Functional Analysis and Integration
Condensed Notes

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Contents

Part I

Definitions

Chapter 1

Functional Analysis

linear functional: A linear mapping $f : V \to \mathbb{R}$ where V is a vector space is called a linear functional.

bounded function: A function $f : X \to Y$ where X and Y are normed vector spaces is bounded if there exists a constant M such that

$$\|f(x)\|_Y \leq M\|x\|_X,$$

where $x \in X$ and $\|\|_X, \|\|_Y$ are the norms on X and Y, respectively.

function norm: The norm of a bounded mapping, $f : X \to Y$, where X and Y are normed vector spaces, is defined as

$$\|G\| = \sup_{\|x\|_X=1} \|G(x)\|_Y.$$

An equivalent definition takes the supremum over all $x \in X$.

rare, nowhere dense: A subset M of a metric space X is called rare or nowhere dense in X if the closure $\bar{M}$ has no interior points.

meager, first category: A subset M of a metric space X is called meager or of the first category in X if M is the union of countably many sets which are each rare in X.

nonmeager, second category: A subset M of a metric space X is called nonmeager or of the second category in X if it is not meager in X.

strong convergence: A sequence (x_n) in a normed space X is called strongly convergent (or convergent in the norm) if there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. This is denoted $x_n \to x$. x is called the strong limit of x_n.

weak convergence: A sequence (x_n) in a normed space X is called weakly convergent if there is an $x \in X$ such that for every $f \in X'$, $\lim_{n\to\infty} f(x_n - x) = 0$. This is denoted $x_n \xrightarrow{w} x$. x is called the weak limit of x_n.

open mapping: Let X, Y be metric spaces. The map $T : D(T) \to Y$ with $D(T) \subset X$ is called an open mapping if the image the of every open set in $D(T)$ is an open set in Y.

closed linear operator: Let X, Y be normed spaces and $T : D(T) \to Y$ a linear operator with $D(T) \subset X$. T is called a closed linear operator if its graph is closed in the normed space $X \times Y$, with the norm $\|(x, y)\| = \|x\| + \|y\|$.

An equivalent definition of a closed linear operator is that for X, Y, and T as above, T has the property that if $x_n \to x$ where $x_n \in D(T)$ and $Tx_n \to y$, then $x \in D(T)$ and $Tx = y$.

Chapter 2

Measure Theory

length: The length of an interval $(a, b]$ is $b - a$. The length of the intervals $(-\infty, B]$ and $(-\infty, \infty)$ is defined to be ∞. Note that the length is the same even for different types of intervals (i.e., closed intervals, open intervals, and the two types of half-open intervals). The length of the union of a finite number of disjoint intervals is the sum of the corresponding lengths.

characteristic function: Let $E \in \mathbf{X}$. The characteristic function of E, χ_E, is defined by

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E \end{cases}$$

step function: A step function is a finite linear combination of characteristic functions of intervals.

extended real number system: The extended real number system is the set $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, where the additional values are defined so that $-\infty < x < \infty$ for any $x \in \mathbb{R}$.

limit superior, limit inferior, limit: The limit superior and limit inferior of a sequence (x_n) of extended real numbers are defined by

$$\limsup x_n = \inf_m \left(\sup_{n \geq m} x_n \right),$$

$$\liminf x_n = \sup_m \left(\inf_{n \geq m} x_n \right).$$

If the limit inferior and limit superior agree, their value is called the limit of the sequence.

For a sequence (A_n) of sets, the limit superior and limit inferior are defined as

$$\limsup A_n = \bigcap_{m=1}^{\infty} \left(\bigcup_{n=m}^{\infty} A_n \right),$$

$$\liminf A_n = \bigcup_{m=1}^{\infty} \left(\bigcap_{n=m}^{\infty} A_n \right).$$

Note the consistency between the definitions for a real-valued sequence and a sequence of sets with the usual partial ordering.

σ-algebra: A family $\mathbf{X}$ of subsets of a set X with the properties

1. $\emptyset, X \in \mathbf{X}$,

2. $A \in \mathbf{X} \Rightarrow X - A \in \mathbf{X}$,
3. $(A_n) \in \mathbf{X} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathbf{X}$,

is called a σ-algebra or σ-field if it

measurable space: An ordered pair, $(X, \mathbf{X})$ consisting of a set X and a σ-algebra $\mathbf{X}$ of subsets of X is called a measurable space.

measurable set: A member, X, of a σ-algebra, $\mathbf{X}$, is called an $\mathbf{X}$-measurable set. If $\mathbf{X}$ is understood, it is called a measurable set.

σ-algebra generated by A: Let $\mathbf{A}$ be a given σ-algebra. The intersection of all σ-algebras containing $\mathbf{A}$ is called the σ-algebra generated by $\mathbf{A}$. It is the smallest σ-algebra containing $\mathbf{A}$.

Borel algebra: The σ-algebra generated by all open intervals in $\mathbb{R}$ is called the Borel algebra and denoted $\mathbf{B}$. The Borel algebra is also generated by all closed intervals of $\mathbb{R}$.

