Numerically deciding the arithmetically Cohen-Macaulayness of a projective scheme

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Abstract

In numerical algebraic geometry, a witness point set $W$ is a key object for performing numerical computations on a projective scheme $X$ of pure dimension $d > 0$ defined over $\mathbb{C}$. If $X$ is arithmetically Cohen-Macaulay, $W$ can also be used to obtain information about $X$, such as the initial degree of the ideal generated by $X$ and its Castelnuovo-Mumford regularity. Due to this relationship, we develop a new numerical algebraic geometric test for deciding if $X$ is arithmetically Cohen-Macaulay using points which (approximately) lie on a general curve section $C$ of $X$. For any curve, we also compute other information such as the arithmetic genus and index of regularity. Several examples are presented showing the effectiveness of this method, even when defining equations for $X$ are unknown.

Key words and phrases. Numerical algebraic geometry, witness set, arithmetically Cohen-Macaulay, Castelnuovo-Mumford regularity, arithmetic genus

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1 Introduction

Let $\mathbb{P}^n$ denote the projective space of dimension $n$ over $\mathbb{C}$ and let $X \subset \mathbb{P}^n$ be a pure-dimensional projective scheme of dimension $d > 0$. A fundamental goal in computational algebraic geometry is to compute information about $X$, especially when defining equations are not known. If $X$ is arithmetically Cohen-Macaulay (aCM), information about $X$, such as the initial degree of the ideal generated by $X$, Castelnuovo-Mumford regularity, Hilbert function, Hilbert polynomial, and Hilbert series, can be recovered from general hyperplane (and hypersurface) sections of $X$. Therefore, a procedure for deciding the arithmetically Cohen-Macaulayness of a scheme is a key problem in computational algebraic geometry. Since defining equations for $X$ may not be known, e.g., $X$ can be a pure-dimensional component of some other scheme $Y$ or the image of an algebraic set under an algebraic map, we propose a test for deciding if $X$ is aCM given the ability to sample points lying (approximately) on a general curve section of $X$. For any curve (a pure-dimensional projective scheme of dimension 1), invariants such as the arithmetic genus, Castelnuovo-Mumford regularity, and index of regularity are computed.

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One key fact of schemes of dimension at least 2 is arithmetically Cohen-Macaulayness is preserved under slicing by a general hyperplane (or hypersurface). In particular, a pure-dimensional scheme $X$ of positive dimension is aCM if and only if a general curve section of $X$ is aCM. A numerical test is provided in [15] that determines if a curve $C$ is aCM. This test relies upon computing Hilbert functions of zero schemes defined by intersecting $C$ with general hypersurfaces of various degrees. Due to the increasingly higher degree zero schemes under consideration, this test becomes impractical for curves of even moderate degree. Section 5.3 presents an example that compares the approach of [15] with our approach.

Two important topics related to symbolic computations in algebraic geometry are minimal free resolutions and complexity of Gröbner basis computations. The arithmetically Cohen-Macaulayness of a scheme is related to the length a minimal free resolution via the relationship between codimension and depth in the Auslander-Buchsbaum formula (see [6] for a general overview). Over fields of characteristic zero, e.g., $\mathbb{Q}$ and $\mathbb{C}$, the Castelnuovo-Mumford regularity of the ideal $I$ defining a scheme $X$ is equal to the maximum degree of the elements in a Gröbner basis when working with generic coordinates in the reverse lexicographic ordering (see [4] for more information). Thus, the Castelnuovo-Mumford regularity provides a measure of complexity for performing symbolic computations on $I$.

The arithmetic and geometric genus are two invariants of a curve $C$ of particular interest in computational algebraic geometry. These genera must be equal if $C$ is smooth. A numerical algebraic geometric procedure for computing the geometric genus is presented in [3] which was extended in [17] to curves which arise as the image of an algebraic set under a polynomial map. The geometric genus of a general four-bar coupler curve was verified to be one in [3] with the arithmetic genus of such a curve computed in Section 5.1.

Even though it is not directly related to deciding the arithmetically Cohen-Macaulayness of a projective scheme, we note that a symbolic-numeric approach for computing Hilbert functions and Hilbert polynomials in local rings is described in [20]. This approach is based on computing the Macaulay dual space of an ideal at a point that (approximately) lies in the solution set of the ideal. There are no assumptions related to the point, e.g., multiple components could pass through the point including embedded components. The practicality of this approach, especially for high dimensional components, is limited by the stopping criterion which requires that the Macaulay dual space is computed in degree up to twice the maximum degree of a “g-corner.”

The rest of this article is organized as follows. Section 2 provides the necessary background information to describe the numerical algebraic geometry methods used throughout. Section 3 develops an algorithm for deciding the arithmetically Cohen-Macaulayness of a curve with Section 4 considering the general case. Several examples are presented in Section 5.

2 Background

2.1 Arithmetically Cohen-Macaulay

A positive dimensional projective scheme $X$ with ideal sheaf $\mathcal{I}_X$ is said to be arithmetically Cohen-Macaulay (aCM) if

$$H^i_*(\mathcal{I}_X) = 0 \quad \text{for} \quad 1 \leq i \leq \dim X$$

(1)

where $H^i_*(\mathcal{I}_X)$ denotes the $i$th cohomology module of $\mathcal{I}_X$. Equivalently, a projective scheme $X$ is aCM if and only if its coordinate ring has Krull dimension equal to its depth [21] (in this case, the coordinate ring is called a Cohen-Macaulay ring). All zero-dimensional schemes are aCM and a consequence of the above definition is that all aCM schemes must be pure-dimensional.
Example 2.1 Consider the curves in \( \mathbb{P}^3 \):
\[
C = \{(s^3, s^2t, st^2, t^3) \mid (s, t) \in \mathbb{P}^1\} \quad \text{and} \quad Q = \{(s^4, s^3t, st^3, t^4) \mid (s, t) \in \mathbb{P}^1\}
\]
with corresponding ideals
\[
I(C) = (xz - y^2, yw - z^2, xw - yz) \quad \text{and} \quad I(Q) = \langle xw - yz, x^2 z - y^3, xz^2 - y^2 w, z^3 - yw^2 \rangle.
\]
The curve \( C \) is the twisted cubic curve while \( Q \) is a smooth rational quartic curve. The twisted cubic curve \( C \) is well-known to be aCM while [8, Ex. 1.7] shows that \( Q \) is not. We verify this statement using the definition by comparing the Krull dimension and depth. Clearly, both coordinates rings have Krull dimension 2. Computations using the \texttt{Depth} package of \texttt{Macaulay2} [10] find that the depth of \( C \) is 2 and the depth of \( Q \) is 1.

The cohomology characterization presented in [1] imposes conditions on the Hilbert function, which is defined next. For a curve, Corollary 3.2 presents an effective test of arithmetically Cohen-Macaulayness that can be performed using numerical algebraic geometric computations.

2.2 Hilbert functions, genus, and regularity

Let \( X \subset \mathbb{P}^n \) be a projective scheme with corresponding homogeneous ideal \( I \subset \mathbb{C}[x_0, \ldots, x_n] \).
Let \( \mathbb{C}[x_0, \ldots, x_n]_t \) denote the vector space of homogeneous polynomials of degree \( t \), which has dimension \( \binom{n+t}{t} \), and \( I_t = I \cap \mathbb{C}[x_0, \ldots, x_n]_t \). The Hilbert function of \( X \) is defined as
\[
HF_X(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\binom{n+t}{t} - \dim I_t & \text{otherwise.}
\end{cases}
\]

The initial degree of \( X \) is the smallest \( t \) such that \( \dim I_t > 0 \). If \( X = \mathbb{P}^n \), that is, \( I = (0) \), then the initial degree is defined as \( -\infty \). If \( X = 0 \), that is, \( I = (1) \), then the initial degree is 0.

For all other schemes \( X \subset \mathbb{P}^n \), the initial degree is a positive integer.

