Chapter 1

Matrices

We review the basic matrix operations.

1.1 What is a Matrix?

An array of numbers

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

with m rows and n columns is a $m \times n$ matrix. Element a_{ij} in located in position (i, j). The elements a_{ij} are *scalars*, namely real or complex numbers. The set of real numbers is $\mathbb{R}$, and the set of complex numbers is $\mathbb{C}$.

We write $A \in \mathbb{R}^{m \times n}$ if A is a $m \times n$ matrix whose elements are real numbers, and $A \in \mathbb{C}^{m \times n}$ if A is a $m \times n$ matrix whose elements are complex numbers. Of course, $\mathbb{R}^{m \times n} \subset \mathbb{C}^{m \times n}$. If $m = n$ then we say that A is a *square matrix* of order n.

For instance

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

is a 2×4 matrix with elements $a_{13} = 3$ and $a_{24} = 8$.

Vectors. A *row vector* $y = \begin{pmatrix} y_1 & \dots & y_m \end{pmatrix}$ is a $1 \times m$ matrix, i.e. $y \in \mathbb{C}^{1 \times m}$. A *column vector*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a $n \times 1$ matrix, i.e. $x \in \mathbb{C}^{n \times 1}$ or shorter, $x \in \mathbb{C}^n$. If the elements of x are real then $x \in \mathbb{R}^n$.

Submatrices. Sometimes we need only those elements of a matrix that are situated in particular rows and columns.

Definition 1.1. *Let $A \in \mathbb{C}^{m\times n}$ have elements a_{ij}. If $1 \le i_1 < i_2 < \ldots < i_k \le m$ and $1 \le j_1 < j_2 < \ldots < j_l \le n$ then the $k \times l$ matrix*

$$\begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_l} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_1,j_l} \\ \vdots & \vdots & & \vdots \\ a_{i_k,j_1} & a_{i_k,j_2} & \cdots & a_{i_k,j_l} \end{pmatrix}$$

is called a submatrix *of A. The submatrix is a* principal submatrix *if it is square and its diagonal elements are diagonal elements of A, that is, $k = l$ and $i_1 = j_1, i_2 = j_2, \ldots, i_k = j_k$.*

Example. If

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

then the following are submatrices of A,

$$\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}, \qquad \begin{pmatrix} a_{21} & a_{23} \end{pmatrix} = \begin{pmatrix} 4 & 6 \end{pmatrix}, \qquad \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 6 \\ 8 & 9 \end{pmatrix}.$$

The submatrix

$$\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$$

is a principal matrix of A, as are the diagonal elements a_{11}, a_{22}, a_{33}, and A itself.
$\square$

Notation. Most of the time we will use following notation:

- Matrices: upper case Roman or Greek letters, e.g. A, Λ
- Vectors: lower case Roman letters, e.g. x, y
- Scalars: lower case Greek letters, e.g. α;
  or lower case Roman with subscripts, e.g. x_i, a_{ij}
- Running variables: i, j, k, l, m, and n.

The elements of the matrix A are called a_{ij} or A_{ij}, and the elements of the vector x are called x_i.

Zero Matrices. The *zero matrix* $0_{m\times n}$ is the $m \times n$ matrix all of whose elements are zero. When m and n are clear from the context, we also write 0. We say $A = 0$ if all elements of the matrix A are equal to zero. The matrix A is nonzero, $A \neq 0$, if at least one element of A is nonzero.

Identity Matrices. The *identity matrix* of order n is the real square matrix

$$I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix},$$

with ones on the diagonal and zeros everywhere else (instead of writing many zeros, we often write blanks). In particular, $I_1 = 1$. When n is clear from the context, we also write I.

The columns of the identity matrix are also called *canonical vectors* e_i. That is, $I_n = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix}$, where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \cdots, \qquad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Exercises

(i) Hilbert Matrix.
A square matrix of order n whose element in position (i,j) is $\frac{1}{i+j-1}$, $1 \le i,j \le n$, is called a *Hilbert* matrix.
Write down a Hilbert matrix for $n = 5$.

(ii) Toeplitz Matrix.
Given $2n-1$ scalars α_k, $-n+1 \le k \le n-1$, a matrix of order n whose element in position (i,j) is α_{j-i}, $1 \le i,j \le n$, is called a *Toeplitz* matrix.
Write down the Toeplitz matrix of order 3 when $\alpha_i = i$, $-2 \le i \le 2$.

(iii) Hankel Matrix.
Given $2n-1$ scalars α_k, $0 \le k \le 2n-2$, a matrix of order n whose element in position (i,j) is α_{i+j-2}, $1 \le i,j \le n$, is called a *Hankel* matrix.
Write down the Hankel matrix of order 4 for $\alpha_i = i$, $0 \le i \le 6$.

