1}\lambda), \quad b_i(V_i(\lambda)) \subset W_i(\varepsilon^{-1}\lambda).$$

Data B, a, b and their composite never map $V_i(\lambda)$ to $V_j(\mu)$ unless λ/μ is a power of ε. Thus we have the decomposition

$$V = \bigoplus_n V(\varepsilon^n\lambda) \oplus \bigoplus_n V(\varepsilon^n\lambda') \oplus \cdots, \qquad W = \bigoplus_n W(\varepsilon^n\lambda) \oplus \bigoplus_n W(\varepsilon^n\lambda') \oplus \cdots$$

with $\lambda \neq \varepsilon^n\lambda'$ for any n, etc, preserved by B, a, b. We have the corresponding factorization $\mathfrak{M}^\bullet(\rho) = \mathfrak{M}^\bullet(\rho_\lambda) \times \mathfrak{M}^\bullet(\rho_{\lambda'}) \times \cdots$ where ρ_λ corresponds to $\bigoplus_n V(\varepsilon^n\lambda)$ and $\bigoplus_n W(\varepsilon^n\lambda)$ and so on. Therefore it is enough to study all eigenvalues are related by powers of ε. Moreover, the variety itself is unchanged even if replace λ by any other scalar, especially by 1. Therefore we may assume $V = \bigoplus V(\varepsilon^n)$, $W = \bigoplus W(\varepsilon^n)$. According to either ε is a root of unity or not, V, W are graded by a finite cyclic group or $\mathbb{Z}$. This is the reason why we call the fixed point set as cyclic/graded quiver varieties.

From now we change our notation slightly. We take V, W as $I \times \mathbb{C}^*$-graded vector spaces

$$V = \bigoplus_{i\in I, \lambda\in\mathbb{C}^*} V_i(\lambda), \qquad W = \bigoplus_{i\in I, \lambda\in\mathbb{C}^*} W_i(\lambda),$$

and define their dimension vectors as

$$\dim V = (\dim V_i(\lambda)), \dim W = (\dim W_i(\lambda)) \in \mathbb{Z}_{\geq 0}^{I\times\mathbb{C}^*}.$$

We denote them by $\mathbf{v}$, $\mathbf{w}$ as before. We denote the corresponding variety $\mathfrak{M}^\bullet(\mathbf{v}, \mathbf{w})$.

16.2.3. Properties of graded/cyclic quiver varieties.

————- To be written. ————-

We decompose the complex $\mathscr{C}_i$ in (14.22) according to the eigenvalue λ:

$$\mathscr{C}^\bullet_{i,\lambda} : V_i(\lambda q) \xrightarrow{\sigma_{i,\lambda}} \bigoplus_{h:\mathrm{i}(h)=i} V_{\mathrm{o}(h)}(\lambda) \oplus W_i(\lambda) \xrightarrow{\tau_{i,a}} V_i(\lambda q^{-1}), \tag{16.8}$$

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