Since \( HF_X(t) = 0 \) for \( t < 0 \), we will express \( HF_X \) via the list \( HF_X(0), HF_X(1), HF_X(2), \ldots \).

The generating function of \( HF_X \) is called the Hilbert series of \( X \), namely
\[
HS_X(t) = \sum_{j=0}^{\infty} HF_X(j) \cdot t^j.
\]

The Hilbert function of \( X \) becomes polynomial in \( t \) for \( t \gg 0 \). That is, there exists a polynomial \( HP_X \), called the Hilbert polynomial of \( X \), such that \( HF_X(t) = HP_X(t) \) for all \( t \gg 0 \). The Hilbert polynomial has rational coefficients with highest degree term \( \deg X / (\dim X)! \cdot t^{\dim X} \).

When \( X \) is a curve, the Hilbert polynomial of \( X \) has the form
\[
HP_X(t) = \deg X \cdot t + (1 - g_X)
\]
where \( g_X \) is the arithmetic genus of \( X \).

Example 2.2 Consider the quartic curve \( Q \subset \mathbb{P}^3 \) from Ex. 2.1. From the generators of \( I(Q) \), it is easy to compute, e.g., via \texttt{Macaulay2} [10], that
\[
HF_Q = 1, 4, 9, 13, 17, 21, 25, \quad HS_Q(t) = (1 + 2t + 2t^2 - t^3)/(1 - t)^2, \quad \text{and} \quad HP_Q(t) = 4t + 1.
\]
From \( HF_Q \), the initial degree of \( Q \) is 2. From \( HP_Q \) and (3), the arithmetic genus of \( Q \) is \( g_Q = 0 \).
We will discuss two types of regularity for $X$. The index of regularity of $X$ is the smallest integer $\rho_X$ such that $HF_X(t) = HP_X(t)$ for all $t \geq \rho_X$. Let $I_X$ be the sheafification of the ideal $I$ corresponding to $X$. The Castelnuovo-Mumford regularity of $X$ is

$$\operatorname{reg} X = \min\{m \mid H^i(I_X(m-i)) = 0 \text{ for all } i > 0\}.$$ 

If $X$ is aCM, $\rho_X$, $\operatorname{reg} X$, and $\dim X$ are related as follows.

**Proposition 2.3** Suppose that $X \subset \mathbb{P}^n$ is an aCM scheme.

1. $\operatorname{reg} X = \rho_X + \dim X + 1$.

2. Let $\mathcal{L} \subset \mathbb{P}^n$ be a general linear space with $\operatorname{codim} \mathcal{L} \leq \dim X$ and $Z = X \cap \mathcal{L}$. Then, $\operatorname{reg} Z = \operatorname{reg} X$ and $\rho_Z = \rho_X + \operatorname{codim} \mathcal{L}$.

**Proof.** See, for example, [5, Remark 2.5a] for Item 1. Item 2 follows immediately by combining [21, pg. 30] and Item 1. □

For aCM schemes, this proposition shows that the index of regularity increases under intersection with a general hyperplane. Thus, the index of regularity can be negative so that the Hilbert polynomial has roots at negative integers. Section 5.2 presents an example of this.

The following will be used in Section 3 for computing $\operatorname{reg} C$ where $C \subset \mathbb{P}^n$ is a curve, that is, a union of irreducible one-dimensional projective schemes.

**Proposition 2.4** Let $C \subset \mathbb{P}^n$ be a curve and $\mathcal{H} \subset \mathbb{P}^n$ be a general hyperplane. If $W = C \cap \mathcal{H}$, $\operatorname{reg} C = \max\{\rho_C + 1, \operatorname{reg} W, 1\}$. (4)

**Proof.** By [5, Lemma 2.6], $\operatorname{reg} C = \max\{\rho_C + 1, \operatorname{reg} W\}$. Since $\dim W = 0$, $W$ is aCM yielding $\operatorname{reg} W = \rho_W + 1$ by Item 1 of Prop. 2.3. □

For example, using the notation of Prop. 2.4 if $C$ is also aCM, then $\rho_W = \rho_C + 1$ so that $\operatorname{reg} W = \operatorname{reg} C = \rho_W + 1 = \rho_C + 2$. (5)

**Example 2.5** From $HF_Q$ and $HP_Q$ presented in Ex. 2.2, we have $\rho_Q = 2$. If $W = Q \cap \mathcal{H}$ for a general hyperplane $\mathcal{H} \subset \mathbb{P}^3$, one can use [11] to compute $HF_W = 1, 3, 4, 4, \ldots$ and $\rho_W = 2$. Hence, (4) yields $\operatorname{reg} Q = 3$ and, since (5) does not hold, this again shows $Q$ is not aCM.

### 2.3 Witness sets

Both Prop. 2.3 and 2.4 consider general linear sections of schemes. This is amenable to numerical algebraic geometry where the fundamental data structure of an irreducible algebraic set, a witness set, is based on linear sections of complimentary dimension. When defining equations are not known, computations involving witness sets are performed using pseudowitness sets [16, 17].

Let $f$ be a system of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ and $V \subset \mathbb{P}^n$ be an algebraic set of dimension $d$ such that $V$ is an irreducible component of $V(f) = \{x \in \mathbb{P}^n \mid f(x) = 0\}$. Then, a witness set for $V$ is a triple $\{f, \mathcal{L}, W\}$ where $\mathcal{L} \subset \mathbb{P}^n$ is a general linear space of codimension $d$ and $W = V \cap \mathcal{L} \subset \mathbb{P}^n$, a witness point set consisting of $\deg V$ points.
Example 2.6 Let $C \subset \mathbb{P}^3$ be the twisted cubic curve, as described in Ex. [2.1] and

$$f(x, y, z, w) = \begin{bmatrix} xz - y^2 \\ yw - x^2 \end{bmatrix}.$$ 

Clearly, $\mathcal{V}(f) = C \cup \mathcal{V}(x, y)$. So, a witness set for $C$ is $\{f, \mathcal{L}, W\}$ where $\mathcal{L} = \mathcal{V}(x+ y + z + 2w)$ and $W = \{(8, -4, 2, -1), (2, 1 + \sqrt{-3}, -1 + \sqrt{-3}, -2), (2, 1 - \sqrt{-3}, -1 - \sqrt{-3}, -2)\} \subset \mathbb{P}^3$.

We note that $\mathcal{L}$ is defined via a linear polynomial with integer coefficients only for illustration.

Rather than consider $V$ as an algebraic subset of $\mathcal{V}(f) \subset \mathbb{P}^n$, one can consider $V$ as a scheme contained in $f^{-1}(0)$. If the multiplicity of $V$ with respect to $f$ is greater than 1, one can use isosingular deflation \cite{13} to replace $f$ with another system $f'$ of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ such that $V$ has multiplicity 1 with respect to $f'$.

One key operation in numerical algebraic geometry, called sampling, is the ability to produce a collection of arbitrarily close numerical approximations of arbitrarily many general points of $V$ given a numerical approximation of one sufficiently general point of $V$. In particular, if $x^* \in V$ is sufficiently general, let $\mathcal{L}, \mathcal{L}^* \subset \mathbb{P}^n$ be general linear spaces of codimension $d$ such that $x^* \in \mathcal{L}^*$. This setup defines a path $x(t) : [0, 1) \to V$ where $x(1) = x^*$ and $x(t) \in V \cap (t \cdot \mathcal{L}^* + (1 - t) \cdot \mathcal{L})$ is a smooth point of $V$. By construction, $x(0)$ is a general point of $V$. See \cite{22} for more details.

### 2.4 Macaulay dual spaces

One way to computationally understand the local structure of a scheme is via Macaulay dual spaces. Since this will be exploited in Section \ref{2.5}, the following provides a brief overview with expanded details presented in the books \cite{19, 23}.

