Test 3 key

1. Let $f(x) = (x-7)^8$. We want to find the linearization of f at $a = 6$. The formula is $L(x) = f'(a)(x-a) + f(a)$ so we need to calculate $f'(a)$ and $f(a)$.

$$f(6) = (6-7)^8 = (-1)^8 = 1$$

$$f'(6) = 8(6-7)^7 = 8(-1) = -8$$

Using our formula for $L(x)$

$$L(x) = f'(a)(x-a) + f(a) = 8(x-6) + 1 = 8x - 48 + 1 = \boxed{8x - 47}$$

2. For the function $y = xe^{2x}$ we want to find dy. Since $\frac{dy}{dx} = f'(x)$ we need to find the derivative.

$$\begin{aligned} \frac{dy}{dx} &= e^{2x} + xe^{2x}2 \\ dy &= (e^{2x} + 2xe^{2x})dx \quad \text{now plug in x=-1 and dx=2} \\ dy &= (e^{-2} - 2e^{-2})2 = -2e^{-2} = \boxed{\tfrac{-2}{e^2}} \end{aligned}$$

3. Evaluate the follwing limits

a) $\lim_{x\to 0^+} \frac{ln(4x)}{(lnx)^2}$ first notice that

$$\lim_{x\to 0^+} \frac{ln(4x)}{(lnx)^2} \to \frac{ln(0)}{ln(0)^2} \to \frac{\infty}{\infty}.$$

Therefore we can use L'Hospital's Rule.

$$\begin{aligned} \lim_{x\to 0^+} \frac{ln(4x)}{(lnx)^2} &= \lim_{x\to 0^+} \frac{\frac{1}{4x}(4)}{2lnx(\frac{1}{x})} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{2lnx(\frac{1}{x})} = \lim_{x\to 0^+} \frac{1}{2lnx} = \frac{1}{\infty} = \boxed{0} \end{aligned}$$

b) $\lim_{x\to -\infty} (x^2-7)e^x$ first notice that

$$\lim_{x\to -\infty} (x^2-7)e^x \to \infty \cdot 0.$$

We need to rewrite the equation as $\lim_{x\to -\infty} \frac{(x^2-7)}{e^{-x}}$. Now notice that

$$\lim_{x\to -\infty} \frac{x^2-7}{e^{-x}} \to \frac{\infty}{\infty}.$$

Therefore we can use L'Hospital's Rule.

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^2-7}{e^{-x}} &= \lim_{x\to-\infty}\frac{2x}{e^{-x}(-1)} \quad \text{notice that this limit also} \to \tfrac{\infty}{\infty} \\ &= \lim_{x\to-\infty}\frac{2x}{e^{-x}(-1)} = \lim_{x\to-\infty}\frac{2}{e^{-x}} = \frac{2}{\infty} = \boxed{0}\end{aligned}$$

c) $\lim_{x\to\infty}\frac{e^x}{x^2-8}$ first notice that

$$\lim_{x\to\infty}\frac{e^x}{x^2-8} \to \frac{\infty}{\infty}.$$

Therefore we can use L'Hospital's Rule.

$$\begin{aligned}\lim_{x\to\infty}\frac{e^x}{x^2-8} &= \lim_{x\to\infty}\frac{e^x}{2x} \quad \text{notice that this limit also} \to \tfrac{\infty}{\infty} \\ &= \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} \to \boxed{\infty}\end{aligned}$$

4. For $f(x) = (x^2+3x+2)^{\frac{1}{3}}$ we want to find all critcal points.
To find the critical points we first take the derivative.

$$f'(x) = \frac{1}{3}(x^2+3x+2)^{\frac{-2}{3}}(2x+3) = \frac{2x+3}{3(x^2+3x+2)^{\frac{2}{3}}}$$

To find all critical point we check two conditions $f'(x) = 0$ and $f'(x)$ DNE.