(iv) Vandermonde Matrix.
Given n scalars α_i, $1 \le i \le n$, a matrix of order n whose element in position (i,j) is α_i^{j-1}, $1 \le i,j \le n$, is called a *Vandermonde* matrix. Here we interpret $\alpha_i^0 = 1$ even for $\alpha_i = 0$. The numbers α_i are also called *nodes* of the Vandermonde matrix.
Write down the Vandermonde matrix of order 4 when $\alpha_i = i$, $1 \le i \le 3$, and $\alpha_4 = 0$.

(v) Is a square zero matrix a Hilbert, Toeplitz, Hankel or Vandermonde matrix?

(vi) Is the identity matrix a Hilbert, Toeplitz, Hankel or Vandermonde matrix?

(vii) Is a Hilbert matrix a Hankel matrix or a Toeplitz matrix?

1.2 Scalar Matrix Multiplication

Each element of the matrix is multiplied by a scalar. If $A \in \mathbb{C}^{m\times n}$ and λ a scalar, then the elements of the *scalar matrix product* $\lambda A \in \mathbb{C}^{m\times n}$ are

$$(\lambda A)_{ij} \equiv \lambda a_{ij}.$$

Multiplying the matrix $A \in \mathbb{C}^{m\times n}$ by the scalar zero produces a zero matrix,

$$0\, A = 0_{m\times n},$$

where the first zero is a scalar, while the second zero is a matrix with the same number of rows and columns as A. Scalar matrix multiplication is associative,

$$(\lambda\mu)\, A = \lambda\, (\mu A).$$

Scalar matrix multiplication by -1 corresponds to negation,

$$-A \equiv (-1)\, A.$$

Exercises

(i) Let $x \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$. Prove: $\alpha x = 0$ if and only if $\alpha = 0$ or $x = 0$.

1.3 Matrix Addition

Corresponding elements of two matrices are added. The matrices must have the same number of rows and the same number of columns. If A and $B \in \mathbb{C}^{m\times n}$ then the elements of the *sum* $A + B \in \mathbb{C}^{m\times n}$ are

$$(A + B)_{ij} \equiv a_{ij} + b_{ij}.$$

Properties of Matrix Addition.

- Adding the zero matrix does not change anything. That is, for any $m \times n$ matrix A,
$$0_{m\times n} + A = A + 0_{m\times n} = A.$$
- Matrix addition is commutative,
$$A + B = B + A.$$
- Matrix addition is associative,
$$(A + B) + C = A + (B + C).$$

- Matrix addition and scalar multiplication are distributive,

$$\lambda\,(A+B) = \lambda A + \lambda B, \qquad (\lambda+\mu)\,A = \lambda A + \mu A.$$

One can use the above properties to save computations. For instance, computing $\lambda A + \lambda B$ requires twice as many operations as computing $\lambda(A+B)$. In the special case $B = -C$ computing $(A+B)+C$ requires two matrix additions, while $A + (B+C) = A + 0 = A$ requires no work.

A special type of addition is the sum of scalar vector products.

Definition 1.2. *A* linear combination *of $m \geq 1$ column (or row) vectors $v_1, \ldots, v_m$ is*

$$\alpha_1 v_1 + \cdots + \alpha_m v_m,$$

where the scalars $\alpha_1, \ldots, \alpha_m$ are the coefficients.

Example. Any vector in $\mathbb{R}^n$ or $\mathbb{C}^n$ can be represented as a linear combination of canonical vectors,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n.$$

□

1.4 Inner Product (Dot Product)

The product of a row vector times an equally long column vector produces a single number. If

$$x = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

then the *inner product* of x and y is the scalar

$$xy = x_1 y_1 + \cdots + x_n y_n.$$

Example. A sum of n scalars a_i, $1 \leq i \leq n$, can be represented as an inner product of two vectors with n elements each,

$$\sum_{j=1}^{n} a_j = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

□

Example. A polynomial $p(\alpha) = \sum_{j=0}^{n} \lambda_j \alpha^j$ of degree n can be represented as an inner product of two vectors with $n+1$ elements each,

$$p(\alpha) = \begin{pmatrix} 1 & \alpha & \dots & \alpha^n \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^n \end{pmatrix}.$$

□

Exercises

(i) Let $n \geq 1$ be an integer. Represent $n(n+1)/2$ as an inner product of two vectors with n elements each.

1.5 Matrix Vector Multiplication

The product of a matrix and a vector is again a vector. There are two types of matrix vector multiplications: matrix times column vector, and row vector times matrix.

Matrix Times Column Vector. The product of matrix times column vector is again a column vector. We present two ways to describe the operations that are involved in a matrix vector product. Let $A \in \mathbb{C}^{m \times n}$ with rows r_j and columns c_j, and $x \in \mathbb{C}^n$ with elements x_j,

$$A = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

1. View: Ax is a column vector of inner products, so that element j of Ax is the inner product of row r_j with x,

$$Ax = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

2. View: Ax is a linear combination of columns

$$Ax = c_1 x_1 + \cdots + c_n x_n.$$

The vectors in the linear combination are the columns c_j of A, and the coefficients are the elements x_j of x.