Borel set: A member of the Borel algebra is called a Borel set.

extended Borel algebra: Let $\mathbf{B}$ be the Borel algebra. For every $E \in \mathbf{B}$, add the sets $E \cup \{\infty\}$, $E \cup \{-\infty\}$, and $E \cup \{-\infty, \infty\}$ to the collection of sets. The resulting σ-algebra is called the extended Borel algebra and is denoted $\bar{\mathbf{B}}$.

measurable function: A function $f : X \to \mathbb{R}$ is called $\mathbf{X}$-measurable or measurable if for every real number, α, the set

$$\{x \in X : f(x) > \alpha\},$$

is in $\mathbf{X}$. The definition applies to extended real-valued functions as well. Complex-valued functions are measurable if both the real and imaginary parts of the function are measurable. A function, $f : \mathbf{X} \to \mathbf{Y}$, where $\mathbf{X}$ $\mathbf{Y}$ are two general measurable spaces, is measurable if for any $E \in \mathbf{Y}$, we have $f^{-1}(E) \in \mathbf{X}$.

positve part, negative part of a function: Given a function $f : X \to \mathbb{R}$, define

$$f^{+}(x) = \sup\{f(x), 0\},$$
$$f^{-}(x) = \sup\{-f(x), 0\}.$$

f^{+} is called the positive part of f and f^{-} is called the negative part of f.

monotone class: A nonempty collection $\mathbf{M}$ of subsets of a set X is called a monotone class if both of the following properties hold:

$$(E_n) \in \mathbf{M} \text{ increasing } \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathbf{M},$$
$$(F_n) \in \mathbf{M} \text{ decreasing } \Rightarrow \bigcap_{n=1}^{\infty} F_n \in \mathbf{M}.$$

countably additive: An extended real-valued function μ defined on a σ-algebra $\mathbf{X}$ is countably additive if

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

for any sequence (E_n) of disjoint sets in $\mathbf{X}$.

measure: A measure is an extended real-valued function μ on a σ-algebra $\mathbf{X}$ such that

1. $\mu(\emptyset) = 0$,
2. $\mu(E) \geq 0$ for all $E \in \mathbf{X}$,
3. μ is countably additive

finite measure: A finite measure is a measure whose range does not include the values ∞ or $-\infty$.

σ-finite measure: A σ-finite measure is a measure for which there exists a sequence (E_n) of sets in $\mathbf{X}$ with $X = \bigcup E_n$ such that $\mu(E_n) < \infty$ for all n.

unit measure: Let $\mathbf{X}$ be any σ-algebra of the subsets of some nonempty set X and $p \in X$ be fixed. The finite measure μ defined as

$$\mu(E) = \begin{cases} 0, & p \notin E, \\ 1, & p \in E. \end{cases}$$

is called the unit measure concentrated at p.

counting measure: Let $\mathbf{X}$ be the σ-algebra of all subsets of $\mathbb{N}$. Define $\mu(E)$ for $E \in \mathbf{X}$ to be the number of elements in E if E is finite or ∞ otherwise. μ is a σ-finite measure called the counting measure.

Lebesgue measure, Lebesgue measurable sets: The extension of the algebra of sets made up of all finite unions of sets of the form

$$(a, b], \quad (-\infty, b], \quad (a, \infty], \quad (-\infty, \infty),$$

with length measure, l, to a σ-algebra with measure l^* is called the collection of Lebesgue measurable sets. The σ-finite measure l^* is called Lebesgue measure.

Borel measure: The restriction of Lebesgue measure to the Borel sets is called Borel measure (or Lebesgue measure).

Borel-Stieltjes measure: Let $g : \mathbb{R} \to \mathbb{R}$ be monotone increasing and continuous on the right at every point. Define

$$\begin{aligned} \mu_g\left((a, b]\right) &= g(b) - g(a), \\ \mu_g\left((-\infty, b]\right) &= g(b) - \lim_{x \to -\infty} g(x), \\ \mu_g\left((a, \infty)\right) &= \lim_{x \to \infty} g(x) - g(a), \\ \mu_g\left((-\infty, \infty)\right) &= \lim_{x \to \infty} g(x) - \lim_{x \to -\infty} g(x). \end{aligned}$$

These sets form an algebra of sets. The extension of the measure, also denoted μ_g, to the algebra of all Borel subsets of $\mathbb{R}$ is called the Borel-Stieltjes measure generated by g. Extending this to a complete σ-algebra which contains the Borel subsets of $\mathbb{R}$, the resulting measure is called the Lebesgue-Stieltjes measure generated by g.

measure space: A 3-tuple, $(X, \mathbf{X}, \mu)$, consisting of a set X, a σ-algebra $\mathbf{X}$ of subsets of X and a measure μ defined on $\mathbf{X}$ is called a measure space.

μ-almost everywhere: A proposition which holds everywhere on a set X except possibly for some $N \in X$ with $\mu(N) = 0$ is said to hold μ-almost everywhere.

charge: A real-valued function λ defined on a σ-algebra which has the same properties as a measure except that it may possibly take on negative values is called a charge. Note: this terminology is not standard, according to Bartle, but is used in that book.

simple function: A function with only a finite number of values is called a simple function.

standard representation: A simple measurable function ϕ can be represented in the form

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i},$$

where $a_j \in \mathbb{R}$ and χ_{E_i} is the characteristic function of a set E_i in $\mathbf{X}$. The unique standard representation of ϕ is the one where the a_i are distinct and E_i are disjoint nonempty subsets of X such that $X = \bigcup_{i=1}^{n} E_i$.

integrable functions: The collection $L = L(X, \mathbf{X}, \mu)$ of integrable functions consists of all real-valued $\mathbf{X}$-measurable functions f defined on X, such that both f^+ and f^- have finite integrals with respect to μ.

Riemann integral:

step functions:

Let ϕ be a step function written in the form

$$\phi = \sum_{i=1}^{n} c_i \chi_{E_i}$$

where E_i is the interval with left, right endpoints a_i, b_i. The integral of ϕ is

$$\int \phi = \sum_{i=1}^{n} c_i (b_i - a_i)$$

Riemann integrable functions:

The lower Riemann integral is the supremum of the integrals of all step functions ϕ such that $\phi(x) \leq f(x)$ for all $x \in [a, b]$ and $\phi(x) = 0$ for $x \notin [a, b]$. The upper Riemann integral is the infimum of the integrals of all step functions ϕ such that $\phi(x) \geq f(x)$ for all $x \in [a, b]$ and $\phi(x) = 0$ for $x \notin [a, b]$. If the lower and upper Riemann integral of f agree, then f is Riemann integrable, and its Riemann integral is equal to the value of its upper and lower Riemann integrals.