$$\begin{aligned}f'(x) &= 0 \\ \frac{1}{3}(x^2+3x+2)^{\frac{-2}{3}}(2x+3) &= 0 \\ 2x+3 &= 0 \\ x &= \frac{-3}{2}\end{aligned}$$

$f'(x)$ DNE when the denominator is equal to zero.
Since $x^2+3x+2 = (x+1)(x+2) = 0$ when $x = -1, -2$, then all critical pts. are the set

$$\{-2, -1, 0\}$$

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5. To set up the problem we draw a rectangle and lable one side x and the other side y. We are given area=75 and we want to minimize cost. Define the notation to be area=A and cost=C. So $A = xy = 75$ and since three sides are brick and one is fence our cost function is

$$C = 10x + 10x + 10y + 5y = 20x + 15y.$$

We use the area to solve for x, $x = \frac{75}{y}$. Then our cost function becomes

$$C = 20(\frac{75}{y}) + 15y = \frac{1500}{y} + 15y.$$

Now we find the local minimum for this function. To do this we need the critical points.

$$C' = \frac{-1500}{y^2} + 15$$

Notice that the derivative C' DNE at y=0 however this is not a critical point because it is not in our domain. If y=0 our area is 0 not 75 so it is not a critical point. The derivative equals zero when

$$\begin{aligned} C' &= 0 \\ \frac{-1500}{y^2} + 15 &= 0 \quad \text{now multiply both sides by } y^2 \\ 1500 + 15y^2 &= 0 \\ 15y^2 &= 1500 \\ y^2 &= 100 \\ y &= 10 \; or - 10 \end{aligned}$$

Notice that -10 is not a critical point because y is a distance. So our only critical pt. is $y = 10$. Now we check to see if $y = 10$ is a minimum. Notice that $1 \leq 10 \leq 75$ and

$$C'(1) = \frac{-1500}{1^2} + 15 = -1485 < 0$$

$$C'(75) = \frac{-1500}{(75)^2} + 15 = \frac{-4}{15} + 15 > 0$$

Since the derivative is negative to the left of 10 and positive to the right of 10 the $y = 10$ is a minimum and the diminsions needed to minimize cost are $\boxed{y = 10 \text{ and } x = \frac{75}{10}}$.

6. Set up notation volume=V and radius=r. We are given that $\frac{dV}{dt} = 2000$ and we want $\frac{dr}{dt}$ when $r = 1000$. The oil slick is in the shape of a circle with a fixed length of 2, this is the shape of a cylinder. The equation for the volume of a cylinder $V = \pi r^2 l = 2\pi r^2$, where l =length= 2,

gives us an equation that links the two variables. Now we take the derivative with respect to t.

$$\frac{dV}{dt} = 2\pi(2r\frac{dr}{dt})$$

All that is left is to substitute our values and solve for $\frac{dr}{dt}$.

$$\begin{aligned}
\frac{dV}{dt} &= 4\pi r\frac{dr}{dt} \\
2000 &= 2\pi(1000)\frac{dr}{dt} \\
\frac{dr}{dt} &= \frac{2000}{4000\pi} \\
\boxed{\frac{dr}{dt}} &= \boxed{\frac{1}{2\pi}}
\end{aligned}$$

7. The equation that has $\sqrt{3}$ as a root is

$$f(x) = x^2 - 3.$$

Newtons method is given by the formula $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$. So we compute the derivative

$$f'(x) = 2x$$

and use the formula to approximate the zeros of f(x).

$$\begin{aligned}
x_1 &= 1 \text{ given} \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-2}{2} = 2 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{1}{4} = \frac{7}{4}
\end{aligned}$$

BONUS
The answer to the bonus is in your class notes. It is the very first thing we did with Newton's method. Start by finding the equation of the tangent line to the curve $y = f(x)$ at the point $(x_1, f(x_1))$. To get the formula for Newton's Method, evaluate the equation of tangent line at the point $(x_2, 0)$(notice since y=0 this is the x-intercept of the tangent line).