Example. Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}.$$

The first view shows that Ae_2 is equal to column 2 of A. That is,

$$Ae_2 = 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

The second view shows that the first and second elements of Ae_2 are equal to zero. That is,

$$Ax = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} e_2 \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} e_2 \\ \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

□

Example. Let A be the Toeplitz matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

The first view shows that the last element of Ax is equal to zero. That is,

$$Ax = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix} x = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ 0 \end{pmatrix}.$$

□

Row Vector Times Matrix. The product of a row vector times a matrix is a row vector. There are again two ways to think about this operation. Let $A \in \mathbb{C}^{m \times n}$ with rows r_j and columns c_j, and $y \in \mathbb{C}^{1 \times m}$ with elements y_j,

$$A = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 & \dots & y_m \end{pmatrix}.$$

View 1: yA is a row vector of inner products, where element j of yA is an inner product of y with the column c_j,

$$yA = \begin{pmatrix} yc_1 & \dots & yc_n \end{pmatrix}.$$

View 2: yA is a linear combination of rows of A,

$$yA = y_1 r_1 + \cdots + y_m r_m.$$

The vectors in the linear combination are the rows r_j of A, and the coefficients are the elements y_j of y.

Exercises

(i) Show that Ae_j is the jth column of the matrix A.

(ii) Let A be a $m \times n$ matrix and e the $n \times 1$ vector of all ones. What does Ae do?

(iii) Let $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$ a linear combination of vectors $v_1, \ldots, v_m$. Prove: If one of the coefficients α_j is non-zero then one of the vectors can be represented as a linear combination of the other vectors.

1. Let $A, B \in \mathbb{C}^{m \times n}$. Prove: $A = B$ if and only if $Ax = Bx$ for all $x \in \mathbb{C}^n$.

1.6 Outer Product

The product of a column vector times a row vector gives a matrix (this is not to be confused with an inner product which produces a single number). If

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 & \ldots & y_n \end{pmatrix},$$

then the *outer product* of x and y is the $m \times n$ matrix

$$xy = \begin{pmatrix} x_1 y_1 & \ldots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \ldots & x_m y_n \end{pmatrix}.$$

The vectors in an outer product are allowed to have different lengths. The columns of xy are multiples of each other, and so are the rows. That is, each column of xy is a multiple of x,

$$xy = \begin{pmatrix} xy_1 & \ldots & xy_n \end{pmatrix},$$

and each row of xy is a multiple of y,

$$xy = \begin{pmatrix} x_1 y \\ \vdots \\ x_m y \end{pmatrix}.$$

Example. A Vandermonde matrix of order n all of whose nodes are the same, e.g. equal to α, can be represented as the outer product

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha & \dots & \alpha^{n-1} \end{pmatrix}.$$

□

Exercises

(i) Write the matrix below as an outer product,

$$\begin{pmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{pmatrix}.$$

1.7 Matrix Multiplication

The product of two matrices A and B is defined if the number of columns in A is equal to the number of rows in B. Specifically, if $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ is $AB \in \mathbb{C}^{m\times p}$. We can describe matrix multiplication in four different ways. Let $A \in \mathbb{C}^{m\times n}$ with rows a_j, and $B \in \mathbb{C}^{n\times p}$ with columns b_j,

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & \dots & b_p \end{pmatrix}.$$

View 1: AB is a block row vector of matrix vector products. The columns of AB are matrix vector products of A with columns of B,

$$AB = \begin{pmatrix} Ab_1 & \dots & Ab_p \end{pmatrix}.$$

View 2: AB is a block column vector of matrix vector products, where the rows of AB are matrix vector products of the rows of A with B.

$$AB = \begin{pmatrix} a_1 B \\ \vdots \\ a_m B \end{pmatrix}.$$

View 3: The elements of AB are inner products, where element (i,j) of AB is an inner product of row i of A with column j of B,

$$(AB)_{ij} = a_i b_j, \qquad 1 \le i \le m, \quad 1 \le j \le p.$$

View 4: If we denote by c_i the columns of A and r_i the rows of B,

$$A = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}, \qquad B = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix},$$

then AB is a sum of outer products $AB = c_1 r_1 + \ldots + c_n r_n$.

Properties of Matrix Multiplication.

- Multiplying by the identity matrix does not change anything. That is, for a $m \times n$ matrix A
$$I_m\, A = A\, I_n = A.$$
- Matrix multiplication is associative,
$$A\,(BC) = (AB)\,C.$$
- Matrix multiplication and addition are distributive,
$$A\,(B + C) = AB + AC, \qquad (A + B)\,C = AC + BC.$$
- Matrix multiplication is **not** commutative.
For instance, if
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$
then
$$AB = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = BA.$$

Example. Associativity can save work. If

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

then multiplying

$$(AB)\,C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \\ 5 & 10 & 15 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

requires more operations than

$$A\,(BC) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \cdot 10.$$

□

Warning. Don't misuse associativity. For instance, if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 5 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

it looks as if we could compute

$$A\,(BC) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 5 & 5 \end{pmatrix} \cdot 10.$$

However the product ABC is not defined because AB is not defined (here we have to view BC as a 1×1 matrix rather than just a scalar). In a product ABC, all adjacent products AB and BC have to be defined. Hence the above option $A\,(BC)$ is not defined either.