For $\alpha = (\alpha_0, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{Z}_{\geq 0}$, define

$$|\alpha| = \alpha_0 + \cdots + \alpha_n, \quad \alpha! = \alpha_0! \cdots \alpha_n!, \quad \text{and} \quad x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}.$$ 

For a fixed $x^* \in \mathbb{C}^{n+1}$, let $\partial_\alpha : \mathbb{C}[x_0, \ldots, x_n] \to \mathbb{C}$ be the operator defined by

$$f \mapsto \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \bigg|_{x=x^*} \quad \text{so that} \quad \partial_\alpha \left( \sum_{\beta \in (\mathbb{Z}_{\geq 0})^{n+1}} c_\beta (x - x^*)^\beta \right) = c_\alpha.$$ 

For convenience, we often write $\partial_\alpha$ as $\partial_{x^\alpha}$ which simply computes the coefficient of $(x - x^*)^\alpha$ in a Taylor series expansion centered at $x^*$. For $j \geq 0$, consider the linear space of all such differential operators spanned by elements of the form $\partial_\alpha$ with $|\alpha| \leq j$, namely

$$D_{x^*}^j = \left\{ \sum_{|\alpha| \leq j} c_\alpha \partial_\alpha \bigg| c_\alpha \in \mathbb{C} \right\}.$$ 

Then, for an ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$, the $j^{th}$ Macaulay dual space of $I$ at $x^*$ is

$$D_{x^*}^j[I] = \left\{ \partial \in D_{x^*}^j \bigg| \partial(g) = 0 \text{ for all } g \in I \right\}.$$ 

Numerical approaches for computing $D_{x^*}^j[I]$ using closedness subspaces are presented in \cite{13, 24}.
Example 2.7 For \( I = (x^2 - z^2, xz - z^2, xy - z^2) \subset \mathbb{C}[x, y, z] \) and \( x^* = 0 \in \mathbb{C}^3 \), one has

\[
\begin{align*}
D_0^0[I] &= \text{span}\{\partial_1\}, \\
D_1^0[I] &= \text{span} (D_0^0[I] \cup \{\partial_x, \partial_y, \partial_z\}), \\
D_2^0[I] &= \text{span} (D_1^0[I] \cup \{\partial_x + \partial_y + \partial_x z + \partial_y z, \partial y^2, \partial yz\}), \\
D_3^0[I] &= \text{span} (D_2^0[I] \cup \{\partial_x + \partial_x y + \partial_x z + \partial y^2 + \partial yz + \partial yz^2, \partial y^3, \partial yz + \partial yz^2\}).
\end{align*}
\]

The particular case of interest is when \( I \subset \mathbb{C}[x_0, \ldots, x_n] \) is a homogeneous ideal such that \( \hat{x} \in \mathbb{P}^n \) is an isolated point of \( \mathcal{V}(I) = \{ x \in \mathbb{P}^n | f(x) = 0 \text{ for all } f \in I \} \). By treating \( \hat{x} \in \mathbb{P}^n \) as a line through the origin in \( \mathbb{C}^{n+1} \), a random affine linear space \( A \subset \mathbb{C}^{n+1} \) will intersect this line in one point, say \( x^* \in \mathbb{C}^{n+1} \). If \( p \) is a linear polynomial such that \( A = \mathcal{V}(p) \), consider the ideal \( J = I + \langle p \rangle \subset \mathbb{C}[x_0, \ldots, x_n] \). Since \( x^* \) is an isolated point in \( \mathcal{V}(J) \subset \mathbb{C}^{n+1} \),

\[
D_{x^*}[J] = \bigcup_{j=0}^{\infty} D_{x^*}^j[J]
\]

is a vector space of dimension equal to the multiplicity of \( x^* \) with respect to \( J \). In particular, there must exist \( j^* \geq 0 \) such that \( D_{x^*}^{j^*}[J] = D_{x^*}^j[J] \) for all \( j \geq j^* \). The vector space \( D_{x^*}[J] = D_{x^*}^j[J] \) is called the Macaulay dual space of \( J \) at \( x^* \).

Example 2.8 Reconsider \( I \subset \mathbb{C}[x, y, z] \) from Ex. 2.7 with \( \hat{x} = (0, 1, 0) \in \mathbb{P}^2 \). By taking \( A = \mathcal{V}(p) \) where \( p(x, y, z) = 2x + 3y - 4z - 6 \), we have

\[
J = \langle x^2 - z^2, xz - z^2, xy - z^2, 2x + 3y - 4z - 6 \rangle \text{ and } x^* = (0, 2, 0) \in \mathbb{C}^3.
\]

One can easily compute

\[
\begin{align*}
D_0^{x^*}[J] &= \text{span}\{\partial_1\}, \\
D_1^{x^*}[J] &= \text{span} (D_0^{x^*}[J] \cup \{4\partial_y + 3\partial_z\}), \\
D_2^{x^*}[J] &= D_1^{x^*}[J]
\end{align*}
\]

so that \( D_{x^*}[J] = D_1^{x^*}[J] \). Thus, \( x^* \) and \( \hat{x} \) have multiplicity 2 with respect to \( J \) and \( I \), respectively.

2.5 Interpolation

The key computational step in our approach for deciding the arithmetically Cohen-Macaulayness of a curve is the computation of the Hilbert function of the curve and its general hyperplane section in particular degrees. For zero-dimensional schemes, such as the general hyperplane section of a curve, the Hilbert function and index of regularity can be computed using the approach of [11] after fixing a general affine patch. For curves (and higher dimensional schemes), the Hilbert function in a particular degree can be computed by considering a sufficiently large zero-dimensional subscheme. This computation is summarized in Algorithm 7 which uses the ability to sample points on each irreducible component of a curve and compute Macaulay dual spaces.

Theorem 2.9 Subject to genericity, Algorithm 7 is an algorithm that computes \( HF_C(t) \).

Proof. Clearly, \( 0 \leq h_1 \leq \binom{n+t}{t} \). Since either \( h_1 \) must increase by at least one or remain the same during each loop, Algorithm 7 terminates in at most \( \binom{n+t}{t} \) loops. Since \( I(C) \subset I(S) \), we always have \( h_1 = HF_S(t) \leq HF_C(t) \leq \binom{n+t}{t} \). Thus, if \( h_1 = \binom{n+t}{t} \), then \( HF_C(t) = \binom{n+t}{t} \).
Algorithm 1 Numerical computation of $HF_C(t)$

**Input:** A curve $C \subseteq \mathbb{P}^n$ with irreducible decomposition $C = C_1 \cup \cdots \cup C_r$ presented via a collection of $r$ witness sets $\{f_i, L_i, W_i\}$ and an integer $t \geq 1$.

**Output:** $HF_C(t)$.

1. Initialize $h_0 := -1$, $h_1 := 0$, and $S := \emptyset$.
2. while $h_0 \neq h_1$ and $h_1 < \binom{n+t}{t}$ do
   3. Set $h_0 := h_1$.
   4. for $i = 1, \ldots, r$ do
      5. Use sampling to compute a point $\hat{c} \in \mathbb{P}^n$ contained in $C_i \cap H$ where $H \subseteq \mathbb{P}^n$ is a random hyperplane. Let $\ell$ be a linear form so that $H = \mathbb{V}(\ell)$.
      6. Pick a random affine patch of $\mathbb{P}^n$ via a random affine linear polynomial $p$. Compute the point in $\mathbb{C}^{n+1}$ corresponding to $\hat{c}$ in $\mathbb{V}(p)$. Denote it by $c^* \in \mathbb{C}^{n+1}$.
      7. Compute a basis $B_{c^*}$ for $D_{c^*}(\{f_i, \ell, p\})$ and append $\{c^*, B_{c^*}\}$ to $S$.
      8. Compute $h_1 := HF_S(t) = \binom{n+t}{t} - \dim S$ via \[\] 9. return $HF_C(t) := h_1$.

