
A Preconditioned Power Method for Computing Stationary Vectors of Markov Chains, with Application to Internet Search Engines

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Overview

- Simple Web Model
- Power Method
- Exploiting Structure
- ILU Preconditioned Power Method
- Google Matrix

Simple Web Model

Determine importance of a web page

PageRank of page:

Probability that surfer visits page

Page i has d outgoing links

If page i has no link to page j then $p_{ij} = 0$
else $p_{ij} = 1/d$

P is stochastic matrix

$$0 \leq p_{ij} \leq 1 \quad P\mathbf{1} = \mathbf{1}$$

Eigenvectors of P

$P\mathbf{1} = \mathbf{1} \Rightarrow P$ has eigenvalue 1

left eigenvector: $\pi^T P = \pi^T, \pi \geq 0, \|\pi\|_1 = 1$

i th entry of π : probability that surfer visits page i
PageRank of page i

PageRank \doteq largest left eigenvector of P

Power Method

Initial vector $v_{(0)} > 0$, $\|v_{(0)}\|_1 = 1$

After k iterations: $v_{(k)}^T \leftarrow v_{(0)}^T P^k$

If P is primitive then:

$\pi^T P = \pi^T$ with $\pi > 0$, $\|\pi\|_1 = 1$, unique

Power method converges: $v_{(k)} \rightarrow \pi$

Convergence rate $|\lambda_2| < 1$



Primitive Matrices

Matrix $P \geq 0$ is **primitive** if $P^m > 0$ for some $m \geq 1$

P is irreducible: $P \not\approx \begin{pmatrix} X & X \\ 0 & X \end{pmatrix}$

If P stochastic then eigenvalue 1 is distinct

$\rho(P)$ produced by single eigenvalue

If P stochastic then eigenvalues $\neq 1$ have magnitude < 1

$(P^m > 0 \Rightarrow \rho(P^m) \text{ simple})$

Power Method

Initial vector $v_{(0)} > 0$, $\|v_{(0)}\|_1 = 1$

After k iterations: $v_{(k)}^T \leftarrow v_{(0)}^T P^k$

If P is primitive then:

$\pi^T P = \pi^T$ with $\pi > 0$, $\|\pi\|_1 = 1$, unique

Power method converges: $v_{(k)} \rightarrow \pi$

Convergence rate $|\lambda_2| < 1$

Slow convergence if $|\lambda_2| \approx 1$

Can we exploit the stochastic structure of P ?

Exploiting Structure in P

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \quad I - P_{11} \text{ nonsingular}$$

Factor $I - P = LDU$

$$L = \begin{pmatrix} I & \\ * & I \end{pmatrix} \quad U = \begin{pmatrix} I & * \\ & I \end{pmatrix} \quad D = \begin{pmatrix} I - P_{11} & \\ & I - S \end{pmatrix}$$

where

$$S = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}$$

stochastic complement

LDU in Eigenvalue Problem

Use $I - P = LDU$:

$$\pi^T P = \pi^T \Leftrightarrow \pi^T (I - P) = 0 \Leftrightarrow \pi^T L D = 0$$

$$\begin{pmatrix} \pi_1^T & \pi_2^T \end{pmatrix} \begin{pmatrix} I - P_{11} & \\ -P_{21} & I - S \end{pmatrix} = 0$$

$$\pi_2^T (I - S) = 0 \quad \pi_1^T = \pi_2^T P_{21} (I - P_{11})^{-1}$$

π_2 is eigenvector for smaller matrix S

Smaller Eigenvalue Problem

$$S = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}$$

If P is stochastic and irreducible then

- S is stochastic and irreducible
- $\sigma^T S = \sigma^T$ with $\sigma > 0$, $\|\sigma\|_1 = 1$, unique

If S is primitive then

- Power method converges
- Convergence rate $|\lambda_2(S)| < 1$

Idea

- Exact: $\pi_2^T = \pi_2^T S$

$$\pi^T = \left(\pi_2^T P_{21} (I - P_{11})^{-1} \quad \pi_2^T S \right)$$

- Approximation: **Any** $\tilde{\pi}_2 > 0$

$$\tilde{\pi}^T = \left(\tilde{\pi}_2^T P_{21} (I - P_{11})^{-1} \quad \tilde{\pi}_2^T S \right)$$

