# Asymptotic Capacity of Multi-User MIMO Communications

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## Abstract

This poster introduces two theorems from Random Matrix Theory to derive capacity expressions of a class of multi-user communication schemes, among which single-user MIMO with multi-cell interference presented here. A Kronecker channel model is assumed between the base stations and the terminals. New asymptotic capacity formulas, independent of the instantaneous channel realization, are provided for single-user decoding and MMSE decoding at the terminal.

## Introduction

- We consider a downlink scenario with multi-cell interference with
  - $K$ base-stations, among which $K-1$ called interferers
  - $n_{\mathrm{T}_k},\ n_{\mathrm{R}} \gg K$ antennas at the transmitter $k$ and the receiver
  - Kronecker channel model $\mathbf{H}_k = \mathbf{R}_k^{\frac{1}{2}}\mathbf{X}_k\mathbf{T}_k^{\frac{1}{2}}$, due to
    - limited distance between antennas
    - privileged directions of energy propagation
- Assuming large numbers of antennas, we study the asymptotic
  - single user (SU) decoding capacity, with uniform and optimal power allocation
  - MMSE decoder capacity
- Our objective is to

**provide deterministic equivalents to SU and MMSE capacities independently of the instantaneous channel realization**

## System Model

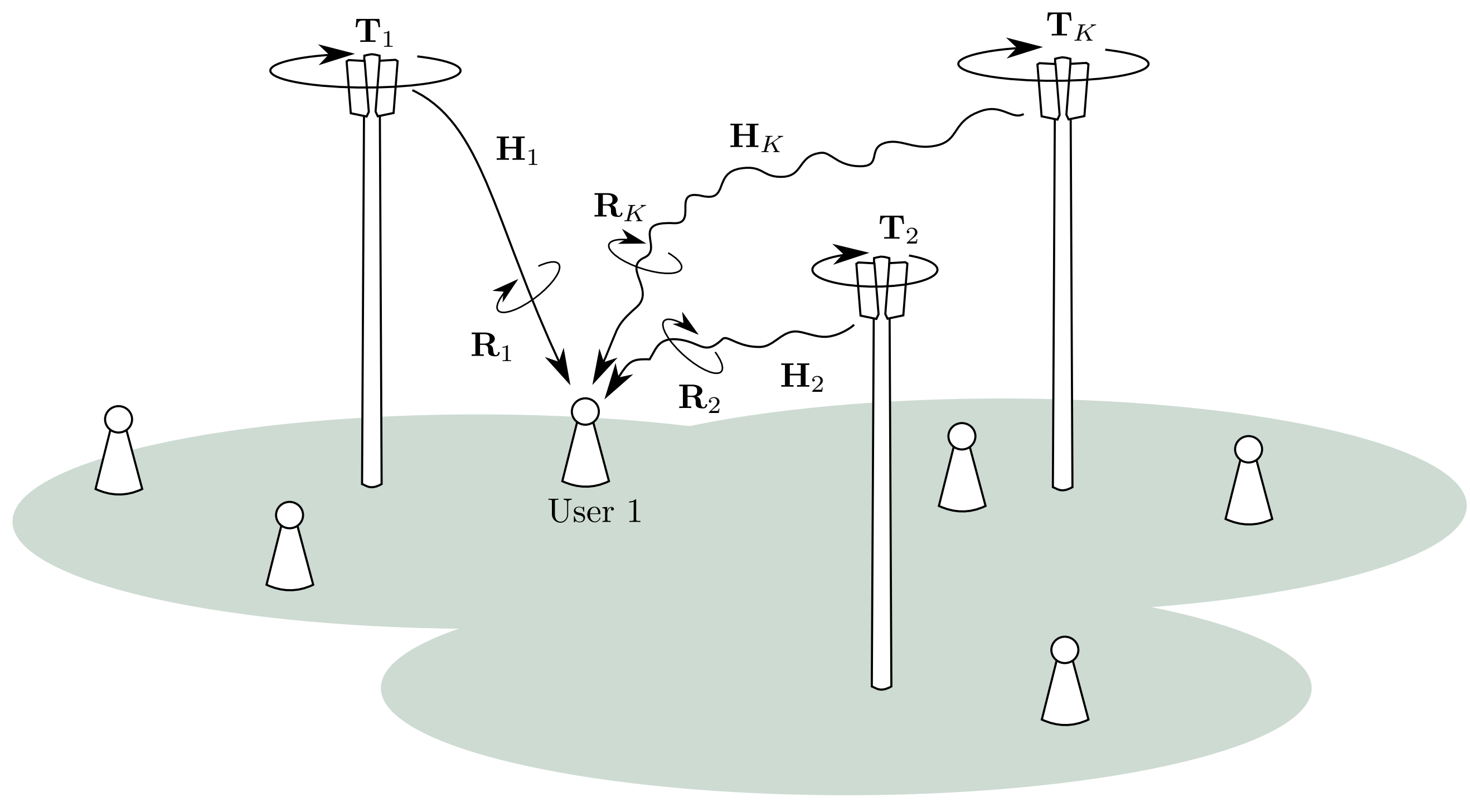

Figure: Single-user MIMO in multi-cell interference

$$\mathbf{y} = \mathbf{H}_1\mathbf{s}_1 + \sum_{j=2}^{K}\mathbf{H}_j\mathbf{s}_j + \mathbf{n} \qquad (1)$$