Suppose that $r = 1$, that is, $C$ is irreducible and assume that we have reached where $h_0 = h_1$. If $f \in I(C)_t$, then clearly $f \in I(S)_t$. If $f \in I(S)_t$, then, since $d_1 \in D_c(\{f_i, \ell, p\})$, standard interpolation theory provides that $f \in \sqrt{I(C)}$. Since $I(C)$ is a saturated homogeneous primary ideal, if $f \notin I(C)$, then the multiplicity of $C$ must be smaller with respect to $I(C) + \langle f \rangle$ than with respect to $I(C)$. However, this is impossible since $h_0 = h_1$ ensures that the multiplicity has not decreased, i.e., $I(S)_t = I(C)_t$.

When $r > 1$, since the algorithm uses points on every irreducible component of $C$, the algorithm is simply performing the intersection $I(C_1)_t \cap \cdots \cap I(C_r)_t = I(C)_t$. \[\]

**Example 2.10** Reconsider $I = \langle x^2 - z^2, xz - z^2, xy - z^2 \rangle$ from Ex. 2.7 now as an ideal in $\mathbb{C}[x, y, z, w]$. Thus, $I$ defines a curve $C \subseteq \mathbb{P}^3$ having two irreducible components $C_1$ and $C_2$ with

$$I(C_1) = \langle x - z, y - z \rangle, I(C_2) = \langle x, z^2 \rangle,$$

and $I = I(C) = I(C_1) \cap I(C_2)$.

Since $C_1$ has multiplicity one, applying Algorithm 1 to $C_1$ is simply computing $HF_{C_1}(t)$ by performing standard interpolation at arbitrarily many points in $C_1$. For example, the corresponding values of $h_1$ after each loop for computing $HF_{C_1}(1)$ are 1, 2, 2 with $HF_{C_1}(1) = 2$.

Since $C_2$ has multiplicity two, each new point under consideration in Algorithm 1 applied to $C_2$ imposes two conditions. In this case, the corresponding values of $h_1$ after each loop for computing $HF_{C_2}(1)$ are 2, 3, 3 thereby computing $HF_{C_2}(1) = 3$.

Finally, each loop through Algorithm 1 applied to $C$ imposes 3 conditions, one from $C_1$ and two from $C_2$. The corresponding values of $h_1$ for $HF_C(2)$ are 3, 6, 7, 7 showing $HF_C(2) = 7$.

3 Computations for a curve

The following considers curves with Section 4 exploring higher-dimensional cases.

3.1 Testing arithmetic Cohen-Macaulayness of a curve

Let $C \subseteq \mathbb{P}^n$ be a curve, that is, $C$ is a union of one-dimensional irreducible schemes. The defining equations for $C$ may be unknown, but we assume that we have either a witness set or a pseudow-
For example, the values of \( t \)itness set for each irreducible component of \( C \), thereby providing the ability to sample points from each irreducible component of \( C \). We also need the ability to compute \( HF_C(t) \) for specified values of \( t \), which, for example using Algorithm 1 only additionally requires the ability to compute the corresponding Macaulay dual spaces for components of multiplicity greater than one.

One key operation on Hilbert functions is taking differences, e.g., the first difference of \( HF_C \) is

\[
\Delta HF_C(t) = HF_C(t) - HF_C(t - 1) \quad \text{for all } t \in \mathbb{Z}.
\]

By [2], we know \( \Delta HF_C(t) = 0 \) for \( t < 0 \) and \( \Delta HF_C(0) = 1 \). One can also iterate this process. For example, the \( k^{\text{th}} \) difference of \( HF_C \) is

\[
\Delta^k HF_C(t) = \Delta \circ \cdots \circ \Delta HF_C(t).
\]

The following tests the arithmetically Cohen-Macaulayness of a curve.

**Theorem 3.1** Let \( C \subset \mathbb{P}^n \) be a curve, \( \mathcal{H} \subset \mathbb{P}^n \) be a general hyperplane, and \( W = C \cap \mathcal{H} \). Then, \( C \) is aCM if and only if \( \Delta HF_C(t) = HF_W(t) \) for all \( t \geq 0 \).

**Proof.** Since \( C \) is a curve, \( C \) is aCM if and only if \( H^1(\mathcal{I}_C) = 0 \). By [21] Prop. 1.3.4, this is equivalent to \( J = I(C) + \langle \ell \rangle \) being a saturated ideal in \( \mathbb{C}[x_0, \ldots, x_n] \) where \( \mathcal{H} = \mathcal{V}(\ell) \). That is, \( C \) is aCM if and only if \( J = I(W) \). Since \( J \subset I(W) \), this is equivalent to \( HF_J(t) = HF_W(t) \) for all \( t \geq 0 \). The result now follows since \( HF_J(t) = \Delta HF_C(t) \) because \( C \) is a curve. \( \square \)

The following is a so-called effective version of Theorem 3.1.

**Corollary 3.2** Let \( C \subset \mathbb{P}^n \) be a curve, \( \mathcal{H} \subset \mathbb{P}^n \) be a general hyperplane, and \( W = C \cap \mathcal{H} \). Then, \( C \) is aCM if and only if \( \Delta HF_C(t) = HF_W(t) \) for all \( 1 \leq t \leq \rho_W + 1 \).

**Proof.** Clearly, \( \Delta HF_C(0) = HF_W(0) = 1 \) and \( HF_W(\rho_W) = HF_W(\rho_W + t) \) for all \( t \geq 0 \). Also,

\[
\operatorname{reg} C = \max\{\rho_C + 1, \rho_W + 1\} = \min\{t \geq \rho_W + 1 \mid \Delta HF_C(t) = HF_W(\rho_W)\} \quad (6)
\]

by Prop. 2.4 and [3] § 3. If \( \Delta HF_C(\rho_W + 1) = HF_W(\rho_W + 1) = HF_W(\rho_W) \), then \( \rho_C + 1 \leq \rho_W + 1 \) so that \( \rho_C \leq \rho_W \). Hence, \( HF_C(\rho_W + t) = HF_C(\rho_W + t) \) for all \( t \geq 0 \) which yields

\[
\Delta HF_C(\rho_W + t) = HF_W(\rho_W + t) = HF_W(\rho_W)
\]

for all \( t \geq 1 \).

In particular, we have shown that \( \Delta HF_C(t) = HF_W(t) \) for \( 1 \leq t \leq \rho_W + 1 \) is equivalent to \( \Delta HF_C(t) = HF_W(t) \) for all \( t \geq 0 \). Therefore, the statement holds by Theorem 3.1. \( \square \)

Section 5.3 provides an example that is not aCM such that \( \Delta HF_C(t) = HF_W(t) \) for all \( 1 \leq t \leq \rho_W \). Hence, the effective upper bound \( \rho_W + 1 \) provided in Corollary 3.2 is sharp.

Corollary 3.2 immediately yields an algorithm for determining the arithmetically Cohen-Macaulayness of a curve \( C \). As discussed in Section 2.3, 11 can be used to compute both \( HF_W \) and \( \rho_W \), where \( W \) is a general hyperplane section of \( C \), upon fixing a general affine patch. Additionally, Algorithm 1 can be used to compute \( HF_C(1), \ldots, HF_C(\rho_W + 1) \) with \( HF_C(0) = 1 \). Thus, \( C \) is aCM if and only if \( HF_C(t) - HF_C(t - 1) = HF_W(t) \) for \( t = 1, \ldots, \rho_W + 1 \).
**Example 3.3** Recall the curves $C$ and $Q$ in $\mathbb{P}^3$ introduced in Ex. 2.1. Let $\mathcal{H}$ be a general hyperplane, $W_C = C \cap \mathcal{H}$, and $W_Q = Q \cap \mathcal{H}$. For the twisted cubic curve $C$, we compute

$$HF_{W_C} = 1, 3, 3, \quad \rho_{W_C} = 1, \quad HF_C = 1, 4, 7, \quad \Delta HF_C = 1, 3, 3.$$ 

Since $\Delta HF_C(t) = HF_{W_C}(t)$ for $1 \leq t \leq \rho_{W_C} + 1 = 2$, Corollary 3.2 shows $C$ is aCM.

