TEST THREE, MA 305, DR. JING'S SECTION
NOVEMBER 9, 2006. 3:00–4:15

Print Your Name: **Signature**

1. (18 pts) Judge if the following statements are true (T) or false (F).

(a) For any matrix $A \in \mathbb{R}^{m\times n}$ one has $rank(A) = rank(A^T A)$.

(b) If a square matrix A has orthogonal column vectors, then A is called an orthogonal matrix.

(c) The linear system $A^T Ax = A^T b$ always has a solution.

(d) If the matrix P is the transition matrix from an ordered basis B to another ordered basis C, then P^T is the transition matrix from C to B.

(e) A 2×2 matrix always has two distinct eigenvalues.

(f) If the linear transformation T has the standard matrix $A \in \mathbb{R}^{m\times n}$, then $T(2x) = 2Ax$ for $x \in \mathbb{R}^n$.

2. (20 pts) Let T be the linear transformation of $\mathbb{R}^2$ that reflects about the line $y = x$ and then rotates 30 degress counterclockwise. Find the matrix representation of T with repect to the ordered basis $\{(1,3)^T, (2, 1^T)\}$.

3. (20 pts) Find a complete set of linearly independent eigenvectors for the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. (20 pts) Let $T(x) = (2x_1 - x_2, x_1 + x_3, x_2 - 5x_3)$ be a linear transformation on $\mathbb{R}^3$. Let $B = \{(1,0,2),(1,1,2),(3,2,1)\}$ $C = \{(1,1,1),(1,1,0),(1,0,0)\}$ be two ordered bases in $\mathbb{R}^3$.

(a) Find the transition matrix from C to B.

(b) Find the matrix representation $[T]_B$.

(c) Find the matrix representation $[T]_C$

(d) What is the matrix equation relating the two matrices $[T]_B$ and $[T]_C$?

5. (22 pts) Let A be the following matrix.

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix}$$

(a) Find an orthonormal basis for the column space CS(A)

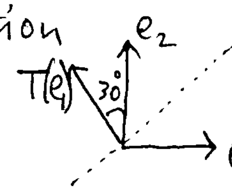
(b) Find a QR decomposition for the matrix A.

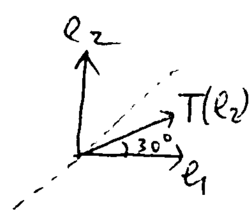
(c) Consider the linear system $Ax = (1, 2, -1, 1)^T$. Use your results in (b) to find the least squares solution.

Solution to Test 3, MA305, Fall 2006

1. (a) T (b) F (c) T (d) F (e) F (f) T

2. We first find the ~~transition~~ standard matrix representation





$$T(e_1) = -\tfrac{1}{2}e_1 + \tfrac{\sqrt{3}}{2}e_2$$
$$T(e_2) = \tfrac{\sqrt{3}}{2}e_1 + \tfrac{1}{2}e_2$$

i.e. $$T(x) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} x$$

Now we compute $$T(1,3) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-1+3\sqrt{3}}{2} \\ \frac{\sqrt{3}+3}{2} \end{pmatrix}$$

$$T(2,1) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-2+\sqrt{3}}{2} \\ \frac{2\sqrt{3}+1}{2} \end{pmatrix}$$