Matrix Powers. A special case of matrix multiplication is the repeated multiplication of a square matrix by itself. If A is a non-zero square matrix we define $A^0 \equiv I$, and for any integer $k > 0$

$$A^k = \overbrace{A \ldots A}^{k \text{ times}} = A^{k-1}\,A = A\,A^{k-1}.$$

Definition 1.3. *A square matrix is*

- *involutory if $A^2 = I$,*
- *idempotent (or: a projector) if $A^2 = A$,*
- *nilpotent if $A^k = 0$ for some integer $k > 0$*

Example. For any scalar α

$$\begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} \quad \text{is involutory,}$$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} \quad \text{is idempotent,}$$

and

$$\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \quad \text{is nilpotent}$$

. □

Exercises

(i) Which is the only matrix that is both idempotent and involutory?

(ii) Which is the only matrix that is both idempotent and nilpotent?

(iii) Let $x \in \mathbb{C}^{n\times 1}$, $y \in \mathbb{C}^{1\times n}$. When is xy idempotent? When is it nilpotent?

(iv) Prove: If A is idempotent, then $I - A$ is also idempotent.

(v) Prove: If A and B are idempotent, and $AB = BA$, then AB is also idempotent.

(vi) Prove: A is involutory if and only if $(I - A)(I + A) = 0$.

(vii) Prove: If A is involutory and $B = \frac{1}{2}(I + A)$ then B is idempotent.

(viii) Let $x \in \mathbb{C}^{n\times 1}$, $y \in \mathbb{C}^{1\times n}$. Compute $(xy)^3 x$ using only inner products and scalar multiplication.

1. Fast Matrix Multiplication.
   One can multiply two complex numbers with only three real multiplications instead of four. Let $\alpha = \alpha_1 + \imath\alpha_2$ and $\beta = \beta_1 + \imath\beta_2$ be two complex numbers, where $\imath^2 = -1$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Writing

   $$\alpha\beta = \alpha_1\beta_1 - \alpha_2\beta_2 + \imath\left[(\alpha_1 + \alpha_2)(\beta_1 + \beta_2) - \alpha_1\beta_1 - \alpha_2\beta_2\right]$$

   shows that the complex product $\alpha\beta$ can be computed with three real multiplications: $\alpha_1\beta_1$, $\alpha_2\beta_2$ and $(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)$.
   Show that this approach can be extended to the multiplication AB of two complex matrices $A = A_1 + \imath A_2$ and $B = B_1 + \imath B_2$, where $A_1, A_2 \in \mathbb{R}^{m\times n}$ and $B_1, B_2 \in \mathbb{R}^{n\times p}$. In particular, show that no commutativity laws are violated.

1.8 Conjugate Transpose and Transpose

Transposing a matrix amounts to turning rows into columns and vice versa. If

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

then its transpose $A^T \in \mathbb{C}^{n\times m}$ is obtained by converting rows to columns

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

There is a second type of transposition that requires more work when the matrix elements are complex numbers. A complex number α is written $\alpha = \alpha_1 + \imath\alpha_2$, where $\imath^2 = -1$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. The complex conjugate of the scalar α is $\overline{\alpha} \equiv \alpha_1 - \imath\alpha_2$.

If $A \in \mathbb{C}^{m\times n}$ is a matrix its conjugate transpose $A^* \in \mathbb{C}^{n\times m}$ is obtained by converting rows to columns and, in addition, taking the complex conjugates of the elements,

$$A^* = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{21} & \dots & \overline{a}_{m1} \\ \overline{a}_{12} & \overline{a}_{22} & \dots & \overline{a}_{m2} \\ \vdots & \vdots & & \vdots \\ \overline{a}_{1n} & \overline{a}_{2n} & \dots & \overline{a}_{mn} \end{pmatrix}.$$

Example. If

$$A = \begin{pmatrix} 1+2\imath & 5 \\ 3-\imath & 6 \end{pmatrix}$$

then

$$A^T = \begin{pmatrix} 1+2\imath & 3-\imath \\ 5 & 6 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 1-2\imath & 3+\imath \\ 5 & 6 \end{pmatrix}.$$

□

Example. We can express the rows of the identity matrix in terms of canonical vectors,

$$I_n = \begin{pmatrix} e_1^T \\ \vdots \\ e_n^T \end{pmatrix} = \begin{pmatrix} e_1^* \\ \vdots \\ e_n^* \end{pmatrix}.$$

□

Fact 1.4 (Properties of Transposition)