Similarly, for the quartic curve $Q$, we compute

$$HF_{W_Q} = 1, 3, 4, 4, \quad \rho_{W_Q} = 2, \quad HF_Q = 1, 4, 9, 13, \quad \Delta HF_Q = 1, 3, 5, 4.$$ 

Since $\Delta HF_Q(2) = 5 \neq 4 = HF_{W_Q}(2)$, Corollary 3.2 shows $Q$ is not aCM.

### 3.2 Computing other invariants for any curve

When a curve $C \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay, invariants such as the Castelnuovo-Mumford regularity, index of regularity, arithmetic genus, Hilbert polynomial, and Hilbert series can be computed from the Hilbert function of a general hyperplane section $W$ of $C$. The following computes these five for any curve $C \subset \mathbb{P}^n$ (in particular, non-aCM curves) using the ability to compute $HF_C(t)$ via Algorithm 1 given $HF_W$ and $\rho_W$, both of which can be computed via [11].

**Castelnuovo-Mumford regularity** The Castelnuovo-Mumford regularity $\text{reg} C$ is derived from (6) by using Algorithm 1 to compute enough terms of $HF_C$.

**Hilbert polynomial, arithmetic genus, and index of regularity** If $\text{reg} C > \rho_W + 1$, then (6) also yields $\rho_C = \text{reg} C - 1$. Thus, $HF_C(\rho_C) = HP_C(\rho_C)$ so that (3) yields

$$g_C = \deg C \cdot \rho_C - HF_C(\rho_C) + 1 = HP_W(\rho_W) \cdot \rho_C - HF_C(\rho_C) + 1 \quad (7)$$

$$HP_C(t) = \deg C \cdot t + (1 - g_C) = HP_W(\rho_W) \cdot t + (HF_C(\rho_C) - HP_W(\rho_W) \cdot \rho_C). \quad (8)$$

If $\text{reg} C \leq \rho_W + 1$, then (6) yields $\rho_C \leq \rho_W$. Since $HF_C(\rho_W) = HP_C(\rho_W)$, (3) yields

$$g_C = \deg C \cdot \rho_W - HF_C(\rho_W) + 1 = HP_W(\rho_W) \cdot \rho_W - HF_C(\rho_W) + 1 \quad (9)$$

$$HP_C(t) = \deg C \cdot t + (1 - g_C) = HP_W(\rho_W) \cdot t + (HF_C(\rho_W) - HP_W(\rho_W) \cdot \rho_W). \quad (10)$$

In this case, $\rho_C = \min \{ -1 \leq t \leq \rho_W \mid HF_C(t) = HP_C(t) \}$.

**Hilbert series** By adapting [21] p. 28] to this situation, we have

$$HS_C(t) = \sum_{j=0}^{\rho_C+1} \frac{\Delta^2 HF_C(j) \cdot t^j}{(1-t)^2}. \quad (11)$$

**Example 3.4** Consider the degree 8 curve in $\mathbb{P}^3$ derived from [3] Ex. 1.7:

$$C = \{(s^8, s^7t, st^7, t^8) \mid (s, t) \in \mathbb{P}^1 \}.$$

It is easy to verify that the corresponding ideal is

$$I(C) = \langle xw-yz, x^6z-y^7, x^5z^2-y^6w, x^4z^3-y^5w^2, x^3z^4-y^4w^3, x^2z^5-y^3w^4, xz^6-y^2w^5, z^7-yw^6 \rangle.$$

9
Let $\mathcal{H}$ be a general hyperplane and $W = C \cap \mathcal{H}$. Using [11] and Algorithm 1, we have

$$HF_W = 1, 3, 5, 7, 8, \quad HF_C = 1, 4, 9, 16, 25, 36, \quad \Delta HF_C = 1, 3, 5, 7, 9, 11.$$  

Hence, $\rho_W = 4$ and $\Delta HF_C(4) = 9 \neq 8 = HF_W(4)$ yields $C$ is not aCM.

In this example, the terms of $HF_C$ up to $\rho_W + 1 = 5$ needed to apply Corollary 3.2 are not enough to use (6) to determine $\text{reg} C$. Using Algorithm 1, we find

$$HF_C = 1, 4, 9, 16, 25, 36, 49, 57, \quad \Delta HF_C = 1, 3, 5, 7, 9, 11, 13, 8, \quad \Delta^2 HF_C = 1, 2, 2, 2, 2, 2, 2,$$  

Hence, $\text{reg} C = 7$ and $\rho_C = \text{reg} C - 1 = 6$. Additionally, (7), (8), and (11) yield

$$g_C = 8 \cdot 6 - 49 + 1 = 0, \quad HP_C(t) = 8t + 1, \quad HS_C(t) = \frac{1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5 + 2t^6 - 5t^7}{(1-t)^2}.$$  

Another invariant often studied in algebraic geometry is the geometric genus of a curve, which is the arithmetic genus of the desingularization of the curve. Since the curve in Ex. 3.4 is smooth, its geometric genus is equal to its arithmetic genus, namely 0. Section 5.1 compares these genera on a nonsmooth curve.

4 Higher-dimensional cases

4.1 Testing arithmetically Cohen-Macaulayness

The key to testing the arithmetically Cohen-Macaulayness of a scheme of dimension at least 2 is to test the arithmetically Cohen-Macaulayness of a general curve section.

**Theorem 4.1** Let $X \subset P^n$ be a pure-dimensional scheme of dimension $d > 1$ and $\mathcal{L} \subset P^n$ be a general linear space of codimension $d - 1$. Then, $X$ is aCM if and only if the curve $X \cap \mathcal{L}$ is aCM.

**Proof.** This is a special case of [21, Thm. 1.3.3].

The combination of Theorem 4.1 and Corollary 3.2 yields a test for deciding the arithmetically Cohen-Macaulayness of a pure-dimensional scheme of dimension at least 2 by determining the arithmetically Cohen-Macaulayness of a general curve section. Additional information about this general curve section can be computed via Section 3.2, such as its arithmetic genus.

**Example 4.2** Let $X \subset P^4$ be the degree 4 surface defined by the ideal

$$I = \langle x_0x_1 - x_2^2, x_0x_3 - x_4^2 \rangle \subset \mathbb{C}[x_0, x_1, x_2, x_3, x_4].$$  

Let $\mathcal{L}$ and $\mathcal{H}$ be general hyperplanes with $C = X \cap \mathcal{L}$ and $W = C \cap \mathcal{H}$. Since

$$HF_W = 1, 3, 4, 4, \quad \rho_W = 2, \quad HF_C = 1, 4, 8, 12, \quad \Delta HF_C = 1, 3, 4, 4,$$  

Corollary 3.2 yields that $C$ is aCM so that $X$ is aCM by Theorem 4.1.

If $X \subset P^n$ is aCM of dimension $d > 1$ and $W \subset P^n$ is a general linear section of complementary dimension, the index of regularity of $X$ and Castelnuovo-Mumford regularity of $X$ can be computed directly from the index of regularity of $W$ via Prop. 2.3. The remainder of this section describes how to compute the Hilbert function, Hilbert series, and Hilbert polynomial of $X$ given the Hilbert function and index of regularity of $W$.
Hilbert function Using [21, Cor. 1.3.8(d)] applied \(d\) times, we have
\[
\Delta^d HF_X(t) = HF_W(t) \text{ for all } t \geq 0.
\]
In particular, unrolling this formula provides
\[
HF_X(t) = \sum_{j_1=0}^{t} \sum_{j_2=0}^{j_1} \cdots \sum_{j_d=0}^{j_{d-1}} HF_W(j_d).
\]

Hilbert series By adapting [21, p. 28] to this situation, we have
\[
HS_X(t) = \frac{\sum_{j=0}^{\rho_W} \Delta^{d+1} HF_X(j) \cdot t^j}{(1-t)^{d+1}} = \frac{\sum_{j=0}^{\rho_W} \Delta HF_W(j) \cdot t^j}{(1-t)^{d+1}}.
\]

Hilbert polynomial Since \(HF_X(t)\) is a polynomial of degree \(d\) with rational coefficients and \(HF_X(\rho_X + j) = HF_X(\rho_X + j)\) for all \(j \geq 0\), standard polynomial interpolation computes \(HF_X\).

