North Carolina State University
MA 242 Section 008 Exam 2 Form 1 Key

Read all directions carefully. **A graphing calculator may NOT be used on this exam.** You must Show All Work for credit and clearly indicate your final answer; no work equals no credit. When you are finished fold your exam, write your name on the outside, turn it in, and then you may leave quietly. Good Luck!

1) For the function $f(x,y,z) = \frac{x}{y} - yz$
a) (12 pts) find the gradient of f at (4,1,1).
ANSWER First find the gradient of f $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \frac{1}{y} i + (\frac{-x}{y^2} - z) j - yk$. Now substitute in the point $(4,1,1)$. $\nabla f(4,1,1) = i + (\frac{-4}{1} - 1) j - k = i - 5j - k.$
b) (12 pts) find the tangent plane to the level surface $4 = f(x,y,z)$ at (4,1,1).
ANSWER Since the gradient of f is normal to the surface $4 = f(x,y,z)$, then the normal vector $\vec{n}$ is just the gradient of f at (4,1,1). So the equation of the tangent plane is

$$1(x-4) - 5(y-1) - 1(z-1) = 0$$

c) (12 pts) find the directional derivative of f in the direction of $\vec{A} = < 2, -1, 2 >$ at the point (4,1,1).
ANSWER The directional derivative in the direction of the unit vector $\vec{u}$ is given by $D_{\vec{u}} f = \nabla f \cdot \vec{u}$. We need a unit vector in the direction of $\vec{A}$ so
$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{<2,-1,2>}{\sqrt{2^2 + (-1)^2 + 2^2}} = < 2/3, -1/3, 2/3 >$. Then using our formula

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = < 1, -5, -1 > \cdot < 2/3, -1/3, 2/3 > = 2/3 + 5/3 - 2/3 = 5/3$$

d) (12 pts) What direction(I want the vector $\vec{u} =$?) gives the greatest rate of change at (4,1,1)?
ANSWER We know the greatest rate of change occurs in the direction of the gradient. Therefore a unit vector in the direction of the gradient is $\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{27}} < 1, -5, -1 >$.

2) (15 pts) Given $z = e^r \cos(\theta)$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ calculate $\frac{\partial z}{\partial x}$.
ANSWER Here we must use the chain rule, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$. Computing each partial derivative we get $\frac{\partial z}{\partial x} = e^r \cos(\theta) \frac{2x}{2\sqrt{x^2+y^2}} - e^r \sin(\theta) \frac{1}{1+(y/x)^2} \frac{-y}{x^2}$. Finally substituting for r and θ in terms of x,y we get

$$\frac{\partial z}{\partial x} = e^{\sqrt{x^2+y^2}} \cos(\tan^{-1}(y/x)) \frac{2x}{2\sqrt{x^2+y^2}} - e^{\sqrt{x^2+y^2}} \sin(\tan^{-1}(y/x)) \frac{1}{1+(y/x)^2} \frac{-y}{x^2}$$

3) (12 pts) Is the function $f(x,y,z) = \sqrt{x + e^y} + z$ differentiable at (3,0,1)? You must justify your answer.

ANSWER If all the partials of f are continuous then this implies f is differentiable. So we check partials. $\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x+e^y}}$, $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x+e^y}}e^y$, and $\frac{\partial f}{\partial z} = 1$. A constant function is always continuous so 1 is continuous for all (x,y,z). Exponential functions are continuous so e^y is continuous for all (x,y,z). Products of continuous functions are continuous so all we have to show is that $\frac{1}{2\sqrt{x+e^y}}$ is continuous at (3,0,1). Note that $\frac{1}{2\sqrt{x+e^y}}$ is a composition of three functions $r(t) = \frac{1}{t}$, $h(k) = \sqrt{k}$, and $c(x,y) = x + e^y$. $x + e^y$ is the sum of continuous functions so it is continuous. The other two functions are functions of 1 variable and are continuous on their domains. Since compositions of continuous functions are continuous it is enough to show that $x + e^y > 0$ at (3,0,1). Since $3 + e^1 > 0$ then all first partials are continuous and therefore f is differentiable. NOTE: Just showing a function exists at a point is not enough to prove something is continuous.

4) (10 pts) Suppose $f_x = \frac{1}{x} - g(y)$ and $f_y = \frac{-1}{y} + h(x)$. What can we say about any non-zero critical point of z=f(x,y)(is it a local max, local min, saddle)?

ANSWER Once we have a critical point we check whether we have a local max, local min, or saddle by looking at the sign of $D = f_{xx}f_{yy} - f_{xy}^2$. Since $f_{xx} = \frac{-1}{x^2}$ and $f_{yy} = \frac{1}{y^2}$ then $f_{xx}f_{yy} = \frac{-1}{x^2y^2}$ is always negative if x and y not equal to zero. Therefore $D = f_{xx}f_{yy} - f_{xy}^2 < 0$ this means our critical point is a saddle.

5) (15 pts) Integrate the following integral over the region $R = \{(x,y)|0 \le x \le 2,\ 0 \le y \le 1\}$.

$$\int\int_R \frac{xy}{x^2+1}dA$$

ANSWER $\int\int_R \frac{xy}{x^2+1}dA = \int_0^2\int_0^1 \frac{xy}{x^2+1}dydx = \int_0^2 \frac{x}{x^2+1}y^2|_0^2\ dx = \int_0^2 \frac{4x}{x^2+1}\ dx$. For this integral use a u substitution $u = x^2 + 1$ to get $\int_0^2 \frac{4x}{x^2+1}\ dx = \int_{x=0}^{x=2} \frac{2du}{u} = 2\ln(x^2+1)|_0^2 = 2\ln(5) - 2\ln(1) = 2\ln(5)$

BONUS

(+5 pts) Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere. Hint: the normal line to a surface at a point is perpendicular to the tangent plane at that same point.