- For real matrices, the conjugate transpose and the transpose are identical. That is, if $A \in \mathbb{R}^{m\times n}$ then $A^* = A^T$.
- Transposing a matrix twice gives back the original,

$$(A^T)^T = A, \qquad (A^*)^* = A.$$

- Transposition does not affect a scalar, while conjugate transposition conjugates the scalar,

$$(\lambda A)^T = \lambda A^T, \qquad (\lambda A)^* = \overline{\lambda} A^*,$$

- The transpose of a sum is the sum of the transposes,

$$(A+B)^T = A^T + B^T, \qquad (A+B)^* = A^* + B^*.$$

- The transpose of a product is the product of the transposes with the factors in reverse order,

$$(AB)^T = B^T A^T, \qquad (AB)^* = B^* A^*.$$

Example. Why do we have to reverse the order of the factors when the transpose is pulled inside the product AB? Why isn't $(AB)^T = A^T B^T$? One of the reasons is that one of the products may not defined. If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then

$$(AB)^T = \begin{pmatrix} 2 & 2 \end{pmatrix}$$

while the product $A^T B^T$ is not defined. $\square$

Exercises

(i) Let A be a $n \times n$ matrix, and let Z be the matrix with $z_{j,j+1} = 1$, $1 \leq j \leq n-1$, and all other elements zero. What does ZAZ^T do?

1.9 Inner and Outer Products, Again

Transposition comes in handy for the representation of inner and outer products. If

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

then

$$x^* y = \overline{x}_1 y_1 + \cdots + \overline{x}_n y_n, \qquad y^* x = \overline{y}_1 x_1 + \cdots + \overline{y}_n x_n.$$

Example. Let $\alpha = \alpha_1 + \imath \alpha_2$ be a complex number, where $\imath^2 = -1$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. With

$$x \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

the absolute value of α can be represented as the inner product, $|\alpha| = \sqrt{x^* x}$. $\square$

Fact 1.5 (Properties of Inner Products) Let $x, y \in \mathbb{C}^n$.

1. $y^* x$ is the complex conjugate of $x^* y$, i.e. $y^* x = \overline{(x^* y)}$
2. $y^T x = x^T y$
3. $x^* x = 0$ if and only if $x = 0$.
4. If x is real then: $x^T x = 0$ if and only if $x = 0$.

Proof. Let $x = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}^T$ and $y = \begin{pmatrix} y_1 & \ldots & y_n \end{pmatrix}^T$. For the first equality write $y^* x = \sum_{j=1}^n \overline{y}_j x_j = \sum_{j=1}^n x_j \overline{y}_j$. Since complex conjugating twice gives back the original, we get $\sum_{j=1}^{n} x_j\overline{y}_j = \sum_{j=1}^{n} \overline{\overline{x}}_j\overline{y}_j = \sum_{j=1}^{n} \overline{\overline{x}_j y_j} = \overline{\sum_{j=1}^{n} \overline{x}_j y_j} = \overline{x^* y}$, where the long overbar denotes complex conjugation over the whole sum.

As for the third statement, $0 = x^*x = \sum_{j=1}^{n} \overline{x}_j x_j = \sum_{j=1}^{n} |x_j|^2$ if and only $x_j = 0$, $1 \leq j \leq n$, if and only if $x = 0$. $\square$

Example. The identity matrix can be represented as the outer product

$$I_n = e_1 e_1^T + e_2 e_2^T + \cdots + e_n e_n^T.$$

$\square$

Exercises

(i) Let x be a column vector. Give an example to show that $x^T x = 0$ can happen for $x \neq 0$.

(ii) Let $x \in \mathbb{C}^n$ and $x^*x = 1$. Show that $I_n - 2xx^*$ is involutory.

(iii) Let A be the square matrix with $a_{j,j+1} = 1$ and all other elements zero. Represent A as a sum of outer products.

1.10 Symmetric and Hermitian Matrices

We look at matrices that remain unchanged by transposition.

Definition 1.6. *A matrix $A \in \mathbb{C}^{n\times n}$ is:*

- *symmetric if $A^T = A$,*
- *Hermitian if $A^* = A$,*
- *skew-symmetric if $A^T = -A$,*
- *skew-Hermitian if $A^* = -A$.*

The identity matrix I_n is symmetric and Hermitian. The square zero matrix $0_{n\times n}$ is symmetric, skew-symmetric, Hermitian, and skew-Hermitian.

Example. Let $\imath^2 = -1$.

$$\begin{pmatrix} 1\imath & 2\imath \\ 2\imath & 4 \end{pmatrix} \text{ is symmetric,} \qquad \begin{pmatrix} 1 & 2\imath \\ -2\imath & 4 \end{pmatrix} \text{ is Hermitian}$$

$$\begin{pmatrix} 0 & 2\imath \\ -2\imath & 0 \end{pmatrix} \text{ is skew} - \text{symmetric,} \qquad \begin{pmatrix} 1\imath & 2\imath \\ 2\imath & 4\imath \end{pmatrix} \text{ is skew} - \text{Hermitian.}$$

$\square$

Example. Let $\imath^2 = -1$.

$$\begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}$$

is symmetric and skew-Hermitian, while

$$\begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}$$

is Hermitian and skew-symmetric. $\square$

Fact 1.7 If $A \in \mathbb{C}^{m\times n}$ then AA^T and A^TA are symmetric, while AA^* and A^*A are Hermitian.

