North Carolina State University
Algebra Seminar

COMBINATORIAL IDENTITIES RELATED TO REPRESENTATIONS OF $U_q(\widetilde{\mathfrak{gl}_2})$

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Some time ago N. Jing discovered the following combinatorial identity

$$\sum_{k=0}^{\ell} \prod_{s=0}^{k-1} \frac{\eta^\ell - \eta^s}{1-\eta^{s+1}} \sum_{\sigma\in \mathbf{S}_\ell} \Bigl(\prod_{1\leqslant a\leqslant k} (t_{\sigma_a}-1) \prod_{k<b\leqslant \ell} (t_{\sigma_b}-\eta^{\ell-1}) \prod_{1\leqslant a<b\leqslant \ell} \frac{t_{\sigma_a}-\eta t_{\sigma_b}}{t_{\sigma_a}-t_{\sigma_b}} \Bigr) = 0 .$$

from the validity of the Serre relations in some vertex representations of quantum Kac-Moody algebras. I will give elementary enough proof of the identity and describe its generalizations. For example, the following identity holds

$$\sum_{k=0}^{\ell} (1-\eta^{2k}\beta) \prod_{s=0}^{k-1} \frac{(\eta^\ell-\eta^s)(1-\eta^s\beta)}{(1-\eta^{s+1})(1-\eta^{s+\ell+1}\beta)} \times$$
$$\times \sum_{\sigma\in \mathbf{S}_\ell} \Bigl(\prod_{1\leqslant a\leqslant k} (t_{\sigma_a}-1)(t_{\sigma_a}-\eta^{2a+\ell-1}\beta) \prod_{k<b\leqslant \ell} (t_{\sigma_b}-\eta^{\ell-1})(t_{\sigma_b}-\eta^{2b-1}\beta) \times$$
$$\times \prod_{1\leqslant a<b\leqslant \ell} \frac{t_{\sigma_a}-\eta t_{\sigma_b}}{t_{\sigma_a}-t_{\sigma_b}} \Bigr) = 0 .$$

The former identity can be recovered from the latter one either in the limit $\beta\to 0$ or in the limit $\beta\to\infty$. Both identities can be extended to polynomials of higher degree, and the latter identity can be also extended from polynomials to elliptic functions. The identities have natural interpretation in terms of tensor products of evaluation representations of the quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_2})$ or the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$.