with $\mathbf{H}_k = \mathbf{R}_k^{\frac{1}{2}}\mathbf{X}_k\mathbf{T}_k^{\frac{1}{2}} \in \mathbb{C}^{n_{\mathrm{R}}\times n_{\mathrm{T}_k}}$, $\mathbf{y} \in \mathbb{C}^{n_{\mathrm{R}}}$ received signal by user 1, $\mathbf{n} \in \mathbb{C}^{n_{\mathrm{R}}}$ AWGN, $\mathbf{s}_k \in \mathbb{C}^{n_{\mathrm{T}_k}}$ symbol of base station $k$.

## Single-user and MMSE decoding

- (Per-antenna) single-user decoding capacity

$$C_{\mathrm{SU}}(\sigma^2) = \frac{1}{n_{\mathrm{R}}}\log_2\left|\mathbf{I}_{n_{\mathrm{R}}} + \frac{1}{\sigma^2}\sum_{j=1}^{K}\mathbf{H}_j\mathbf{H}_j^{\mathrm{H}}\right| - \frac{1}{n_{\mathrm{R}}}\log_2\left|\mathbf{I}_{n_{\mathrm{R}}} + \frac{1}{\sigma^2}\sum_{j=2}^{K}\mathbf{H}_j\mathbf{H}_j^{\mathrm{H}}\right| \qquad (2)$$

- MMSE decoder capacity

$$C_{\mathrm{MMSE}}(\sigma^2) = \frac{1}{n_{\mathrm{R}}}\sum_{i=1}^{n_{\mathrm{T}_1}}\log_2(1+\gamma_i) \qquad (3)$$

with

$$\gamma_i = \mathbf{h}_i^{\mathrm{H}}\left(\sum_{j=1}^{K}\mathbf{H}_j\mathbf{H}_j^{\mathrm{H}} - \mathbf{h}_i\mathbf{h}_i^{\mathrm{H}} + \sigma^2\mathbf{I}_{n_{\mathrm{R}}}\right)^{-1}\mathbf{h}_i \qquad (4)$$

$$\xrightarrow{\text{a.s.}} \frac{T_{1_{ii}}}{n_{\mathrm{T}_1}}\operatorname{tr}\mathbf{R}_1\left(\sum_{j=1}^{K}\mathbf{H}_j\mathbf{H}_j^{\mathrm{H}} + \sigma^2\mathbf{I}_{n_{\mathrm{R}}}\right)^{-1} \qquad (5)$$

where $\mathbf{h}_j \in \mathbb{C}^{n_{\mathrm{T}_j}}$ is the $j^{\text{th}}$ column of $\mathbf{H}_1$

## Theorem 1. *Deterministic equivalent of the Stieltjes Transform*

Let $K,\ N,\ \{n_k\}_{k=1..K} \in \mathbb{N}$, $c_k = n_k/N$, and let

$$\mathbf{B}_N = \sum_{k=1}^{K}\mathbf{R}_k^{\frac{1}{2}}\mathbf{X}_k\mathbf{T}_k\mathbf{X}_k^{\mathrm{H}}\mathbf{R}_k^{\frac{1}{2}} \in \mathbb{C}^{N\times N} \qquad (6)$$