If $A \in \mathbb{C}^{n\times n}$ then $A + A^T$ is symmetric, and $A + A^*$ is Hermitian.

Exercises

(i) Is a Hankel matrix symmetric, Hermitian, skew-symmetric or skew-Hermitian?
(ii) Which matrix is both symmetric and skew-symmetric?
(iii) Prove: If A is a square matrix then $A - A^T$ is skew-symmetric and $A - A^*$ is skew-Hermitian.
(iv) Which elements of a Hermitian matrix cannot be complex?
(v) What can you say about the diagonal elements of a skew-symmetric matrix?
(vi) What can you say about the diagonal elements of a skew-Hermitian matrix?
(vii) If A is symmetric and λ is a scalar, does this imply that λA is symmetric? If yes, give a proof. If no, give an example.
(viii) If A is Hermitian and λ is a scalar, does this imply that λA is Hermitian? If yes, give a proof. If no, give an example.
(ix) Prove: If A is skew-symmetric and λ is a scalar then λA is skew-symmetric.
(x) Prove: If A is skew-Hermitian and λ is a scalar then λA is, in general, not skew-Hermitian.
(xi) Prove: If A is Hermitian then $\imath A$ is skew-Hermitian, where $\imath^2 = -1$.
(xii) Prove: If A is skew-Hermitian then $\imath A$ is Hermitian, where $\imath^2 = -1$.
(xiii) Prove: If A is a square matrix then $\imath(A - A^*)$ is Hermitian, where $\imath^2 = -1$.

1. Prove: Every square matrix A can be written $A = A_1 + A_2$, where A_1 is symmetric and A_2 is skew-symmetric.
2. Prove: Every square matrix A can be written $A = A_1 + \imath A_2$, where A_1 and A_2 are Hermitian and $\imath^2 = -1$.

1.11 Inverse

We want to determine an inverse with respect to matrix multiplication. Inversion of matrices is more complicated than inversion of scalars. There is only one scalar that does not have an inverse: 0, while there are many matrices without inverses.

Definition 1.8. *A matrix $A \in \mathbb{C}^{n\times n}$ is* nonsingular *(or* invertible*) if A has an* inverse*; that is, if there is a matrix A^{-1} so that $AA^{-1} = I = A^{-1}A$. If A does not have an inverse, it is* singular.

Example.

- A 1×1 matrix is invertible if it is non-zero.
- An involutory matrix is its own inverse: $A^2 = I$.

□

Fact 1.9 The inverse is unique.

Proof. Let $A \in \mathbb{C}^{n\times n}$, and $AB = BA = I_n$ and $AC = CA = I_n$ for matrices $B, C \in \mathbb{C}^{n\times n}$. Then

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

□

It is often easier to determine that a matrix is singular than it is to determine that a matrix is nonsingular. The fact below illustrates this.

Fact 1.10 Let $A \in \mathbb{C}^{n\times n}$ and $x, b \in \mathbb{C}^n$.

- If $x \neq 0$ and $Ax = 0$ then A is singular.
- If $x \neq 0$ and A is nonsingular then $Ax \neq 0$.
- If $Ax = b$ where A is nonsingular and $b \neq 0$ then $x \neq 0$.

Proof. To prove the first statement, assume to the contrary that A is nonsingular and has an inverse A^{-1}. Then $0 = Ax$ implies $0 = A^{-1}Ax = I_nx = x$, hence $x = 0$, which contradicts the assumption $x \neq 0$. Therefore A must be singular.

The proofs for the other two statements are similar. □

Fact 1.11 An idempotent matrix is the identity or it is singular.

Proof. If A is idempotent then $A^2 = A$. Hence $0 = A^2 - A = A(A - I)$. Either $I - A = 0$, in which case A is the identity; or else $I - A \neq 0$, in which case it has a nonzero column and Fact 1.10 implies that A is singular. □

Now we show that inversion and transposition can be exchanged.

Fact 1.12 If A is invertible then A^T and A^* are also invertible, and

$$(A^*)^{-1} = (A^{-1})^*, \qquad (A^T)^{-1} = (A^{-1})^T.$$

Proof. Show that $(A^{-1})^*$ fulfills the conditions for an inverse of A^*:

$$A^*(A^{-1})^* = (A^{-1}A)^* = I^* = I,$$

and

$$(A^{-1})^*A^* = (AA^{-1})^* = I^* = I.$$

The proof for A^T is similar. $\square$

Because inverse and transpose can be exchanged, we can simply write A^{-*} and A^{-T}.

The expression below is useful because it can break apart the inverse of a sum.

Fact 1.13 (Sherman-Morrison Formula) If $A \in \mathbb{C}^{n\times n}$ is nonsingular, and $V \in \mathbb{C}^{m\times n}$, $U \in \mathbb{C}^{n\times m}$ are such that $I + VA^{-1}U$ is nonsingular, then

$$(A + UV)^{-1} = A^{-1} - A^{-1}U\left(I + VA^{-1}U\right)^{-1}VA^{-1}$$

Here is an explicit expression for the inverse of a partitioned matrix.