The matrix $[T]_{\{(1,3),(2,1)\}}$ is given by

$$\left(\begin{array}{cc|cc} 1 & 2 & \frac{-1+3\sqrt{3}}{2} & \frac{-2+\sqrt{3}}{2} \\ 3 & 1 & \frac{\sqrt{3}+3}{2} & \frac{2\sqrt{3}+1}{2} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & \frac{-1+3\sqrt{3}}{2} & \frac{-2+\sqrt{3}}{2} \\ 0 & -5 & \frac{6-8\sqrt{3}}{2} & \frac{7-\sqrt{3}}{2} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & \frac{-1+3\sqrt{3}}{2} & \frac{-2+\sqrt{3}}{2} \\ 0 & 1 & \frac{-6+8\sqrt{3}}{10} & \frac{7-\sqrt{3}}{10} \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & \frac{7-\sqrt{3}}{10} & \frac{-24+7\sqrt{3}}{10} \\ 0 & 1 & \frac{-3+4\sqrt{3}}{5} & \frac{7-\sqrt{3}}{10} \end{array}\right) \quad \text{so } [T] = \begin{pmatrix} \frac{7-\sqrt{3}}{10} & \frac{-24+7\sqrt{3}}{10} \\ \frac{-3+4\sqrt{3}}{5} & \frac{7-\sqrt{3}}{10} \end{pmatrix}.$$

3. Sol'n ~~Use Gram Schmidt process~~ (a) Characteristic poly

$$p(\lambda) = |\lambda I - A| = (\lambda-2)^2(\lambda-1)$$

(b) $\lambda = 2$: $$A - 2I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(c) $\lambda = 1$: $$A - I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

4. (a) Transition matrix from C to B

$$\begin{array}{c}\quad B \qquad\qquad C\\ \left(\begin{array}{ccc|ccc}1&1&3&1&1&1\\0&1&2&1&1&0\\2&2&1&1&0&0\end{array}\right)\end{array} \xrightarrow{R_3-2R_1} \begin{pmatrix}1&1&3&1&1&1\\0&1&2&1&1&0\\0&0&-5&-1&-2&-2\end{pmatrix} \xrightarrow{R_1-R_2} \begin{pmatrix}1&0&1&0&0&1\\0&1&2&1&1&0\\0&0&-5&-1&-2&-2\end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5}R_3} \begin{pmatrix}1&0&1&0&0&1\\0&1&2&1&1&0\\0&0&1&\frac{1}{5}&\frac{2}{5}&\frac{2}{5}\end{pmatrix} \xrightarrow[R_1-R_3]{R_2-2R_3} \begin{pmatrix}1&0&0&-\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\\0&1&0&\frac{3}{5}&\frac{1}{5}&-\frac{4}{5}\\0&0&1&\frac{1}{5}&\frac{2}{5}&\frac{2}{5}\end{pmatrix} \quad \therefore P=\begin{pmatrix}-\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\\\frac{3}{5}&\frac{1}{5}&-\frac{4}{5}\\\frac{1}{5}&\frac{2}{5}&\frac{2}{5}\end{pmatrix}$$

(b)
$$\left(\begin{array}{ccc|ccc}1&1&3&2&1&4\\0&1&2&3&3&4\\2&2&1&-10&-9&-3\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&0&0&-\frac{19}{5}&-\frac{21}{5}&-\frac{11}{5}\\0&1&0&-\frac{13}{5}&-\frac{7}{5}&-\frac{2}{5}\\0&0&1&\frac{14}{5}&\frac{11}{5}&\frac{11}{5}\end{array}\right) \quad [T]_B=\begin{pmatrix}-\frac{19}{5}&-\frac{21}{5}&-\frac{11}{5}\\-\frac{13}{5}&-\frac{7}{5}&-\frac{2}{5}\\\frac{14}{5}&\frac{11}{5}&\frac{11}{5}\end{pmatrix}$$

(c)
$$\left(\begin{array}{ccc|ccc}1&1&1&1&1&2\\1&1&0&2&1&1\\1&0&0&-4&1&0\end{array}\right) \to \left(\begin{array}{ccc|ccc}0&1&1&5&0&2\\0&1&0&6&0&1\\1&0&0&-4&1&0\end{array}\right) \to \left(\begin{array}{ccc|ccc}0&0&1&-1&0&1\\0&1&0&6&0&1\\1&0&0&-4&1&0\end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc}1&0&0&-4&1&0\\0&1&0&6&0&1\\0&0&0&-1&0&1\end{array}\right) \qquad \therefore [T]_C=\begin{pmatrix}-4&1&0\\6&0&1\\-1&0&1\end{pmatrix}$$