Example 4.3 Let \(X \subset \mathbb{P}^4\) be the surface from Ex. 4.2 which is aCM. From Prop. 2.3 and (5),
\[
\rho_C = 1, \quad \rho_X = 0, \quad \text{reg } X = 3.
\]
Following (12) and (13) with data from Ex. 4.2 we have
\[
HF_X(t) = 1, 5, 13, 25, 41, 61, \ldots \text{ and } HF_S(t) = \frac{1 + 2t + t^2}{(1-t)^3}.
\]
Since \(\rho_X = 0\), one can easily verify that \(HF_X(t) = 2t^2 + 2t + 1\) with \(HF_X(t) = HF_X(t)\) for \(t \geq 0\).

4.2 Minimal generators
Let \(I \subset \mathbb{C}[x_0, \ldots, x_n]\) be a homogeneous ideal. For each \(j \geq 0\), there exists \(d_j(I) \geq 0\) such every minimal generating set consisting of homogeneous polynomials for \(I\) consists of exactly \(d_j(I)\) polynomials of degree \(j\). For a scheme \(X \subset \mathbb{P}^n\), \(d_j(X)\) is defined as \(d_j(I)\) where \(I\) is the corresponding homogeneous ideal. In fact, \(d_j(X) = 0\) for \(j > \text{reg } X\). When \(X\) is arithmetically Cohen-Macaulay, the following provides an approach to compute \(d_j(X)\).

Proposition 4.4 Let \(X \subset \mathbb{P}^n\) be an arithmetically Cohen-Macaulay scheme of dimension \(d > 0\), \(\mathcal{L} \subset \mathbb{P}^n\) be a general linear space of codimension \(0 < \ell \leq d\), and \(Z = X \cap \mathcal{L} \subset \mathcal{L}\). Then, \(d_j(X) = d_j(Z)\) for all \(j\). In particular, the initial degree of \(X\) is the initial degree of \(Z\).

Proof. By treating \(Z\) as a subscheme of \(\mathcal{L}\), the result now follows from [21, Thm. 1.3.6]. \(\square\)

Example 4.5 Let \(X \subset \mathbb{P}^4\) be the aCM surface introduced in Ex. 4.2. By looking at the generating set of the ideal, one sees \(d_2(X) = 2\) with \(d_j(X) = 0\) for all other \(j\). Thus, by Prop. 4.4 we have \(d_2(W) = 2\) with \(d_j(W) = 0\) for all other \(j\). This can be verified directly by performing computations on \(W\) as follows. Clearly, \(d_0(W) = 0\) and, since \(\text{reg } W = 3\), \(d_j(W) = 0\) for \(j \geq 4\). Since \(HF_W(1) = \binom{2+3}{1} - 3\) and \(HF_W(2) = \binom{2+2}{2} - 2\), we know \(d_1(W) = 0\) and \(d_2(W) = 2\). Using linear algebra, it is easy to verify this two dimensional space of quadratic polynomials generates a six dimensional space of cubic polynomials. Since \(HF_W(3) = \binom{2+3}{3} - 6\), \(d_3(W) = 0\).
5 Examples

5.1 The coupler curve of a planar four-bar linkage

Since the curve in Ex. 3.4 was smooth, the arithmetic genus and geometric genus are equal. Here, we investigate a nonsmooth curve arising in kinematics. In particular, the coupler curve of a planar four-bar linkage describes the motion allowed by a mechanism consisting of four hinged bars arranged as a quadrilateral in the plane. The arrangement of the mechanism is described by ten parameters (\(p, \bar{p}, q, \bar{q}, s, \bar{s}, t, \bar{t}, r, R\)) \(\in \mathbb{C}^{10}\). If
\[
\begin{align*}
    a_1 &= s(\bar{z} - \bar{p}), & \bar{a}_1 &= \bar{s}(z - p), & \alpha_1 &= (z - p)(\bar{z} - \bar{p}) + s\bar{s} - r, \\
    a_2 &= t(\bar{z} - \bar{q}), & \bar{a}_2 &= \bar{t}(z - q), & \alpha_2 &= (z - q)(\bar{z} - \bar{q}) + \bar{t} - R,
\end{align*}
\]
the coupler curve is the set of points \((z, \bar{z})\) in the complex plane \(\mathbb{C}^2\) satisfying
\[
\left|\begin{array}{c}
    \bar{a}_1 & a_1 \\
    \bar{a}_2 & a_2
\end{array}\right| - \left|\begin{array}{c}
    \bar{a}_1 & \alpha_1 \\
    \bar{a}_2 & \alpha_2
\end{array}\right| = 0. \tag{14}
\]
By fixing random values for the parameters and homogenizing \(\bar{a}_1\), \(\bar{a}_2\), \(a_1\), \(a_2\), \(\alpha_1\), \(\alpha_2\), we will treat a general coupler curve \(C\) as a projective scheme on \(\mathbb{P}^2\). The degree of \(C\) is 6, and the numerical algebraic geometry approach of \([3]\) verified that the geometric genus is 1.

Let \(W = C \cap \mathcal{H}\) where \(\mathcal{H} \subset \mathbb{P}^2\) is a random hyperplane. Then,
\[
HF_W = 1, 2, 3, 4, 5, 6, 6, \quad HF_C = 1, 3, 6, 10, 15, 21, 27, \quad \Delta HF_C = 1, 2, 3, 4, 5, 6, 6
\]
shows that \(C\) is aCM by Corollary 3.2. In particular, \([9]\) yields the arithmetic genus is \(g_C = 10\).

5.2 A secant variety example

Consider the fourth secant variety of the Segre product for \(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\), namely
\[X = \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3) \subset \mathbb{P}^{35}\].

In \([2]\), numerical computations showed that \(X\) was set-theoretically defined by 10 polynomials of degree 6 and 20 polynomials of degree 9. This result was shown without the use of numerical computations in \([7]\). Here, we show that \(X\) is aCM and use this to show that \(I(X)\) is minimally generated by 10 polynomials of degree 6 and 20 polynomials of degree 9.

Rather than start with known polynomials vanishing on \(X\), we derive our results from a parameterization of \(X\). In particular, consider the map \(\pi : \mathbb{C}^{40} \to \mathbb{C}^{36}\) defined by
\[(a, b, c) \mapsto \sum_{i=1}^{4} a_{i\ell}b_{j\ell}c_{k\ell} \text{ for } 1 \leq i, j, k \leq 3 \text{ and } 1 \leq \ell \leq 4.\]

If \(Y = \pi(\mathbb{C}^{40}) \subset \mathbb{C}^{36}\), then \(X\) is the projectivization of \(Y\), namely \(X = \mathbb{P}(Y) \subset \mathbb{P}^{35}\). Using \(\pi\), it is easy to verify that \(X\) is non-defective with \(\dim X = 31\). After selecting a random linear space \(\mathcal{L} \subset \mathbb{P}^{35}\) of codimension 30 and random hyperplane \(\mathcal{H} \subset \mathbb{P}^{35}\), consider the curve \(C = X \cap \mathcal{L}\) and witness point set \(W = C \cap \mathcal{H}\). We used Bertini \([1]\) to compute \(W\) and a pseudowitness set \([16]\) for \(C = X \cap \mathcal{L}\). This computation, in particular, verified that \(\deg X = 345\) as reported in \([2]\).