Fact 1.14 Let $A \in \mathbb{C}^{n\times n}$ and

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

If A_{11} and A_{22} are nonsingular then

$$A^{-1} = \begin{pmatrix} S_1^{-1} & -A_{11}^{-1}A_{12}S_2^{-1} \\ -A_{22}^{-1}A_{21}S_1^{-1} & S_2^{-1} \end{pmatrix},$$

where $S_1 = A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $S_2 = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Matrices of the form S_1 and S_2 are called *Schur complements*.

Exercises

(i) Prove: If A and B are invertible then $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) Prove: If $A, B \in \mathbb{C}^{n\times n}$ are nonsingular then $B^{-1} = A^{-1} - B^{-1}(B - A)A^{-1}$.

(iii) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$ be such that $I + BA$ is invertible. Show that $(I + BA)^{-1} = I - B(I + AB)^{-1}A$.

(iv) Let $A \in \mathbb{C}^{n\times n}$ be nonsingular, $u \in \mathbb{C}^{n\times 1}$, $v \in \mathbb{C}^{1\times n}$ and $vA^{-1}u \neq -1$. Show that

$$(A + uv)^{-1} = A^{-1} - \frac{A^{-1}uvA^{-1}}{1 + vA^{-1}u}.$$

(v) The following expression for the partitioned inverse requires only A_{11} to be nonsingular but not A_{22}.

Let $A \in \mathbb{C}^{n\times n}$ and

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Show: If A_{11} is nonsingular and $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ then

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{pmatrix}.$$

(vi) Let $x \in \mathbb{C}^{1\times n}$ and $A \in \mathbb{C}^{n\times n}$. If $x \neq 0$ and $xA = 0$ then A is singular.

(vii) Prove: The inverse, if it exists, of a Hermitian (symmetric) matrix is also Hermitian (symmetric).

(viii) Prove: If A is involutory then $I - A$ or $I + A$ must be singular.

(ix) Let A be a square matrix so that $A + A^2 = I$. Prove: A is invertible.

(x) Prove: A nilpotent matrix is always singular.

1. Let $S \in \mathbb{R}^{n\times n}$. Show: If S is skew-symmetric then $I - S$ is nonsingular. Give an example to illustrate that $I - S$ can be singular if $S \in \mathbb{C}^{n\times n}$.
2. Let x be a non-zero column vector. Determine a row vector y so that $yx = 1$.
3. Let A be a square matrix and α_j are scalars, at least two of them nonzero, such that $\sum_{j=0}^{k} \alpha_j A^j = 0$. Prove: If $\alpha_0 \neq 0$ then A is nonsingular.
4. Prove: If $(I - A)^{-1} = \sum_{j=0}^{k} A^j$ for some integer $k \geq 0$ then A is nilpotent.
5. Let $A, B \in \mathbb{C}^{n\times n}$. Prove: If $I+BA$ is invertible, then $I+AB$ is also invertible.

1.12 Unitary and Orthogonal Matrices

These are matrices whose inverse is a transpose.

Definition 1.15. *A matrix $A \in \mathbb{C}^{n\times n}$ is*

- *unitary if $AA^* = A^*A = I$,*
- *orthogonal if $AA^T = A^TA = I$.*

The identity matrix is orthogonal as well as unitary.

Example 1.16 Let c and s be scalars with $|c|^2 + |s|^2 = 1$. The matrices

$$\begin{pmatrix} c & s \\ -\bar{s} & \bar{c} \end{pmatrix}, \qquad \begin{pmatrix} c & s \\ \bar{s} & -\bar{c} \end{pmatrix}$$

are unitary. ∎

The first matrix above gets its own name.

Definition 1.17. *If $c, s \in \mathbb{C}$ so that $|c|^2 + |s|^2 = 1$ then the unitary 2×2 matrix*

$$\begin{pmatrix} c & s \\ -\bar{s} & \bar{c} \end{pmatrix}$$

is called a Givens rotation*. If c and s are also real then the Givens rotation* $\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ *is orthogonal.*

When a Givens rotation is real, then both diagonal elements are the same. When a Givens rotation is complex, then the diagonal elements are complex conjugates of each other. A unitary matrix of the form

$$\begin{pmatrix} -c & s \\ \overline{s} & \overline{c} \end{pmatrix}$$

where the real parts of the diagonal elements have different signs is a *reflection*; it is *not* a Givens rotation.

An orthogonal matrix that can reorder the rows or columns of a matrix is called a *permutation matrix*. It is an identity matrix whose rows have been reordered (permuted). One can also think of a permutation matrix as an identity matrix whose columns have been reordered. Here is the official definition.