Lebesgue integral:

simple functions:

Let $\phi \in M^+(X, \mathbf{X})$ be a simple function in standard representation, the integral of ϕ with respect to μ is the extended real number

$$\int \phi \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i).$$

positive functions: Let $f \in M^+(X, \mathbf{X})$. The integral of f with respect to μ is the extended real number

$$\int f \, d\mu = \sup \int \phi \, d\mu,$$

where the supremum is taken over all simple functions $\phi \in M^+(X, \mathbf{X})$ satisfying $\phi(x) \leq f(x)$ for all $x \in X$. If $E \in \mathbf{X}$, then $f_{\chi_E} \in M^+(X, \mathbf{X})$ and the integral of f over E with respect to μ is the extended real number

$$\int_E f \, d\mu = \int f_{\chi_E} \, d\mu,$$

integrable functions:

Let f be an integrable function defined on a set X in a measure space $(\mathbf{X}, \mu)$. The integral of f with respect to μ is

$$\int f \, d\mu = \int f^{+} \, d\mu - \int f^{-} \, d\mu,$$

where the integrals on the right are the integrals defined on positive measurable functions. If $E \in \mathbf{X}$, then

$$\int_E f \, d\mu = \int_E f^{+} \, d\mu - \int_E f^{-} \, d\mu,$$

The function $\lambda(E) = \int_E f \, d\mu$ is called the indefinite integral of f with respect to μ.

μ-equivalence: Two functions in $L = L(X, \mathbf{X}, \mu)$ are called μ-equivalent if they are equal μ-almost everywhere. The equivalence class determined by f in L is denoted $[f]$ and consists of all functions in L which are μ-equivalent to f.

conjugate indices: Two numbers, p and q, satisfying $\frac{1}{p} + \frac{1}{q} = 1$ are called conjugate indices.

essentially bounded function: An element of L_∞ is called an essentially bounded function.

uniform convergence: (f_n) converges uniformly to f if for every $\epsilon > 0$, there exists $N(\epsilon)$ such that if $n \geq N(\epsilon)$ and $x \in X$, then $\|f_n(x) - f(x)\| < \epsilon$.

pointwise convergence: (f_n) converges pointwise to f if for every $\epsilon > 0$ and every $x \in X$, there exists $N(\epsilon, x)$ such that if $n \geq N(\epsilon, x)$ and $x \in X$, then $\|f_n(x) - f(x)\| < \epsilon$.

almost everywhere convergence: (f_n) converges almost everywhere to f if there exists a set $M \in \mathbf{X}$ with $\mu(M) = 0$ such that for every $\epsilon > 0$ and $x \in X - M$ there exists $N(\epsilon, x)$ such that if $n \geq N(x, \epsilon)$, then $\|f_n(x) - f(x)\| < \epsilon$.

convergence in L_p: (f_n) converges in L_p to $f \in L_p$ if for every $\epsilon > 0$ there exists $N(\epsilon)$ such that if $n \geq N(\epsilon)$ then $\|f_n - f\|_p < \epsilon$, where $\|\|_p$ is the L_p norm. Another way to say this is that f converges to f in mean of order p.

convergence in measure: (f_n), a sequence of measurable real-valued functions, converges in measure to a measurable real-valued function f if

$$\lim_{n \to \infty} \mu\left(\{x \in X : \|f_n(x) - f(x)\| \geq \alpha\}\right) = 0$$

for each $\alpha > 0$.

almost uniform convergence: (f_n), a sequence of measurable functions, is almost uniformly convergent to a measurable function f if for each $\delta > 0$ there is a set $E_\delta \in \mathbf{X}$ with $\mu(E_\delta) < \delta$ such that (f_n) converges uniformly to f on $X - E_\delta$.

Cauchy in measure: (f_n), a sequence of measurable real-valued functions, is Cauchy in measure if

$$\lim_{m,n \to \infty} \mu\left(\{x \in X : \|f_m(x) - f_n(x)\| \geq \alpha\}\right) = 0$$

for each $\alpha > 0$.

almost uniformly Cauchy: (f_n), a sequence of measurable functions, is almost uniformly Cauchy if for each $\delta > 0$ there is a set $E_\delta \in \mathbf{X}$ with $\mu(E_\delta) < \delta$ such that (f_n) is almost uniformly convergent on $X - E_\delta$.

positive set: A set P in $\mathbf{X}$ is positive with respect to the charge λ if $\lambda\left(E \cap P\right) \geq 0$ for any E in $\mathbf{X}$.

negative set: A set N in $\mathbf{X}$ is negative with respect to the charge λ if $\lambda\left(E \cap N\right) \leq 0$ for any E in $\mathbf{X}$.

null set: A set M in $\mathbf{X}$ is a null set for the charge λ if $\lambda\left(E \cap M\right) = 0$ for any E in $\mathbf{X}$.