Algorithm \([1]\) and \([11]\) produced
\[
HF_W = 1, 5, 15, 35, 70, 126, 200, 280, 345, 345
\]
\[
HF_C = 1, 6, 21, 56, 126, 252, 452, 732, 1077, 1422
\]
\[
\Delta HF_C = 1, 5, 15, 35, 70, 126, 200, 280, 345, 345
\]
which, by Corollary 3.2 and Theorem 4.1, shows that both $C$ and $X$ are aCM. Since $\rho_W = 8$, we know $\text{reg}
olimits_X = \text{reg}
olimits_C = \text{reg}
olimits_W = 9$, $\rho_C = 7$, and $\rho_X = -23$. In particular, (9) yields $\gamma_C = 1684$ and the strategy outlined in Section 4 provides

$$
HF_X = 1, 36, 666, 8436, 82251, 658008, 4496378, 26977968, 145001853, 708846128, \ldots
$$

$$
HS_X(t) = (1 + 4t + 10t^2 + 20t^3 + 35t^4 + 56t^5 + 74t^6 + 80t^7 + 65t^8)/(1 - t)^{32}
$$

$$
HP_X(t) = 345/31! \cdot t^{31} + \ldots + 299405047890287/7220177646800 \cdot t + 1.
$$

In fact, since $\rho_X = -23$, $HP_X(j) = 0$ for $-23 \leq j \leq -1$ so that $HP_X(t)$ can be written as

$$
HP_X(t) = \frac{G(t)}{31!} \prod_{j=1}^{23} (t + j) \quad \text{where}
$$

$$
G(t) = 345 \cdot t^8 + 13032 \cdot t^7 + 484578 \cdot t^6 + 11904840 \cdot t^5 + 218110185 \cdot t^4 + 2831500368 \cdot t^3 + 24772341372 \cdot t^2 + 131202341280 \cdot t + 318073392000.
$$

We now turn to describing a minimal generating set for $I(X)$ using Prop. 4.4. Since $\text{reg}
olimits_X = 9$, we know that $I(X)$ is minimally generated by polynomials of degree at most 9, that is, $d_j(W) = d_j(X) = 0$ for $j \geq 10$. Moreover, $d_j(W) = d_j(X) = 0$ for $0 \leq j \leq 5$ since $HF_W(t) = (4t^4)^j$ for $0 \leq t \leq 5$. Also, $HF_W(6) = (4+6)^t - 10$ yields that $d_6(W) = d_6(X) = 10$ with the initial degree of $X$ being 6. Using linear algebra, we verified that this 10 dimensional space of sextic polynomials vanishing on $W$ generates a 50 dimensional space of septic polynomials, a 150 dimensional space of octic polynomials, and a 350 dimensional space of nonic polynomials. Since $HF_W(7) = (4+7)^t - 50$, $HF_W(8) = (4+8)^t - 150$, and $HF_W(9) = (4+9)^t - 370$, we know $d_W(7) = d_X(7) = d_W(8) = d_W(8) = 0$ and $d_W(9) = d_X(9) = 20$. Therefore, $I(X)$ is minimally generated by 10 sextic polynomials and 20 nonic polynomials.

5.3 A non-aCM example

Consider the map $\pi : \mathbb{C}^{16} \rightarrow \mathbb{C}^{16}$ defined by

$$(s, t, u, v) \mapsto s_{ik}t_{ij} + u_{ik}v_{j\ell} \quad \text{for} \quad i, j, k, \ell = 1, 2.$$

Let $Y = \pi(\mathbb{C}^{16}) \subseteq \mathbb{C}^{16}$. Our object of interest is the projectivization of $Y$, which we will denote by $X = \mathbb{P}(Y) \subseteq \mathbb{P}^{15}$. Using $\pi$, it is easy to compute that $\dim X = 13$. After selecting a random linear space $L \subseteq \mathbb{P}^{15}$ of codimension 12 and random hyperplane $H \subseteq \mathbb{P}^{15}$, consider the curve $C = X \cap L$ and witness point set $W = C \cap H$. We used Bertini to compute $W$ and a pseudowitness set [10] for $C$ which yields that $\deg X = 28$. Algorithm [1] and [11] produced

$$HF_W = 1, 3, 6, 10, 15, 21, 28, 28$$

$$HF_C = 1, 4, 10, 20, 35, 56, 84, 120$$

$$\Delta HF_C = 1, 3, 6, 10, 15, 21, 28, 36.$$

In particular, $\rho_W = 6$ with $\Delta HF_C(7) = 36 \neq 28 = HF_W(7)$ yielding that $C$ is not aCM. Therefore, by Theorem 4.1, $X$ is not aCM.

For this non-aCM example, the terms of $HF_C$ computed while testing $C$ for arithmetically Cohen-Macaulayness are not enough to determine $\rho_C$. Since $\text{reg}
olimits_C > \rho_W + 1$, we use (9) to compute $\text{reg}
olimits_C$ with $\rho_C = \text{reg}
olimits_C - 1$. The additional terms of $HF_C$ needed are

$$HF_C = 1, 4, 10, 20, 35, 56, 84, 120, 165, 196, 224, \quad \Delta HF_C = 1, 3, 6, 10, 15, 21, 28, 36, 45, 31, 28$$

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showing that \( \text{reg } C = 10 \) with \( \rho_C = 9 \). Using (7), the arithmetic genus of \( C \) is \( g_C = 57 \) with
\[
HF_C(t) = 28t - 56
\]
\[
HS_C(t) = (1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 7t^6 + 8t^7 + 9t^8 - 14t^9 - 3t^{10})/(1 - t)^2.
\]

For comparison, consider [15, Alg. 2.4] for numerically testing the arithmetically Cohen-Macaulayness of \( C \). This test requires an \( a \text{ priori} \) bound on \( \text{reg } C \). One could use (6) to compute \( \text{reg } C \) exactly. However, this computation provides enough data needed to use Corollary 3.2 to decide the arithmetically Cohen-Macaulayness of \( C \). Alternatively, one could bound \( \text{reg } C \), for example, by using [12] to conclude that \( \text{reg } C \leq 28 + 2 - 3 = 27 \). In any event, if \( r \geq \text{reg } C \) is the selected bound, [15, Alg. 2.4] requires computing \( HF_C(r - 1) \). Additionally, [14, Alg. 2.4] also requires computing \( HF_{C \cap F}(r - 1) \) where \( F \) is a general form of degree at most \( r - 1 \). Using \( r = 27 \) from [12], one at least needs to compute \( HF_C(26) \) and \( HF_{C \cap F}(26) \) where \( F \) is a general form of degree 26, that is, \( C \cap F \) is a zero scheme of degree \( 28 \cdot 26 = 728 \). Two advantages of using Corollary 3.2 is that the zero scheme under consideration arises as a general hyperplane section of \( C \) rather than a general hypersurface section of \( C \) of possibly high degree and that \( HF_C(t) \) is only needed up to \( \rho_W + 1 \) with \( \rho_W + 1 \leq \text{reg } C \leq r \).

5.4 An application from physics

A question arising in theoretical physics is the nature of the vacuum space in the Minimal Supersymmetric Standard Model. This gives rise to a family of problems that can be written as polynomial images of algebraic sets [9]. The following considers one such problem.

Let \( F : \mathbb{C}^{16} \to \mathbb{C}^{16} \) and \( \pi : \mathbb{C}^{16} \to \mathbb{C}^{25} \) be the polynomial systems defined in Appendix A. Consider the algebraic set \( A = \pi(V(F)) \subset \mathbb{C}^{25} \). Using the approach presented in [14] and Bertini, \( A \) has 11 irreducible components, namely \( Y_1, \ldots, Y_8 \) each of dimension 5 and degree 6, and 3 three-dimensional linear spaces. We take \( Y_1, \ldots, Y_4 \) as the self-conjugate ones whereas \( Y_6 \) and \( Y_8 \) are conjugate to \( Y_5 \) and \( Y_7 \), respectively. For \( j = 1, \ldots, 8 \), let \( X_j \subset \mathbb{P}^{25} \) be the closure of the image of \( Y_j \) under the map \( \mathbb{C}^{25} \to \mathbb{P}^{25} \) defined by \( x \to (1, x) \).