$\mathbf{T}_k \in \mathbb{C}^{n_k\times n_k}$, $\mathbf{R}_k \in \mathbb{C}^{N\times N}$ nonnegative definite, $\frac{1}{\sqrt{n_k}}\mathbf{X}_k \in \mathbb{C}^{N\times n_k}$ standard Gaussian. Denote, for $z \in \mathbb{C}\setminus\mathbb{R}^+$, $m_N(z) = \frac{1}{N}\operatorname{tr}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}$. Then, as $n_k,\ N \to \infty$ ($c_k$ fixed),

$$m_N(z) - m_N^{(0)}(z) \xrightarrow{\text{a.s.}} 0 \qquad (7)$$

where

$$m_N^{(0)}(z) = \frac{1}{N}\operatorname{tr}\left(\sum_{k=1}^{K}\int\frac{\tau_k dF^{\mathbf{T}_k}(\tau_k)}{1+\frac{\tau_k}{c_k}e_k(z)}\mathbf{R}_k - z\mathbf{I}_N\right)^{-1} \qquad (8)$$

and the $\{e_i(z)\}$ form the unique solution to the equations

$$e_i(z) = \frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\sum_{k=1}^{K}\int\frac{\tau_k dF^{\mathbf{T}_k}(\tau_k)}{1+\frac{\tau_k}{c_k}e_k(z)}\mathbf{R}_k - z\mathbf{I}_N\right)^{-1} \qquad (9)$$

## Theorem 2. *Deterministic equivalent of the Shannon Transform*

If $\frac{1}{n_k}\operatorname{tr}(\mathbf{T}_k) = \frac{1}{N}\operatorname{tr}(\mathbf{R}_k) = 1$, then $\mathcal{V}(x) - \mathcal{V}^{(0)}(x) \xrightarrow{\text{a.s.}} 0$, where

$$\mathcal{V}(x) = \frac{1}{N}\log\det\left(\mathbf{I}_N + \frac{1}{x}\mathbf{B}_N\right) \qquad (10)$$

and

$$\mathcal{V}^{(0)}(x) = \frac{1}{N}\log\det\left(\mathbf{I}_N + \frac{1}{x}\sum_{k=1}^{K}\mathbf{R}_k\int\frac{\tau_k dF^{\mathbf{T}_k}(\tau_k)}{1+c_k e_k(-x)\tau_k}\right) + \sum_{k=1}^{K}\frac{1}{c_k}\int\log\left(1+c_k e_k(-x)\tau_k\right)dF^{\mathbf{T}_k}(\tau_k) + x\cdot m_N^{(0)}(-x) - 1 \qquad (11)$$

## Capacity with uniform/optimal power allocation

- Uniform power allocation
  - $C_{\mathrm{SU}}(\sigma^2)$ obtained directly from Theorem 2.
  - $C_{\mathrm{MMSE}}(\sigma^2) = \frac{1}{n_{\mathrm{R}}}\sum_{i=1}^{n_{\mathrm{T}_1}}\log_2\left(1+\frac{1}{c_1}T_{1_{ii}}e_1(-\sigma^2)\right)$, with $e_1$ defined in Theorem 1.
- Optimal power allocation for SU decoding

$$C_{\mathrm{SU}}(\sigma^2) = \frac{1}{n_{\mathrm{R}}}\log\left|\mathbf{I} + \frac{1}{\sigma^2}\mathbf{A}^{-\frac{1}{2}}\mathbf{R}_1^{\frac{1}{2}}\mathbf{X}_1\mathbf{T}_1^{\frac{1}{2}}\mathbf{P}_1\mathbf{T}_1^{\frac{1}{2}}\mathbf{X}_1^{\mathrm{H}}\mathbf{R}_1^{\frac{1}{2}}\mathbf{A}^{-\frac{1}{2}}\right|,\quad \mathbf{A} = \mathbf{I}_{n_{\mathrm{R}}} + \frac{1}{\sigma^2}\sum_{j>1}\mathbf{H}_j\mathbf{P}_j\mathbf{H}_j^{\mathrm{H}} \qquad (12)$$

Optimal Strategy: align eigenvectors of $\mathbf{P}_1$ to $\mathbf{T}_1$ and $\{p_j\} = \operatorname{diag}(\mathbf{P}_1)$ as

$$\begin{cases} p_i = 0 & (1/\alpha_i) - 1 \leq \frac{1}{n_{\mathrm{T}_1}}\sum_{l=1}^{n_{\mathrm{T}_1}}(1-\alpha_l) \\ p_i = (1-\alpha_i)\left(\frac{1}{n_{\mathrm{T}_1}}\sum_{l=1}^{n_{\mathrm{T}_1}}(1-\alpha_l)\right)^{-1} & \text{otherwise} \end{cases} \qquad (13)$$

for $\alpha_i \xrightarrow{\text{a.s.}} (1 + T_{1_{ii}}e_1(-\sigma^2))^{-1}$.