Hahn decomposition: Given a charge λ on $\mathbf{X}$ the Hahn decomposition of X with respect to λ is a pair of disjoint sets $P, N \in \mathbf{X}$ with $X = P \cup N$ such that P is positive and N is negative with respect to λ.

positive and negative variations: λ is a charge on $\mathbf{X}$ and P, N is a Hahn decomposition for λ. The positive and negative variations of λ are the finite measures

$$\lambda^+(E) = \lambda(E \cap P), \qquad \lambda^-(E) = -\lambda(E \cap N).$$

total variation: The total variation of the charge λ is the measure $|\lambda|$ defined for E in $\mathbf{X}$ by

$$|\lambda|(E) = \lambda^+(E) + \lambda^-(E).$$

absolutely continuous measure: A measure λ on $\mathbf{X}$ is absolutely continuous with respect to a measure μ on $\mathbf{X}$ if $E \in \mathbf{X}$ and $\mu(E) = 0$ imply that $\lambda(E) = 0$. This is written $\lambda \ll \mu$.

absolutely continuous charge: A charge λ on $\mathbf{X}$ is absolutely continuous with respect to a charge μ on $\mathbf{X}$ if the total variation $|\lambda|$ of λ is absolutely continuous with respect to $|\mu|$.

Radon-Nikodym derivative: A function $f \in M^+(X, \mathbf{X})$ such that

$$\lambda(E) = \int_E f \, d\mu, \qquad E \in \mathbf{X},$$

where λ and μ are σ-finite measures on $\mathbf{X}$, is called the Radon-Nikodym derivative of λ with respect to μ.

mutually singular: Two measures λ, μ on $\mathbf{X}$ are called mutually singular if there are disjoint sets $A, B \in \mathbf{X}$ such that $X = A \cup B$ and $\lambda(A) = \mu(B) = 0$. This is written $\lambda \perp \mu$. This is sometimes referred to by calling λ singular with respect to μ.

algebra, field: Let $\mathbf{A}$ be a family of subsets of X with the properties

1. $\emptyset, X \in \mathbf{A}$.
2. $E \in \mathbf{A} \Rightarrow X - E \in \mathbf{A}$.
3. $E_1, \ldots, E_n \in \mathbf{A} \Rightarrow \bigcup_{i=1}^n E_i \in \mathbf{A}$.

$\mathbf{A}$ is called an algebra or a field (of sets).

measure: A measure on an algebra of subsets, $\mathbf{A}$, is an extended real-valued function μ defined on $\mathbf{A}$ such that

1. $\mu(\emptyset) = 0$.
2. $\mu(E) \geq 0, \forall E \in \mathbf{A}$.
3. If $(E_n) \in \mathbf{A}$ is a disjoint sequence of sets with $\bigcup_{n=1}^\infty E_n \in \mathbf{A}$, then

$$\mu\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty \mu(E_n).$$

outer measure: Let $\mathbf{A}$ be the algebra of subsets of X. The outer measure generated by μ is denoted μ^* and defined as

$$\mu^*(B) = \inf \sum_{i=1}^\infty \mu(E_i)$$

where B is an arbitrary subset of X, and the infimimum is taken over all sequences of sets in $\mathbf{A}$ such that $B \subseteq \bigcup_{i=1}^{\infty} E_i$.

countably subadditive: μ^* is countably subadditive if

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$$

where (B_n) is a subsequence of subsets of X.

μ^*-measurable: A subset E of X is called μ^*-measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$

for all subsets A of X. The collection of all μ^*-measurable sets is called the σ-algebra generated by $\mathbf{A}$ and is denoted $\mathbf{A}^*$.

complete σ-algebra: A σ-algebra, $\mathbf{A}^*$, with the property that if $E \in \mathbf{A}^*$ with $\mu^*(E) = 0$ and if $B \subseteq E$, then $B \in \mathbf{A}^*$ and $\mu^*(B) = 0$.

measurable rectangle: For two measurable spaces, $(X, \mathbf{X}), (Y, \mathbf{Y})$, a set of the form $A \times B$ with $A \in \mathbf{X}, B \in \mathbf{Y}$ is called a measurable rectangle or rectangle in $X \times Y$.

measurable set: $\mathbf{Z} = \mathbf{X} \times \mathbf{Y}$ is the σ-algebra of subsets of $Z = X \times Y$, where $(X, \mathbf{X}), (Y, \mathbf{Y})$ are measurable spaces. A member of $\mathbf{Z}$ is called a measurable set or a $\mathbf{Z}$-measurable set.

product measure: A measure, π, on the σ-algebra $\mathbf{Z} = \mathbf{X} \times \mathbf{Y}$ such that $\pi(A \times B) = \mu(A)\nu(B)$, where μ is a measure on $\mathbf{X}$ and ν is a measure on $\mathbf{Y}$, is called a product of μ and ν.

x-section, y-section: The x-section of $E \subset X \times Y$, where $x \in X$ is $E_x = \{y \in Y : (x, y) \in E\}$. The y-section of E with $y \in Y$ is $E^y = \{x \in X : (x, y) \in E\}$. The x-section of a function f defined on $X \times Y$ to $\mathbb{R}$ is the image $f(\{x\} \times E_x)$ and is denoted $f_x(y)$. Similarly, the y-section of a function f is the image $f(E^y \times \{y\})$ and is denoted $f^y(x)$.

Part II

Theorems and Formulas

Chapter 3

Functional Analysis

3.1 Normed and Banach Spaces

Zorn's Lemma

Assumptions

- $M \neq \emptyset$.
- M is a partially ordered set.
- Every chain $C \subset M$ has an upper bound.

Conclusions

- M has at least one maximal element.

Hahn-Banach Theorem for Normed Spaces

Assumptions

- Z is a subspace of a normed space X.
- f is a bounded linear functional on Z.