We first investigate the arithmetically Cohen-Macaulayness of each \( X_j \). After selecting a random linear space \( L \subset \mathbb{P}^{25} \) of codimension 4 and random hyperplane \( \mathcal{H} \subset \mathbb{P}^{25} \), we computed the following for each \( C_j = X_j \cap L \) and \( W_j = C_j \cap \mathcal{H} \):
\[
HF_W \equiv 1, 5, 6, 6, \quad HF_{C_j} \equiv 1, 6, 12, 18, \quad \Delta HF_{C_j} \equiv 1, 5, 6, 6.
\]
Thus, Corollary 3.2 and Theorem 1.1 yield \( C_j \) and \( X_j \) are aCM for each \( j = 1, \ldots, 8 \). In particular, \( \text{reg } X_j = \text{reg } C_j = \text{reg } W_j = 3 \), \( \rho_{C_j} = 1 \), \( \rho_{X_j} = -3 \), and [9] yields \( g_{C_j} = 1 \).

Using Prop. 1.1, we can describe the minimal generators of \( X_j \) via \( W_j \). For \( k > \text{reg } X_j = 3 \), we know \( d_k(X_j) = 0 \). By treating \( W_j \subset L \cap \mathcal{H} \), \( HF_{W_j}(1) = (t^{20+1}) - 5 = 16 \) implies that \( d_1(X_j) = d_1(W_j) = 16 \). By additionally restricting to this 16 dimensional linear space, we know \( d_2(X_j) = d_2(W_j) = 9 \) since \( HF_{W_j}(2) = (t^{14+2}) - 9 = 6 \). Moreover, since these quadratics generate a 29 dimensional space of cubics with \( HF_{W_j}(3) = (t^{4+3}) - 29 = 6 \), \( d_3(X_j) = d_3(W_j) = 0 \). Therefore, each \( X_j \) is minimally generated over \( \mathbb{C} \) by 16 linear and 9 quadratic polynomials with
\[
HF_{C_j}(t) = 6t, \quad HF_X(t) = 1, 10, 46, 146, 371, \ldots, \quad HS_{X_j}(t) = (1 + 4t + t^2)/(1 - t)^2
\]
\[
HF_{X_j}(t) = 1/20 \cdot 5^3 + 1/2 \cdot t^4 + 23/12 \cdot t^5 + 7/2 \cdot t^6 + 91/30 \cdot t + 1 = \frac{3t^2 + 12t + 10}{60} \prod_{j=1}^{8} (t + j).
\]

Next, we investigate the \( \mathbb{R} \)-irreducible components \( X_5 \cup X_6 \) and \( X_7 \cup X_8 \). Since
\[
HF_{W_j \cup W_{j+1}} = 1, 9, 12, 12, \quad HF_{C_j \cup C_{j+1}} = 1, 10, 24, 36, \quad \Delta HF_{C_j \cup C_{j+1}} = 1, 9, 14, 12
\]
for $j = 5$ and $j = 7$, $X_5 \cup X_6$ and $X_7 \cup X_8$ are not aCM.

Finally, we consider the arithmetically Cohen-Macaulayness of $X = X_1 \cup \cdots \cup X_8$ using $C = C_1 \cup \cdots \cup C_8$ and $W = W_1 \cup \cdots \cup W_8$. Since $HF_W(1) = 11 \neq 13 = HF_C(1)$, $X$ is not aCM.

6 Conclusion

A fundamental goal of computational algebraic geometry is to compute information about a scheme, even when defining equations are unknown. We developed an effective test, which can be performed using numerical algebraic geometric techniques without defining equations, for deciding the arithmetically Cohen-Macaulayness of a scheme. If the scheme is aCM, additional information such as the Castelnuovo-Mumford regularity, index of regularity, Hilbert series, and Hilbert polynomial can be computed directly from a (pseudo)witness point set. Also, a numerical algebraic geometric approach for computing the arithmetic genus of any curve is presented (see [3] for a numerical approach to compute the geometric genus). The effectiveness of our methods is demonstrated by performing computations related to schemes arising in various applications.

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References


A Physics system

The polynomial systems $F: \mathbb{C}^{16} \rightarrow \mathbb{C}^{16}$ and $\pi: \mathbb{C}^{16} \rightarrow \mathbb{C}^{25}$ from Section 5.4 are defined by

\[ F_1 = 6x_{11}x_4 + 2x_{14}x_4 - 6x_{10}x_5 - 2x_{12}x_5 - 4x_5x_8 + 4x_4x_9 \]
\[ F_2 = 6x_{10}x_3 + 2x_{12}x_3 - 3x_{14}x_6 - x_{15}x_6 - 4x_8x_6 + 4x_3x_8 + 3x_2x_{10} + 3x_2x_{12} + 8x_2x_8 + 2x_{11}x_9 + 5x_1x_{12} + 10x_1x_8 \]
\[ F_3 = 10x_1x_4 + 8x_2x_5 + 4x_3x_5 \]
\[ F_4 = 6x_{11}x_3 + 2x_{13}x_3 - 3x_{14}x_7 - x_{15}x_7 - 4x_6x_7 + 2x_{11}x_1 + 5x_1x_{13} + 10x_1x_9 + 3x_2x_{11} + 3x_2x_{13} + 8x_2x_9 + 4x_3x_9 \]
\[ F_5 = 10x_1x_4 + 8x_2x_4 + 4x_3x_4 \]
\[ F_6 = 2x_1x_5 + 3x_2x_5 + 6x_1x_5 \]
\[ F_7 = 2x_1x_4 + 3x_2x_4 + 6x_1x_4 \]
\[ F_8 = 5x_1x_5 + 3x_2x_5 + 2x_3x_5 \]
\[ F_9 = 5x_1x_4 + 3x_2x_4 + 2x_3x_4 \]
\[ F_{10} = 2x_5x_{10} + 5x_5x_{12} + 10x_6x_8 - 2x_{4}x_{11} - 5x_4x_{13} - 10x_4x_9 \]
\[ F_{11} = 3x_5x_{10} + 3x_5x_{12} + 8x_5x_8 - 3x_4x_{11} - 3x_4x_{13} - 8x_4x_9 \]
\[ F_{12} = 3x_4x_5 + x_{15}x_5 + 4x_6x_5 \]
\[ F_{13} = 2x_{14} + 6x_2x_4 + 3x_{15} + 4x_{14}x_5 + 8x_2x_5 + 10x_{16} + 16x_{14}x_{16} + x_{15}x_{16} + 3x_2^2 - 3x_5x_6 + 3x_4x_7 \]
\[ F_{14} = 3x_4x_4 + x_{15}x_4 + 4x_6x_4 \]
\[ F_{15} = 3x_{14} + 2x_{14}^2 + 6x_{15} + 16x_{14}x_{15} + 3x_{15}^2 + 2x_6 + x_{14}x_{16} + 4x_{15}x_{16} + 4x_6^2 - x_5x_6 + x_4x_7 \]
\[ F_{16} = 10x_{14} + 8x_2^2 + 2x_{15} + x_{14}x_{15} + 2x_{15}^2 + 4x_6 + 6x_{14}x_{16} + 8x_{15}x_{16} + 27x_6^2 - 4x_5x_6 + 4x_4x_7 \]

\[ \pi_1 = x_{14} \]
\[ \pi_2 = x_{15} \]
\[ \pi_3 = x_{16} \]
\[ \pi_4 = x_7x_8 + x_6x_9 \]
\[ \pi_5 = x_{7}x_{10} - x_5x_{11} \]
\[ \pi_6 = x_{7}x_{12} - x_5x_{13} \]
\[ \pi_7 = x_5x_6 - x_4x_7 \]
\[ \pi_8 = x_1x_9x_{10} - x_1x_8x_{11} \]
\[ \pi_9 = x_2x_9x_{10} - x_2x_8x_{11} \]
\[ \pi_{10} = x_3x_9x_{10} - x_3x_8x_{11} \]
\[ \pi_{11} = x_1x_9x_{12} - x_1x_8x_{13} \]
\[ \pi_{12} = x_2x_9x_{12} - x_2x_8x_{13} \]
\[ \pi_{13} = x_3x_9x_{12} - x_3x_8x_{13} \]