North Carolina State University
MA 242 Section 008 Exam 4 Key

Read all directions carefully. **A graphing calculator may NOT be used on this exam.** You must Show All Work for credit and clearly indicate your final answer; no work equals no credit. When you are finished fold your exam, write your name on the outside, turn it in, and then you may leave quietly. Good Luck!

1) Given two vector fields $\vec{F} = 2xi + z^2j + 2yzk$ and $\vec{G} = ye^{xy}i + xe^{xy}j + x\cos(z)k$

a) (7 pts) Compute Curl$\vec{F}$.

ANSWER Curl$\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ 2x & z^2 & 2yz \end{vmatrix} = (2z - 2z)i - (0)j - (0)k = 0$

b) (7 pts) Compute Curl$\vec{G}$.

ANSWER

Curl$\vec{G} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ ye^{xy} & xe^{xy} & x\cos(z) \end{vmatrix} = (0)i - (\cos(z) - 0)j + (e^{xy} + xye^{xy} - [e^{xy} + xye^{xy}])k = -\cos(z)j$

c) (5 pts) Which vector field is conservative?

ANSWER Vector field $\vec{F}$ is conservative since Curl$\vec{F}$=0.

d) (16 pts) For the conservative vector field find the potential function f.

ANSWER To find the f such that $\nabla f = \vec{F}$ we have to solve the following set of partial differential equations. $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = z^2$, and $\frac{\partial f}{\partial z} = 2yz$. Integrating the first equation we have $f = x^2 + h(y, z)$. Taking the partial with respect to y and setting equal to the second equation we see $h_y(y, x) = z^2$ which gives $h(y, z) = yz^2 + g(z)$. Then $f = x^2 + h(y, z) = x^2 + yz^2 + g(z)$. Finally taking the partial with respect to z and setting equal to the third equation we find $g(z) = k$ where k is some constant. So $f = x^2 + yz^2 + g(z) = x^2 + yz^2 + k$ is the potential.

2) For the function $f(x, y, z) = 3z(e^{2x} - e^{2y})$

a) (5 pts) Compute $\nabla(f)$

ANSWER $\nabla(f) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = 6ze^{2x}i - 6ze^{2y}j + 3(e^{2x} - e^{2y})k$.

b) (5 pts) Compute $\nabla^2(f)$

ANSWER $\nabla^2(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 12ze^{2x} - 12ze^{2y} + 0 = 12z(e^{2x} - e^{2y})$

c) (10 pts) Integrate the line integal $\int_C \nabla f \cdot d\vec{r}$ over C where C is the twisted cubic $\vec{r}(t) = ti + t^2j + t^3k \quad 0 \leq t \leq 1$.

ANSWER We could integrate this one normally but we would need multiple integrations by parts. Instead we can use the fundamental theorem of line integrals(FToL). First compute the end points $\vec{r}(0) =< 0, 0, 0 >$ and $\vec{r}(1) =< 1, 1, 1 >$. Then $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = 3(e^2 - e^2) - 0 = 0$

3) (15 pts) The mass of a wire C is given by the line integral $m = \int_C \delta(x,y,z)ds$. Compute the mass of the wire C given by $\vec{r}(t) =< 3\cos(t), 3\sin(t), t > 0 \le t \le 8\pi$ where $\delta(x,y,z) = xy + z$.

ANSWER First compute the derivative of $\vec{r}(t)$, $\vec{r}\,'(t) =< -3\sin(t), 3\cos(t), 1 >$. Then the magnitude $|\vec{r}\,'(t)| = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2 + 1^2} = \sqrt{10}$. Then the line integral $m = \int_C \delta(x,y,z)ds = \int_0^{8\pi} \delta(\vec{r}(t))|\vec{r}\,'(t)|dt = \int_0^{8\pi}(3\cos(t)3\sin(t) + t)\sqrt{10}dt = \sqrt{10}[9\frac{\sin^2(t)}{2} + \frac{t^2}{2}]|_0^{8\pi} = \sqrt{10}(8\pi)^2/2$