Conclusions

- There exists a bounded linear functional $\tilde{f}$ with the properties
  - $\tilde{f}$ is an extension of f to X.
  - $\|\tilde{f}\|_X = \|f\|_Z$

Chapter 4

Measure Theory

4.1 Measurable Functions

Equivalent Definitions of Measurable Lemma

Assumptions

- $f : X \to \mathbb{R}$.
- $A_\alpha = \{x \in X : f(x) > \alpha\}$.
- $B_\alpha = \{x \in X : f(x) \leq \alpha\}$.
- $C_\alpha = \{x \in X : f(x) \geq \alpha\}$.
- $D_\alpha = \{x \in X : f(x) < \alpha\}$.

Equivalent Statements

- For every $\alpha \in \mathbb{R}$, $A_\alpha \in \mathbf{X}$.
- For every $\alpha \in \mathbb{R}$, $B_\alpha \in \mathbf{X}$.
- For every $\alpha \in \mathbb{R}$, $C_\alpha \in \mathbf{X}$.
- For every $\alpha \in \mathbb{R}$, $D_\alpha \in \mathbf{X}$.

Also Measurable Lemma

Assumptions

- f, g real-valued.
- f, g measurable.
- $c \in \mathbb{R}$.

Conclusions

- cf is measurable.
- f^2 is measurable.
- $f + g$ is measurable.
- fg is measurable.
- $|f|$ is measurable.

Measurable Extended Real-Valued Functions Lemma

Assumptions

- f is an extended real-valued function.
- $A = \{x \in X : f(x) = \infty\}$.
- $B = \{x \in X : f(x) = -\infty\}$.
- $f_1(x) = \begin{cases} f(x), & x \notin A \cup B, \\ 0, & x \in A \cup B \end{cases}.$

Equivalent Statements

- f is measurable.
- f_1 is measurable.

Measurable Sequence Extrema Lemma

Assumptions

- $f_n \in M(X, \mathbf{X})$.
- $f(x) = \inf f_n(x)$.
- $f^*(x) = \liminf f_n(x)$.
- $F(x) = \sup f_n(x)$.
- $F^*(x) = \limsup f_n(x)$.

Conclusions

- $f \in M(X, \mathbf{X})$.
- $f^* \in M(X, \mathbf{X})$.
- $F \in M(X, \mathbf{X})$.
- $F^* \in M(X, \mathbf{X})$.

Measurable Sequence Limits Corollary

Assumptions

- $f_n \in M(X, \mathbf{X})$.
- $f_n \to f$ on X.

Conclusions

- $f \in M(X, \mathbf{X})$.

Simple Function Approximation of Nonnegative Measurable Functions Lemma

Assumptions

- $f \in M(X, \mathbf{X})$.
- f is nonnegative.

Conclusions

- There exists a sequence $(\phi_n) \in M(X, \mathbf{X})$ with the properties
  - $0 \leq \phi_n(x) \leq \phi_{n+1}(x)$ for $x \in X, n \in \mathbb{X}$.
  - $f(x) = \lim \phi_n(X)$ for each $x \in X$.
  - Each ϕ_n is a simple function.

4.2 Measures

Subset Measure Comparison Lemma

Assumptions

- $E, F \in \mathbf{X}$.
- $E \subseteq F$.
- **Option 1:** $\mu(E) < \infty$.

Conclusions

- $\mu(E) \leq \mu(F)$.
- **Option 1:** $\mu(F - E) = \mu(F) - \mu(E)$.

Measure of Sequences Lemma

Assumptions

- E_n is an increasing sequence in $\mathbf{X}$.
- F_n is an decreasing sequence in $\mathbf{X}$.
- $\mu(F_1) < \infty$.

Conclusions

- $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$
- $\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu(F_n)$

4.3 Lebesgue Integration

Simple Function Integration Properties Lemma

Assumptions

- $\phi, \psi \in M^+(X, \mathbf{X})$
- ϕ, ψ are simple functions.
- $c \geq 0$.
- $\lambda : \mathbf{X} \to \mathbb{R}$ is defined as $\lambda(E) = \displaystyle\int \phi \chi_E \, d\mu$.

Conclusions

- $\displaystyle\int c\phi \, d\mu = c \int \phi \, d\mu$.
- $\displaystyle\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$.
- λ is a measure on $\mathbf{X}$.

Comparison of Simple Function Integrals Lemma

Assumptions

- $f, g \in M^+(X, \mathbf{X})$.
- $f \leq g$.
- $E, F \in \mathbf{X}$.
- $E \subseteq F$.

Conclusions

- $\displaystyle\int f \, d\mu \leq \int g \, d\mu$.
- $\displaystyle\int_E f \, d\mu \leq \int_F f \, d\mu$.

Monotone Convergence Theorem

Assumptions

- $f_n \in M^+(X, \mathbf{X})$.
- (f_n) is monotone increasing sequence of functions.
- $f_n \to f$.

Conclusions

- $\displaystyle\int f \, d\mu = \lim \int f_n \, d\mu$.

Positive Measurable Function Integration Properties Lemma

Assumptions

- $f, g \in M^+(X, \mathbf{X})$
- $c \geq 0$.

Conclusions

- $\displaystyle\int cf \, d\mu = c \int f \, d\mu$.
- $\displaystyle\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Fatou's Lemma

Assumptions

- $f_n \in M^+(X, \mathbf{X})$.

Conclusions

- $\displaystyle\int (\liminf f_n) \, d\mu \leq \liminf \int f_n \, d\mu$.

Positive Function Measure Representation Corollary

Assumptions

- $f \in M^+$.
- $\lambda : \mathbf{X} \to \mathbb{R}$ is defined as $\lambda(E) = \displaystyle\int_E f \, d\mu$.

Conclusions

- λ is a measure.

Integration Definiteness Corollary

Assumptions

- $f \in M^+$.