(d) $[T]_B = P^{\cancel{-1}}[T]_C P^{-1}$ i.e. $\begin{pmatrix}-\frac{19}{5}&-\frac{21}{5}&-\frac{11}{5}\\-\frac{13}{5}&-\frac{7}{5}&-\frac{2}{5}\\\frac{14}{5}&\frac{11}{5}&\frac{11}{5}\end{pmatrix}=\begin{pmatrix}-\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\\\frac{3}{5}&\frac{1}{5}&-\frac{4}{5}\\\frac{1}{5}&\frac{2}{5}&\frac{2}{5}\end{pmatrix}\begin{pmatrix}-4&1&0\\6&0&1\\-1&0&1\end{pmatrix}\begin{pmatrix}-\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\\\frac{3}{5}&\frac{1}{5}&-\frac{4}{5}\\\frac{1}{5}&\frac{2}{5}&\frac{2}{5}\end{pmatrix}^{-1}$

or $[T]_C = P^{-1}[T]_B P$

5. (a) $w_1=(1,-1,1,-1)$

$$w_2=(1,2,1,2)-\frac{(1,2,1,2)\cdot(1,-1,1,-1)}{4}(1,-1,1,-1)=(1,2,1,2)+\frac{1}{2}(1,-1,1,-1)=\left(\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)$$

$$w_3=(0,3,1,2)-\frac{(0,3,1,2)\cdot(1,-1,1,-1)}{4}(1,-1,1,-1)-\frac{(0,3,1,2)\cdot(1,1,1,1)}{4}(1,1,1,1)$$

$$=(0,3,1,2)+(1,-1,1,-1)-\frac{3}{2}(1,1,1,1)=\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$$

So $u_1 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$
$u_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is an orthonormal basis for $CS(A)$.
$u_3 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$

(b) Let $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix} = (c_1, c_2, c_3) = QR$

then $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$, $R = \begin{pmatrix} \|w_1\| & c_2 q_1 & c_3 q_1 \\ 0 & \|w_2\| & c_3 q_2 \\ 0 & 0 & \|w_3\| \end{pmatrix} = \begin{pmatrix} 2 & -1 & -2 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

where $\|w_1\| = 2$, $\|w_2\| = 3$, $\|w_3\| = 1$

$c_2 \cdot q_1 = (1,2,1,2)^T(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}) = -1$, $c_3 \cdot q_1 = (0,3,1,2)^T(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) = -2$

$c_3 \cdot q_2 = (0,3,1,2)^T(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 3$

(c) The normal equation $A^TAx = A^Tb$ becomes $R^TRx = (-3, 6, 7)^T$

We solve two (triangular) systems:

$R^Ty = \begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix}$: $\begin{cases} 2y_1 = -3 \\ -y_1 + 3y_2 = 6 \\ -2y_1 + 3y_2 + y_3 = 7 \end{cases}$ $\therefore \begin{cases} y_1 = -\frac{3}{2} \\ y_2 = 2 + \frac{y_1}{3} = \frac{3}{2} \\ y_3 = 7 + 2y_1 - 3y_2 = 7 - 3 - \frac{9}{2} = -\frac{1}{2} \end{cases}$

$Rx = y$ $\begin{cases} 2x_1 - x_2 - 2x_3 = -\frac{3}{2} \\ 3x_2 + 3x_3 = \frac{3}{2} \\ x_3 = -\frac{1}{2} \end{cases}$ $\therefore \begin{cases} x_3 = -\frac{1}{2} \\ x_2 = \frac{1}{2} - x_3 = 1 \\ x_1 = -\frac{3}{4} + \frac{x_2}{2} + x_3 = -\frac{3}{4} + \frac{1}{2} - \frac{1}{2} = -\frac{3}{4} \end{cases}$

Answer: $x_1 = -\frac{3}{4}$, $x_2 = 1$, $x_3 = -\frac{1}{2}$