- Repeat with: $\tilde{\pi}_2 := \tilde{\pi}_2^T S$

The Big Picture

$$\begin{aligned}\tilde{\pi}^T &= \left(\tilde{\pi}_2^T P_{21} (I - P_{11})^{-1} \quad \tilde{\pi}_2^T S \right) \\ &= \left(* \quad \tilde{\pi}_2^T \right) \begin{pmatrix} 0 & 0 \\ P_{21} (I - P_{11})^{-1} & S \end{pmatrix}\end{aligned}$$

$$\begin{pmatrix} 0 & 0 \\ P_{21} (I - P_{11})^{-1} & S \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} L^{-1} P$$

ILU Preconditioned Power method ($I - P = LDU$)

ILU Preconditioned Power Method

- Initial guess: $\pi_{(0)} > 0$, $\|\pi_{(0)}\| = 1$
- Iterate: $\pi_{(k+1)}^T = \rho_k \pi_{(k)}^T \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} L^{-1} P$
 $\|\pi_{(k+1)}\| = 1$

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} L^{-1} P = \begin{pmatrix} 0 & 0 \\ P_{21}(I - P_{11})^{-1} & S \end{pmatrix}$$

Convergence rate $|\lambda_2(S)|$

But . . .

- ILU preconditioned power method requires formation of $S = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}$ in

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} L^{-1}P = \begin{pmatrix} 0 & 0 \\ * & S \end{pmatrix}$$

- Does ILU preconditioned power method converge **faster** than power method?

$$\text{Is } |\lambda_2(S)| < |\lambda_2(P)|?$$

How To Avoid Forming S

$$\tilde{\pi}^T = \rho \left(* \quad \tau^T \right) \begin{pmatrix} 0 & 0 \\ P_{21}(I - P_{11})^{-1} & S \end{pmatrix} \quad \|\tilde{\pi}\|_1 = 1$$

is mathematically equivalent to:

$$\text{Form } A = \begin{pmatrix} P_{11} & P_{12}\mathbf{1} \\ \tau^T P_{21} & \tau^T P_{22}\mathbf{1} \end{pmatrix}$$

Compute $\alpha > 0$ where $\alpha^T A = \alpha^T$, $\|\alpha\|_1 = 1$

$$\text{Partition } \alpha = \left(\omega^T \quad \rho \right)$$

$$\text{Multiply } \tilde{\pi}^T = \left(\omega^T \quad \rho \tau^T \right) P$$

Connections

ILU Preconditioned power method \doteq Iterative Aggregation/Disaggregation

- IAD based on stochastic complementation:
Meyer 1989, Meyer 2004 (SIAM)
Langville & Meyer 2002, 2003, 2004
Billy Stewart 1985, 1992, 1994
Lee, Golub & Zenios 2003
- IAD based on general splittings:
Marek & Szyld 1994, Szyld 2004 (ILAS)
Marek & Mayer 1998, 2001, 2003
- Haveliwala, Kamvar, Klein, Manning & Golub 2003
Kamvar, Haveliwala & Golub 2004
Choi & Saunders (SIAM 2004)

Google Matrix

$$G = cP + (1 - c)\mathbf{1}v^T$$

P is stochastic matrix

$0 < c < 1$, $v > 0$ and $\|v\|_1 = 1$

- Power method for G **always** converges:

$$|\lambda_2(G)| \leq c < 1$$

- ILU preconditioned power method **can** converge **faster** than power method:

If $|\lambda_2(P)| = 1$ then $|\lambda_2(S_G)| \leq |\lambda_2(G)|$

Google Matrix

$$G = cP + (1 - c)\mathbf{1}v^T$$

Under reasonable assumptions:

$$QPQ^T = \begin{pmatrix} 0 & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad Q \text{ permutation}$$

Then:

$$|\lambda_2(S_G)| < |\lambda_2(G)|$$

ILU preconditioned power method **converges faster**
than power method

Summary

- Determine **PageRank** of web pages
- Compute eigenvector for eigenvalue 1 of stochastic matrix
- **ILU preconditioned power method**
- Implemented as:
Iterative Aggregation/Disaggregation method
- Converges **faster** than power method
for Google matrix