Equivalent Statements

- $f(x) = 0$ μ-almost everywhere on X.
- $\displaystyle\int f \, d\mu = 0$.

Absolute Continuity of Positive Function Measure Corollary

Assumptions

- $f \in M^+$.
- $\lambda : \mathbf{X} \to \mathbb{R}$ is defined as $\lambda(E) = \displaystyle\int_E f \, d\mu$.

Conclusions

- $\lambda << \mu$.
- In this case, $E \in \mathbf{X}$ and $\mu(E) = 0$ implies $\lambda(E) = 0$.

μ-A.E. Monotone Convergence Theorem

Assumptions

- $f_n \in M^+(X, \mathbf{X})$.
- (f_n) is monotone increasing sequence of functions.
- $f_n \to f$ μ-almost everywhere.

Conclusions

- $\displaystyle\int f \, d\mu = \lim \int f_n \, d\mu$.

Integration of Series Corollary

Assumptions

- $g_n \in M^+$.

Conclusions

- $\displaystyle\int \left(\sum_{n=1}^{\infty} g_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int g_n \, d\mu \right)$.

Charge Representation Lemma

Assumptions

- $f \in L$.
- $\lambda : \mathbf{X} \to \mathbb{R}$ is defined as
$$\lambda(E) = \int_E f \, d\mu$$
.

Conclusions

- λ is a charge.

Absolute Value of Integrable Functions Theorem

Assumptions

- f is measurable.

Conclusions

- $\left|\int f \, d\mu\right| \leq \int |f| \, d\mu$

Equivalent Statements

- $f \in L$.
- $|f| \in L$.

Forced to Be Integrable Corollary

Assumptions

- f measurable.
- $g \in L$.
- $|f| \leq |g|$.

Conclusions

- $f \in L$.
- $\int |f| \, d\mu \leq \int |g| \, d\mu$.

Linearity of Integration Theorem

Assumptions

- $\alpha \in \mathbb{R}$.
- $f, g \in L$.

Conclusions

- $\alpha f \in L$.
- $f + g \in L$.
- $\displaystyle\int \alpha f \, d\mu = \alpha \int f \, d\mu$.
- $\displaystyle\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Lebesgue Dominated Convergence Theorem

Assumptions

- $f_n \in L$.
- $f : X \to \mathbb{R}$ is measurable.
- $f_n \to f$ almost everywhere.
- $g \in L$.
- $|f_n| \leq g$ for all n.

Conclusions

- $f \in L$.
- $\displaystyle\int f \, d\mu = \lim \int f_n \, d\mu$

Parameter Dependence and Integration Commutation Corollary

Assumptions

- The function $t \to f(x,t)$ is continuous on $[a,b]$ for each $x \in X$.
- $g \in L(X)$.
- $|f(x,t)| \leq g(x)$.
- $F(t) = \displaystyle\int f(x,t)\,d\mu(x)$.

Conclusions

- F is continuous on $[a,b]$.
- $\displaystyle\int_a^b F(t)\,dt = \int_a^b \left(\int f(x,t)\,d\mu(x)\right)\,dt = \int \left(\int_a^b f(x,t)\,dt\right)\,d\mu(x)$ where the integrals with respect to t are Riemann integrals.

Integration and Differentiation Commutation Corollary

Assumptions

- The function $x \to f(x,t_0)$ for some $t_0 \in [a,b]$ is integrable on X.
- $\dfrac{\partial f}{\partial t}$ exists on $X \times [a,b]$.
- $g \in L(X)$.
- $\left|\dfrac{\partial f}{\partial t}(x,t)\right| \leq g(x)$.
- $F(t) = \displaystyle\int f(x,t)\,d\mu(x)$.

Conclusions

- F is differentiable on $[a,b]$.
- $\dfrac{\partial F}{\partial t}(t) = \dfrac{d}{dt}\displaystyle\int f(x,t)\,d\mu(x) = \int \dfrac{\partial f}{\partial t}(x,t)\,d\mu(x)$.

4.4 Modes of Convergence

Uniform Convergence and L_p Convergence Theorem

Assumptions

- $\mu(X) < \infty$.
- $f_n \in L_p$.
- $f_n \to f$ uniformly on X.

Conclusions

- $f \in L_p$.
- $f_n \to f$ in L_p.

Dominated A.E. and L_p Convergence Theorem

Assumptions

- $f_n \in L_p$.
- f measurable.
- $f_n \to f$ almost everywhere.
- $g \in L_p$.
- $|f_n(x)| \leq g(x)$ for all $x \in X, n \in \mathbb{N}$.

Conclusions

- $f \in L_p$.
- $f_n \to f$ in L_p.

Bounded Sequence and L_p Convergence Corollary

Assumptions

- $\mu(X) < \infty$.
- $f_n \in L_p$.
- f measurable.
- $f_n \to f$ almost everywhere.
- There exists K such that $|f_n(x)| \leq K$ for all $x \in X, n \in \mathbb{N}$.

Conclusions

- $f \in L_p$.
- $f_n \to f$ in L_p.

Convergent Subsequence of Cauchy in Measure Sequence Theorem

Assumptions

- f_n are measurable.
- f_n are real-valued.
- f_n are Cauchy in measure.

Conclusions

- There exists a subsequence of f_n which converges almost everywhere and in measure to a measurable real-valued function f.

A.E and Cauchy in Measure Convergence Theorem

Assumptions

- f_n are measurable.
- f_n are real-valued.
- f_n are Cauchy in measure.

Conclusions

- There exists a measurable real-valued function f such that $f_n \to f$ in measure and f is uniquely determined almost everywhere.

Dominated Convergence in Measure Theorem

Assumptions

- $f_n \in L_p$.
- $f_n \to f$ in measure.
- $g \in L_p$.
- $|f_n(x)| \leq g(x)$ almost everywhere.