4) Let S be the closed surface constructed from the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$, and capped by the disk $x^2 + z^2 \le 1$ at $y = 1$.
ANSWER First notice that S is made up of two surfaces, the paraboloid and the "cap" a disk. Call the paraboloid part S_1 and the disk S_2. For both a) and b) we will need a parametrization of each surface so we do that first. Since S_1 is just a surface we can parametrize it easily $\vec{r_1}(u,v) =< u, u^2 + v^2, v >$ where $D_1 = \{u^2 + v^2 \le 1\}$. With this parametrization $\vec{r_1}_u =< 1, 2u, 0 >$, $\vec{r_1}_v =< 0, 2v, 1 >$, and $\vec{r_1}_u \times \vec{r_1}_v =< 2u, -1, 2v >$. Since S_2 is a disk in the plane y=1 we parametrize using polar so $\vec{r_2}(u,v) =< u\cos(v), 1, u\sin(v) >$ where $D_2 = \{0 \le u \le 1,\ 0 \le v \le 2\pi\}$. With this parametrization $\vec{r_2}_u =< \cos(v), 0, \sin(v) >$, $\vec{r_2}_v =< -u\sin(v), 0, u\cos(v) >$ and $\vec{r_2}_u \times \vec{r_2}_v =< 0, -u, 0 >$. NOTE: Since parametrizations are not unique there are many parametrizations that would work. The above is just what I would use. However the answers will be the same for all parametrizations after we evaluate the integrals.
a) (15 pts) Set up but do not evaluate the integral that will give the surface area of S.
ANSWER We still need the magnitudes of both cross products $|\vec{r_1}_u \times \vec{r_1}_v| = \sqrt{4u^2 + 1 + 4v^2}$ and $|\vec{r_2}_u \times \vec{r_2}_v| = u$. Since surface area is given by $\int\int_S dS = \int\int_{S_1} dS + \int\int_{S_2} dS = \int\int_{D_1} |\vec{r_1}_u \times \vec{r_1}_v|dudv + \int\int_{D_2} |\vec{r_2}_u \times \vec{r_2}_v|dudv = \int_0^1 \int_0^{2\pi} \sqrt{r^2 + 1}rdrd\theta + \int_0^1 \int_0^{2\pi} udvdu$, where I used polar coordinates to set up the integral over D_1.
b) (20 pts) For the vector field $\vec{F} = yj - zk$ set up but do not evaluate the integral that will give the flux of $\vec{F}$ across S.
ANSWER The orientation for any closed surface is outward. For the paraboloid S_1 outward has negative j component. Since $\vec{r_1}_u \times \vec{r_1}_v =< 2u, -1, 2v >$ also has negative j component the normal vector $\vec{n_1}$ points in the direction of $\vec{r_1}_u \times \vec{r_1}_v$. For the disk outward has a positive j component. Since $\vec{r_2}_u \times \vec{r_2}_v =< 0, -u, 0 >$ has a negative j component the normal vector $\vec{n_2}$ points in the negative direction of $\vec{r_2}_u \times \vec{r_2}_v$. Now compute the dot products $\vec{F} \cdot (\vec{r_1}_u \times \vec{r_1}_v) = -(u^2 + v^2) - 2v^2$ and for the second surface $\vec{F} \cdot (-\vec{r_2}_u \times \vec{r_2}_v) = u$. Then the Flux $= \int\int_{S_1} \vec{F} \cdot \vec{n_1}dS + \int\int_{S_2} \vec{F} \cdot \vec{n_2}dS = \int\int_{D_1} \vec{F} \cdot (\vec{r_1}_u \times \vec{r_1}_v)dudv + \int\int_{D_2} \vec{F} \cdot (-\vec{r_2}_u \times \vec{r_2}_v)dudv = \int\int_{D_1} -(u^2 + v^2) - 2v^2dudv + \int\int_{D_2} ududv = \int_0^1 \int_0^{2\pi}(-r^2 - 2r^2\sin^2(\theta))rd\theta dr + \int_0^1 \int_0^{2\pi} ududv$, where I have again used polar coords to set up the integral over D_1.

BONUS
(+5 pts) Use Green's theorem to compute the line integral

$$\int_C arctan(\frac{y}{x})dx + \ln(x^2 + y^2)dy$$

where C is the boundary of the region in polar coordinates D=$\{1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$.