Definition 1.18 (Permutation Matrix). *A square matrix is a* permutation matrix *if it contains a single one in each column and in each row, and zeros everywhere else.*

Example. The following are permutation matrices:

- The identity matrix I.
- The exchange matrix

$$J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}, \qquad J \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_n \\ \vdots \\ x_2 \\ x_1 \end{pmatrix}.$$

- The upper circular shift matrix

$$Z = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}, \qquad Z \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

□

Fact 1.19 (Properties of Permutation Matrices)

1. Permutation matrices are orthogonal and unitary.
   That is, if P is a permutation matrix then $PP^T = P^TP = PP^* = P^*P = I$.
2. The product of permutation matrices is again a permutation matrix.

Exercises

(i) Prove: If A is unitary, then A^*, A^T and $\overline{A}$ are unitary.

(ii) What can you say about an involutory matrix that is also unitary (orthogonal)?

(iii) Which idempotent matrix is unitary and orthogonal?

(iv) Prove: If A is unitary, so is $\imath A$, where $\imath^2 = -1$.

(v) Prove: The product of unitary matrices is unitary.

(vi) Partitioned Unitary Matrices.
Let $A \in \mathbb{C}^{n\times n}$ be unitary and partition $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$, where A_1 has k columns, and A_2 has $n-k$ columns. Show that $A_1^*A_1 = I_k$, $A_2^*A_2 = I_{n-k}$, and $A_1^*A_2 = 0$.

(vii) Let $x \in \mathbb{C}^n$ and $x^*x = 1$. Prove: $I_n - 2xx^*$ is Hermitian and unitary. Conclude that $I_n - 2xx^*$ is involutory.

(viii) Show: If P is a permutation matrix then P^T and P^* are also a permutation matrices.

(ix) Show: If $\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ is a permutation matrix, then $\begin{pmatrix} P_2 & P_1 \end{pmatrix}$ is also a permutation matrix.

1.13 Triangular Matrices

Triangular matrices occur frequently during the solution of systems of linear equations, because linear systems with triangular matrices are easy to solve.

Definition 1.20. *A matrix $A \in \mathbb{C}^{n\times n}$ is* upper triangular *if $a_{ij} = 0$ for $i > j$. That is,*

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{pmatrix}.$$

A matrix $A \in \mathbb{C}^{n\times n}$ is lower triangular *if A^T is upper triangular.*

Fact 1.21 Let A and B be upper triangular, with diagonal elements a_{jj} and b_{jj}, respectively.

- $A + B$ and AB are upper triangular.
- The diagonal elements of AB are $a_{ii}b_{ii}$.
- If $a_{jj} \neq 0$ for all j then A is invertible, and the diagonal elements of A^{-1} are $1/a_{jj}$.

Definition 1.22. *A triangular matrix A is* unit triangular *if it has ones on the diagonal, and* strictly triangular *if it has zeros on the diagonal.*

Example. The identity matrix is unit upper triangular and unit lower triangular. The square zero matrix is strictly lower triangular and strictly upper triangular.

□

Exercises

(i) What does an idempotent triangular matrix look like? What does an involutory triangular matrix look like?

1. Prove: If A is unit triangular then A is invertible, and A^{-1} is unit triangular. If A and B are unit triangular then so is the product AB.
2. Show that a strictly triangular matrix is nilpotent.
3. Explain why the matrix $I - \alpha e_i e_j^T$ is triangular. When does it have an inverse? Determine the inverse in those cases where it exists.
4. Prove:

$$\begin{pmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^n \\ & 1 & \alpha & \ddots & \vdots \\ & & \ddots & \ddots & \alpha^2 \\ & & & 1 & \alpha \\ & & & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\alpha & & & \\ & 1 & -\alpha & & \\ & & \ddots & \ddots & \\ & & & 1 & -\alpha \\ & & & & 1 \end{pmatrix}.$$

5. Uniqueness of LU Factorization.
   Let L_1, L_2 be unit lower triangular, and U_1, U_2 nonsingular upper triangular. Prove: If $L_1 U_1 = L_2 U_2$ then $L_1 = L_2$ and $U_1 = U_2$.
6. Uniqueness of QR Factorization.
   Let Q_1, Q_2 be unitary (or orthogonal), and R_1, R_2 upper triangular with positive diagonal elements. Prove: If $Q_1 R_1 = Q_2 R_2$ then $Q_1 = Q_2$ and $R_1 = R_2$.

1.14 Diagonal Matrices

Diagonal matrices are special cases of triangular matrices; they are upper and lower triangular at the same time.

Definition 1.23. *A matrix $A \in \mathbb{C}^{n\times n}$ is* diagonal *if $a_{ij} = 0$ for $i \neq j$. That is,*

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}.$$

The identity matrix and the square zero matrix are diagonal.

Exercises

(i) Prove: Diagonal matrices are symmetric. Are they also Hermitian?

(ii) Diagonal matrices commute.
Prove: If A and B are diagonal, then AB is diagonal, and $AB = BA$.

(iii) Represent a diagonal matrix as a sum of outer products.

(iv) Which diagonal matrices are involutory, or idempotent, or nilpotent?

(v) Prove: If a matrix is unitary and triangular, it must be diagonal. What are its diagonal elements?

1. Let D be a diagonal matrix. Prove: If $D = (I + A)^{-1}A$ then A is diagonal.