Conclusions

- $f \in L_p$.
- $f_n \to f$ in L_p.

Almost Uniformly Cauchy and A.E / A.U. Convergence Lemma

Assumptions

- f_n is almost uniformly Cauchy.

Conclusions

- There exists a measurable function f such that $f_n \to f$ almost uniformly and almost everywhere.

Almost Uniformly and In Measure Convergence Theorem

Assumptions

- $f_n \to f$ almost uniformly.

Conclusions

- $f_n \to f$ in measure.

Assumptions

- $f_n \to f$ in measure.

Conclusions

- $f_{n_k} \to f$ almost uniformly for some index set n_k.

Egoroff's Theorem

Assumptions

- $\mu(X) < \infty$.
- f, f_n measurable.
- f, f_n real-valued.
- $f_n \to f$ almost everywhere.

Conclusions

- $f_n \to f$ almost uniformly and in measure.

Vitali Convergence Theorem

Assumptions

- $1 \le p < \infty$.
- $f_n \in L_p(X, \mathbf{X}, \mu)$.

Equivalent Statements

- $f_n \to f$ in L_p
- All of the following hold
  - $f_n \to f$ in measure.
  - For each $\epsilon > 0$ there exists $E_\epsilon \in \mathbf{X}$ with $\mu(E_\epsilon) < \infty$ such that if $F \in \mathbf{X}$ and $F \cap E_\epsilon = \emptyset$ then
    $$\int_F |f_n|^p \, d\mu < \epsilon^p \qquad \text{for all } n \in \mathbb{N}.$$
  - For each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $E \in \mathbf{X}$ and $\mu(E) < \delta(\epsilon)$ then
    $$\int_E |f_n|^p \, d\mu < \epsilon^p \qquad \text{for all } n \in \mathbb{N}.$$

4.5 Decomposition of Measures

Hahn Decomposition Theorem

Assumptions

- $\mathbf{X}$ is a measure space.
- λ is a charge on $\mathbf{X}$.

Conclusions

- There exists sets P and N in $\mathbf{X}$ with the properties
  - P is positive with respect to λ.
  - N is negative with respect to λ.
  - $P \cup N = \mathbf{X}$.
  - $P \cap N = \emptyset$.

Hahn Decomposition Practically Unique Lemma

Assumptions

- P_1, N_1 and P_2, N_2 are Hahn decompositions for λ.
- $E \in \mathbf{X}$.

Conclusions

- $\lambda(E \cap P_1) = \lambda(E \cap P_2)$.
- $\lambda(E \cap N_1) = \lambda(E \cap N_2)$.

Jordan Decomposition Theorem

Assumptions

- λ is a charge on $\mathbf{X}$.
- μ, ν finite measures on $\mathbf{X}$.
- $\lambda = \mu - \nu$.

Conclusions

- $\lambda = \lambda^+ - \lambda^-$, the positive and negative variations of λ.
- $\mu(E) \geq \lambda^+(E)$ and $\nu(E) \geq \lambda^-(E)$.

Charge Representation Theorem

Assumptions

- $f \in L(X, \mathbf{X}, \mu)$.
- $\lambda(E) = \int_E f \, d\mu$ (Reminder: this defines a charge).

Conclusions

- $$\lambda^+(E) = \int_E f^+ \, d\mu$$
- $$\lambda^-(E) = \int_E f^- \, d\mu$$
- $$|\lambda|(E) = \int_E |f| \, d\mu$$

Absolutely Continuous Intuition Lemma

Assumptions

- λ, μ finite measures on $\mathbf{X}$.

Equivalent Statements

- $\lambda \ll \mu$.
- For every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

  IF :

  - $E \in \mathbf{X}$.
  - $\mu(E) < \delta(\epsilon)$.

  THEN :

  - $\lambda(E) < \epsilon$.

Radon-Nikodým Theorem

Assumptions

- λ, μ are σ-finite measures on $\mathbf{X}$.
- $\lambda \ll \mu$.

Conclusions

- There exists $f \in M^+(X, \mathbf{X})$ such that for $E \in \mathbf{X}$,
$$\lambda(E) = \int_E f \, d\mu.$$
- f is uniquely determined μ-almost everywhere.
- **Reminder:** f is called the Radon-Nikodým derivative of λ with respect to μ and is denoted $\frac{d\lambda}{d\mu}$.

Lebesgue Decomposition Theorem

Assumptions

- $\mathbf{X}$ is a σ-algebra.
- λ, μ are σ-finite measures on $\mathbf{X}$.

Conclusions

- There exist λ_1 and λ_2 such that
  - $\lambda_1 \perp \mu$
  - $\lambda_2 \ll \mu$
  - $\lambda = \lambda_1 + \lambda_2$.
- λ_1 and λ_2 are unique.

L_p Functional Decomposition Lemma

Assumptions

- G is a bounded linear functional on L_p.

Conclusions

- There exist positive bounded linear functionals G^+ and G^- such that $G(f) = G^+(f) - G^-(f)$ for all $f \in L_p$.

Riesz Representation Theorem

Assumptions

- $(X, \mathbf{X}, \mu)$ is a σ-finite measure space.
- G is a bounded linear functional on $L_1(X, \mathbf{X}, \mu)$.
- **Option 1:** G is positive.

Conclusions

- There exists $g \in L_\infty((X, \mathbf{X}, \mu)$ such that
$$G(f) = \int fg \, d\mu,$$
for all $f \in L_1$.
- $\|G\| = \|g\|_\infty$.
- **Option 1:** $g \geq 0$.