## Simulations

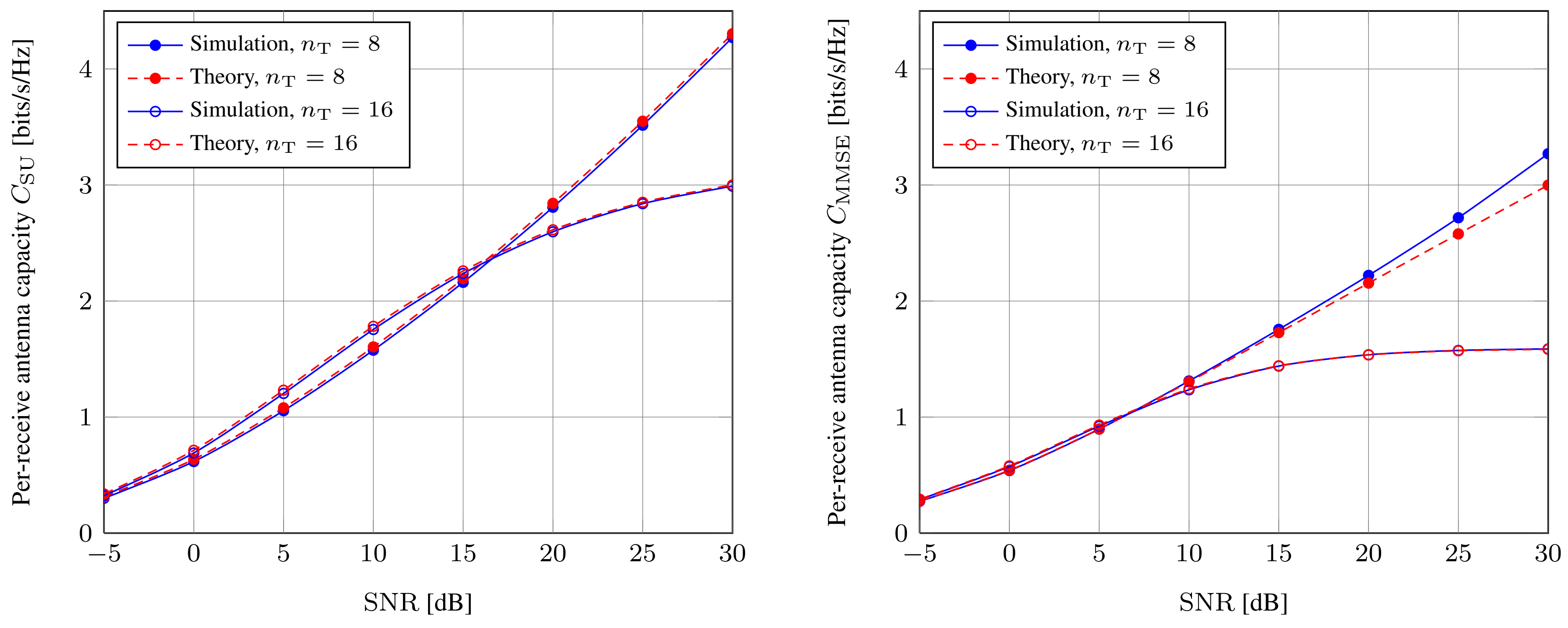

Figure: Capacity of point-to-point MIMO in two-cell downlink, single-user (left) and MMSE (right) decoding, $n_{\mathrm{R}} = 16$, $n_{\mathrm{T}} \in \{8, 16\}$, interferer strength 0.25.

## Conclusion

- deterministic equivalent of the capacity of multi-user correlated MIMO systems: SU decoding with power allocation and MMSE decoder.
- asymptotic independence of channel realization.

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