Riesz Representation Theorem

Assumptions

- $(X, \mathbf{X}, \mu)$ is a measure space.
- G is a bounded linear functional on $L_p(X, \mathbf{X}, \mu)$, $1 < p < \infty$.
- $q = p/(p-1)$.

Conclusions

- There exists $g \in L_q((X, \mathbf{X}, \mu)$ such that
$$G(f) = \int fg \, d\mu,$$
for all $f \in L_p$.
- $\|G\| = \|g\|_q$.

4.6 Generation of Measures

Pre-Lebesgue Measure Lemma

Assumptions

- **F** is the collection of all finite unions of sets of the form

$$(a,b], \quad (-\infty,b], \quad (a,\infty), \quad (-\infty,\infty).$$

Conclusions

- **F** is an algebra of subsets of $\mathbb{R}$.
- Length is a measure on **F**.

Outer Measure Properties Lemma

Assumptions

- μ^* is the outer measure generated by μ.
- $B_n \subseteq X$.
- **Option 1:** $A \subseteq B$.
- **Option 2:** $B \in \mathbf{A}$.

Conclusions

- $\mu^*(\emptyset) = 0$.
- $\mu^*(B) \geq 0$, for $B \subset X$.
- **Option 1:** $\mu^*(A) \leq \mu^*(B)$.
- **Option 2:** $\mu^*(B) = \mu(B)$.
- $\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu^*\left(B_n\right)$.

Carathéodory Extension Theorem

Assumptions

- $\mathbf{A}^*$ is the collection of all μ^* measurable sets.
- (E_n) is a disjoint sequence in $\mathbf{A}^*$.

Conclusions

- $\mathbf{A}^*$ is a σ-algebra containing $\mathbf{A}$.
- $$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu^*(E_n).$$

Hahn Extension Theorem

Assumptions

- $\mathbf{A}$ is an algebra.
- μ is a σ-finite measure on $\mathbf{A}$.

Conclusions

- There is a unique extension of μ to a measure on $\mathbf{A}^*$.

Riesz Representation Theorem

Assumptions

- $J = [a, b]$.
- $C(J)$ is the Banach space of all continuous functions on J to $\mathbb{R}$ with the sup norm.
- G is a bounded positive linear functional on $C(J)$.

Conclusions

- There exists a measure γ defined on the Borel subsets of $\mathbf{R}$ with $G(f) = \int_J f\, d\gamma$, for all $f \in C(J)$.
- $\|G\| = \gamma(J)$.

4.7 Product Measures

Rectangles Are Algebras Lemma

Assumptions

- $(X, \mathbf{X})$ and $(Y, \mathbf{Y})$ are measurable spaces.
- $\mathbf{Z}_0$ is the collection of all finite unions of rectangles in $Z = X \times Y$.

Conclusions

- $\mathbf{Z}_0$ is an algebra of subsets of Z.

Product Measure Theorem

Assumptions

- $(X, \mathbf{X}, \mu)$, $(X, \mathbf{X}, \mu)$ measure spaces.
- **Option 1:** $\mathbf{X}, \mathbf{Y}$ are σ-finite.

Conclusions

- There exists measure π on $\mathbf{Z} = \mathbf{X} \times \mathbf{Y}$ such that $\pi(A \times B) = \mu(A)\nu(B)$.
- **Option 1:** π is the unique measure with this property.

Measurable Sections Lemma

Assumptions

- $E \subset Z$ is measurable.

Conclusions

- Every section of E is measurable.

Assumptions

- $f : Z \to \bar{\mathbb{R}}$ is measurable.

Conclusions

- Every section of f is measurable.

Monotone Class Lemma

Assumptions

- $\mathbf{A}$ an algebra of sets

Conclusions

- $\mathbf{S}$, the σ-algebra generated by $\mathbf{A}$, coincides with $\mathbf{M}$, the monotone class generated by $\mathbf{A}$.

Product Measure Representation Lemma

Assumptions

- $(X, \mathbf{X}, \mu), (Y, \mathbf{Y}, \nu)$ σ-finite measure spaces.
- $E \in \mathbf{Z} = \mathbf{X} \times \mathbf{Y}$.
- $f(x) = \nu(E_x), \quad g(y) = \mu(E^y)$.

Conclusions

- f, g are measurable.
-
$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\nu.$$

Tonelli's Theorem

Assumptions

- $(X, \mathbf{X}, \mu), (Y, \mathbf{Y}, \nu)$ σ-finite measure spaces.
- $Z = X \times Y$.
- $F : Z \to \bar{\mathbb{R}}$ nonnegative and measurable.
- $f(x) = \int_Y F_x \, d\nu, \quad g(y) = \int_X F^y \, d\mu$.

Conclusions

- f, g are measurable.
-
$$\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\nu.$$
-
$$\int_X \left(\int_Y F \, d\nu \right) d\mu = \int_Z F \, d\pi = \int_Y \left(\int_X F \, d\mu \right) d\nu.$$

Fubini's Theorem

Assumptions

- $(X, \mathbf{X}, \mu), (Y, \mathbf{Y}, \nu)$ σ-finite measure spaces.
- $\mathbf{Z} = \mathbf{X} \times \mathbf{Y}$.
- π is the product measure of μ and ν.
- $F : Z \to \mathbb{R}$ integrable with respect to π.
- $f(x) = \int_Y F_x \, d\nu, \quad g(y) = \int_X F^y \, d\mu$ defined almost everywhere, extended real-valued.

Conclusions

- f, g have finite integrals.
- $$\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\nu.$$
- $$\int_X \left(\int_Y F \, d\nu \right) d\mu = \int_Z F \, d\pi = \int_Y \left(\int_X F \, d\mu \right) d\nu.$$